The structure of a DFT algebroid

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General motivation

- String geometry departs from Riemannian geometry, notably in presence of fluxes
  - open strings $\rightsquigarrow$ noncommutativity - Poisson structure - $\star$-product - Kontsevich '97 DQ
    Chu, Ho '99; Seiberg, Witten '99
  - closed strings $\rightsquigarrow$ noncommutativity/nonassociativity - (twisted) Poisson - $\star$-product
    Lüst '10; Blumenhagen, Plauschinn '10; Mylonas, Schupp, Szabo '12; & c.
- Dualities relate different geometries/topologies $\rightsquigarrow$ "non-geometric backgrounds"
- Manifestly duality-invariant theories - double and exceptional field theories
  Hull, Hohm, Zwiebach; Hohm, Samtleben; & c.
- Evidence that the correct language is algebroid/generalized geometry
  Courant; Liu, Weinstein, Xu, Ševera; Roytenberg; Hitchin; Gualtieri; Cavalcanti; Bouwknegt, Hannabuss, Mathai; & c.

Generalized Geometries and Double Field Theory

- Courant Algebroids and Generalized Geometry double the bundle, e.g. $TM \oplus T^*M$
- DFT doubles the base, $\mathcal{M} = M \times \widetilde{M}$ — comes with constraints

- Solving the strong constraint, reduces DFT data to the data of the standard CA
- What is the geometric origin of the DFT data and the strong constraint?
  cf. also Deser, Stasheff '14; Deser, Saemann '16

- CAs provide membrane sigma models $\rightsquigarrow$ describe non-geometric backgrounds
  Roytenberg '06
  Mylonas, Schupp, Szabo '12; ACh, Jonke, Lechtenfeld '15; Bessho, Heller, Ikeda, Watamura '15
- Is there a "DFT algebroid" that could provide a DFT membrane sigma model?
  see talk by Jonke

Basic DFT data

Hohm, Hull, Zwiebach '10

- Doubled coordinates (all fields depend on both) and derivatives

$$(x^I) = (x^i, \tilde{x}_i), \quad (\partial_I) = (\partial_i, \tilde{\partial}^i) , \quad i = 1, \ldots, d, \quad I = 1, \ldots, 2d$$

- $O(d,d)$ structure/(constant) $O(d,d)$-invariant metric — generalised metric

$$\eta = (\eta_{IJ}) = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} , \quad \mathcal{H}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik}g^{kl}B_{lj} & B_{ik}g^{kj} \\ -g^{ik}B_{kj} & g^{ij} \end{pmatrix} .$$

(or a gen. vielbein $\mathcal{H}_{IJ} = \mathcal{E}^A{}_I \mathcal{E}^B{}_J S_{AB}$. Siegel '93; Hohm, Kwak '10; Aldazabal et al. '11; Geissbuhler '11)

- The gauge transformation of $\mathcal{H}$ is given by the generalized Lie derivative

$$\delta_\epsilon \mathcal{H}^{IJ} \quad = \quad \epsilon^K \partial_K \mathcal{H}^{IJ} + (\partial^I \epsilon_K - \partial_K \epsilon^I)\mathcal{H}^{KJ} + (\partial^J \epsilon_K - \partial_K \epsilon^J)\mathcal{H}^{IK} := \mathcal{L}_\epsilon \mathcal{H}^{IJ} .$$

- The identity $\mathcal{L}_{\epsilon_1}\mathcal{L}_{\epsilon_2} - \mathcal{L}_{\epsilon_2}\mathcal{L}_{\epsilon_1} = \mathcal{L}_{[\![\epsilon_1,\epsilon_2]\!]}$, gives the C-bracket

$$[\![\epsilon_1, \epsilon_2]\!]^J = \epsilon_1^K \partial_K \epsilon_2^J - \tfrac{1}{2}\epsilon_1^K \partial^J \epsilon_{2K} - (\epsilon_1 \leftrightarrow \epsilon_2) .$$

- Weak constraint (LMC): $\Delta\cdot := \partial^I\partial_I \cdot = 0$; Strong constraint: $\partial^I\partial_I(\ldots) = 0$.

Definition of a Courant Algebroid

Courant '90; Liu, Weinstein, Xu '97

$(E \xrightarrow{\pi} M, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \rho : E \to TM)$, such that for $A, B, C \in \Gamma(E)$ and $f, g \in C^\infty(M)$:

1. Modified Jacobi identity ($\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by $\langle \mathcal{D}f, A\rangle = \frac{1}{2}\rho(A)f$.)

$$[[A, B], C] + \text{c.p.} = \mathcal{D}\mathcal{N}(A, B, C) \ , \quad \text{where} \quad \mathcal{N}(A, B, C) = \tfrac{1}{3}\langle [A, B], C\rangle + \text{c.p.} \ ,$$

2. Modified Leibniz rule

$$[A, fB] = f[A, B] + (\rho(A)f)B - \langle A, B\rangle \mathcal{D}f \ ,$$

3. Compatibility condition

$$\rho(C)\langle A, B\rangle = \langle [C, A] + \mathcal{D}\langle C, A\rangle, B\rangle + \langle [C, B] + \mathcal{D}\langle C, B\rangle, A\rangle \ ,$$

The structures also satisfy the following properties (they follow directly...):

4. Homomorphism $\quad \rho[A, B] = [\rho(A), \rho(B)]$.
5. "(no need for) strong constraint" $\quad \rho \circ \mathcal{D} = 0 \quad \Leftrightarrow \quad \langle \mathcal{D}f, \mathcal{D}g\rangle = 0$.

Alternative definition of a Courant Algebroid

Ševera '98

Definition in terms of a bilinear, non-skew operation (Dorfman derivative)

$$[A, B] = A \circ B - B \circ A \ ,$$

notably satisfying instead of 1, the Jacobi identity (in Loday-Leibniz form):

$$A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C) \ .$$

Axioms 2 and 3 do not contain $\mathcal{D}$-terms any longer, instead:

$$\begin{aligned} A \circ fB &= f(A \circ B) + (\rho(A)f)B \ , \\ \rho(C)\langle A, B\rangle &= \langle C \circ A, B\rangle + \langle C \circ B, A\rangle \ . \end{aligned}$$

The two definitions are equivalent, as proven by Roytenberg '99

Local expressions for CAs

In a local basis (e^I) of $\Gamma(E)$, $I = 1, \ldots, 2d$, we can write the local form of the operations:

$$
\begin{aligned}
[e^I, e^J] &= \eta^{IK}\eta^{JL} T_{KLM} e^M \ , \\
\langle e^I, e^J \rangle &= \tfrac{1}{2}\eta^{IJ} \ , \\
\rho(e^I) f &= \eta^{IJ}\rho^i{}_J \partial_i f \ , \\
\mathcal{D} f &= \mathcal{D}_I f \ e^I = \rho^i{}_I \partial_i f \ e^I \ ,
\end{aligned}
$$

with $(\rho^i{}_J)$ the anchor components. The axioms and properties of a CA take the form:

$$
\begin{aligned}
&\eta^{IJ}\rho^i{}_I \rho^j{}_J = 0 \ , \\
&\rho^i{}_I \partial_i \rho^j{}_J - \rho^i{}_J \partial_i \rho^j{}_I - \eta^{KL}\rho^i{}_K T_{LIJ} = 0 \ , \\
&4\rho^i{}_{[L}\partial_i T_{IJK]} + 3\eta^{MN} T_{M[IJ} T_{KL]N} = 0 \ .
\end{aligned}
$$

In other words, the defn gives: no strong constraint, GG fluxes, GG Bianchi identities.

Brackets for standard and non-standard CAs

The *standard* CA: $E = TM \oplus T^*M$, $\rho = (\mathrm{id}, 0)$, and $T_{IJK} = (H_{ijk}, 0, 0, 0)$ with $\mathrm{d}H = 0$.

$$\begin{aligned} [A, B]_s &= [A_V, B_V] + \mathcal{L}_{A_V} B_F - \mathcal{L}_{B_V} A_F - \tfrac{1}{2}\mathrm{d}(\iota_{A_V} B_F - \iota_{B_V} A_F) + H(A_V, B_V) \\ &= A^i \partial_i B^j \partial_j + (A^i \partial_i B_j + \tfrac{1}{2} A^i \partial_j B_i)\mathrm{d}x^j - (A \leftrightarrow B) + A^i B^j H_{ijk} \mathrm{d}x^k \ , \end{aligned}$$

where $A = (A_V, A_F) \in \Gamma(E)$, with $A_V \in \Gamma(TM)$ and $A_F \in \Gamma(T^*M)$.

Another simple ex.: $\rho = (0, \Pi^\sharp)$, $T_{IJK} = (0, 0, \partial_i \Pi^{jk}, R^{ijk})$ with $[\Pi, \Pi]_S = [\Pi, R]_S = 0$.

In general, the Courant bracket is given by an expression of the form Liu, Weinstein, Xu

$$\begin{aligned} [A, B] &= [A_V, B_V] + \mathcal{L}_{A_F} B_V - \mathcal{L}_{B_F} A_V + \tfrac{1}{2}\mathrm{d}_*(\iota_{A_V} B_F - \iota_{B_V} A_F) \\ &+ [A_F, B_F] + \mathcal{L}_{A_V} B_F - \mathcal{L}_{B_V} A_F - \tfrac{1}{2}\mathrm{d}(\iota_{A_V} B_F - \iota_{B_V} A_F) + T(A, B) \ , \\ &= \left(\rho^i{}_J (A^J \partial_i B_K - B^J \partial_i A_K) - \tfrac{1}{2} \rho^i{}_K (A^J \partial_i B_J - B^J \partial_i A_J)\right) e^K + A^L B^M T_{LMK} e^K \ . \end{aligned}$$

Pre-Courant algebroids

Vaisman '04 considered the structure $(E \xrightarrow{\pi} M, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \rho : E \to TM)$ without axiom 1.

Relaxation of (modified) Jacobi identity $\rightsquigarrow$ Pre-Courant algebroid

Hansen, Strobl '09 considered 3D σ-models twisted by a 4-form $\mathcal{T}$, and defined a twisted CA,

$$A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C) + \rho^* \mathcal{T}(\rho(A), \rho(B), \rho(C)) ,$$

where $\rho^* : T^*M \to E$ is the transpose map of ρ. (Plus axioms 2 and 3.)

Liu, Sheng, Xu '12 showed that pre-CA = 4-form-twisted CA.

The relation of DFT and CAs

Solving the s.c. by elimination of $\widetilde{x}$, i.e. $\widetilde{\partial}^i = 0$, takes us from DFT to the standard CA.

more generally, Freidel, Rudolph, Svoboda '17

However, CAs double the bundle, DFT doubles the space.

What if we take a CA over doubled space?

- Geometric origin of the DFT operations and the strong constraint?
- Definition of a DFT algebroid and role of pre-CAs?

Our proposal is instead that the DFT geometry should lie “in between” two (pre-)CAs.

⟵ "Large" CA over $M \times \widetilde{M}$

⟵ Projection

⟵ DFT structure

⟵ Strong Constraint

⟵ "Canonical" CA over M

Doubling and rewriting

In order to relate to DFT, we consider a Courant algebroid over the doubled space.

At least locally, we can work with a 2nd order bundle $\mathbb{E} = (T \oplus T^*)\mathcal{M}$, over $\mathcal{M} = T^*M$.

For simplicity start with the standard CA over $\mathcal{M}$. A section $\mathbb{A} \in \mathbb{E}$ is

$$\mathbb{A} := \mathbb{A}_V + \mathbb{A}_F = \mathbb{A}^I \partial_I + \widetilde{\mathbb{A}}_I \mathrm{d}\mathbb{X}^I \, .$$

Now introduce the following combinations: (N.B. η_{IJ} is not the metric of the CA over $\mathcal{M}$)

$$\mathbb{A}^I_{\pm} = \tfrac{1}{2}(\mathbb{A}^I \pm \eta^{IJ} \widetilde{\mathbb{A}}_J) \, .$$

Strategy: rewrite all structural data of $\mathbb{E}$ in terms of $\mathbb{A}_{\pm}$.

Projected sections and bilinear

Starting with sections of the large CA:

$$\mathbb{A} = \mathbb{A}_+^I e_I^+ + \mathbb{A}_-^I e_I^- \ , \quad \text{where} \quad e_I^{\pm} = \partial_I \pm \eta_{IJ} \mathrm{d}\mathbb{X}^J \ ,$$

a projection to the subbundle L_+ spanned by local sections (e_I^+)

$$\begin{aligned} \mathrm{p}_+ : \mathbb{E} \quad &\to \quad L_+ \\ (\mathbb{A}_V, \mathbb{A}_F) \quad &\mapsto \quad \mathbb{A}_+ := A \ , \end{aligned}$$

leads exactly to the form of a DFT $O(d,d)$ vector

$$A = A_i(\mathrm{d}X^i + \tilde{\partial}^i) + A^i(\mathrm{d}\widetilde{X}_i + \partial_i) \ .$$

Projection of the symmetric bilinear of $\mathbb{E}$, leads to the $O(d,d)$ invariant DFT metric:

$$\langle \mathbb{A}, \mathbb{B} \rangle_{\mathbb{E}} = \tfrac{1}{2} \eta_{\hat{I}\hat{J}} \mathbb{A}^{\hat{I}} \mathbb{B}^{\hat{J}} = \eta_{IJ}(\mathbb{A}_+^I \mathbb{B}_+^J - \mathbb{A}_-^I \mathbb{B}_-^J) \quad \mapsto \quad \eta_{IJ} A^I B^J = \langle A, B \rangle_{L_+} \ ,$$

where $\hat{I} = 1, \ldots, 4d$, while $I = 1, \ldots, 2d$.

Projected brackets

Rewriting the Courant bracket on $\mathbb{E}$ in terms of the $\pm$ components:

$$\begin{aligned}[\mathbb{A},\mathbb{B}]_E &= \eta_{IK}(\mathbb{A}_+^K\partial^I\mathbb{B}_+^L - \mathbb{A}_-^K\partial^I\mathbb{B}_+^L - \tfrac{1}{2}(\mathbb{A}_+^K\partial^L\mathbb{B}_+^I - \mathbb{A}_-^K\partial^L\mathbb{B}_-^I) - \{\mathbb{A}\leftrightarrow\mathbb{B}\})e_L^+ + \\ &+ \eta_{IK}(\mathbb{A}_+^K\partial^I\mathbb{B}_-^L - \mathbb{A}_-^K\partial^I\mathbb{B}_-^L + \tfrac{1}{2}(\mathbb{A}_+^K\partial^L\mathbb{B}_+^I - \mathbb{A}_-^K\partial^L\mathbb{B}_-^I) - \{\mathbb{A}\leftrightarrow\mathbb{B}\})e_L^- \ .\end{aligned}$$

The C-bracket of DFT is obtained from the large standard Courant bracket as:

$$[\![A,B]\!] = \mathrm{p}_+\left([\mathrm{p}_+(\mathbb{A}),\mathrm{p}_+(\mathbb{B})]_{\mathbb{E}}\right) \ .$$

(L_+ is not an involutive subbundle, thus neither a Dirac structure of $\mathbb{E}$.)

Projection of the Dorfman derivative on $\mathbb{E}$ to the generalised Lie derivative of DFT:

$$\mathcal{L}_A B = \mathrm{p}_+\left(\mathrm{p}_+(\mathbb{A})\circ\mathrm{p}_+(\mathbb{B})\right) \ .$$

Thus, the map p_+ sends all CA structures to the corresponding DFT structures.

General anchor and flux formulation of DFT

This works for general CAs over $\mathcal{M}$ with anchor $\rho^I{}_{\hat{J}} = (\rho^I{}_J, \widetilde{\rho}^{IJ})$, yielding a C-bracket:

$$[\![A, B]\!]^J = (\rho_+)^L{}_I \left(A^I \partial_L B^J - \tfrac{1}{2} \eta^{IJ} A^K \partial_L B_K - (A \leftrightarrow B) \right) + \hat{T}_{IK}{}^J A^I B^K \ ,$$

in terms of a map $\rho_+ : L_+ \to T\mathcal{M}$ with components $(\rho_+)^I{}_J = \rho^I{}_J \pm \eta_{JK} \widetilde{\rho}^{IK}$.

Taking a parametrization of the ρ_+ components to be $\rho^I{}_J = (\delta^i{}_j, \beta^{ij}, \delta_i^j + \beta^{jk} B_{ki}, B_{ij})$, one can draw a parallel to the flux formulation of DFT. The relevant expressions are:

Geissbuhler, Marques, Nunez, Penas '13

$$\begin{aligned} &\eta^{IJ} \rho^K{}_I \rho^L{}_J = \eta^{KL} \\ &2\rho^L{}_{[I} \partial_{\underline{L}} \rho^K{}_{J]} - \eta^{LM} \rho^K{}_L \hat{T}_{MIJ} = \rho_{L[I} \partial^K \rho^L{}_{J]} \\ &4\rho^M{}_{[L} \partial_{\underline{M}} \hat{T}_{IJK]} + 3\eta^{MN} \hat{T}_{M[IJ} \hat{T}_{KL]N} = \mathcal{Z}_{IJKL} \ . \end{aligned}$$

- Their resemblance to the local expressions of the CA axioms is very suggestive.
- They can be used to reverse-engineer a geometric definition for a DFT algebroid.

Towards a DFT Algebroid structure

Strategy: Replace $[\cdot,\cdot]_E \to [\![\cdot,\cdot]\!]$, $\langle\cdot,\cdot\rangle_E \to \langle\cdot,\cdot\rangle_{L_+}$ and $\rho \to \rho_+$, and also define $\mathcal{D}_+$ as

$$\langle A, \mathcal{D}_+ f\rangle_{L_+} = \tfrac{1}{2}\rho_+(A)f \ ,$$

amd determine one by one the obstructions to the CA axioms and properties.

1. Modified Jacobi identity ($\mathcal{N}(A,B,C) = \frac{1}{3}\langle [\![A,B]\!], C\rangle_{L_+} + \text{c.p.}$) $\rightsquigarrow$ obstructed

   $$[\![[\![A,B]\!],C]\!] + \text{c.p.} = \mathcal{D}_+\mathcal{N}(A,B,C) + \mathcal{Z}(A,B,C) + SC_1(A,B,C) \ ,$$

   where the last term (which vanishes on the strong constraint) is explicitly given by

   $$\begin{aligned} SC_1(A,B,C)^L &= -\tfrac{1}{2}\left(A^I\partial_J B_I \partial^J C^L - B^I\partial_J A_I\partial^J C^L\right) - \\ - &\quad \rho_{I[J}\partial_M \rho^I{}_{N]}\left(A^J B^N \partial^M C^L - \tfrac{1}{2}C^J A^K\partial^M B_K \eta^{NL} + \tfrac{1}{2}C^J B^K \partial^M A_K \eta^{NL}\right) + \\ + &\quad \text{c.p.}(A,B,C) \ . \end{aligned}$$

2. Modified Leibniz rule $\rightsquigarrow$ unobstructed

   $$[\![A, fB]\!] = f[\![A,B]\!] + (\rho_+(A)f)\,B - \langle A,B\rangle_{L_+}\mathcal{D}_+ f \ .$$

Towards a DFT Algebroid structure

3. Compatibility condition $\rightsquigarrow$ unobstructed

$$\langle [\![C,A]\!] + \mathcal{D}_+\langle C,A\rangle_{L_+}, B\rangle_{L_+} + \langle [\![C,B]\!] + \mathcal{D}_+\langle C,B\rangle_{L_+}, A\rangle_{L_+} = \rho_+(C)\langle A,B\rangle_{L_+} \ .$$

Up to now, these would point to a pre-CA, but there are two more properties:

4. Homomorphism $\rightsquigarrow$ obstructed

$$\rho_+[\![A,B]\!] = [\rho_+(A), \rho_+(B)] + SC_2(A,B) \ ,$$

where the last term (which vanishes on the strong constraint) is explicitly given by

$$SC_2(A,B) = \left(\rho_{L[I}\partial^K \rho^L{}_{J]} A^I B^J + \tfrac{1}{2}\left(A^I \partial^K B_I - B^I \partial^K A_I\right)\right)\partial_K \ .$$

5. "(no need for) strong constraint" $\rightsquigarrow$ obstructed

$$\langle \mathcal{D}_+ f, \mathcal{D}_+ g\rangle_{L_+} = \langle \mathrm{d}f, \mathrm{d}g\rangle_{L_+} = \eta^{IJ}\rho^K{}_I \rho^L{}_J \partial_K f \partial_L g = \partial^L f \partial_L g \ .$$

A proposal for the DFT Algebroid structure and its relation to CAs

A DFT algebroid is a quadruple $(L_+, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle_{L_+}, \rho_+)$ satisfying

2. $[\![A, fB]\!] = f[\![A,B]\!] + (\rho_+(A)f)\,B - \langle A,B\rangle_{L_+}\mathcal{D}_+ f$,
3. $\langle [\![C,A]\!] + \mathcal{D}_+\langle C,A\rangle_{L_+}, B\rangle_{L_+} + \langle [\![C,B]\!] + \mathcal{D}_+\langle C,B\rangle_{L_+}, A\rangle_{L_+} = \rho_+(C)\langle A,B\rangle_{L_+}$,

where $\mathcal{D}_+$ is defined through $\langle A, \mathcal{D}_+ f\rangle_{L_+} = \frac{1}{2}\rho_+(A)f$.

When the s.c. is imposed, it reduces to a (pre-)CA and ρ_+ becomes a homomorphism.

Relation to Roytenberg's supermanifold description

- QP2 manifolds $(\mathfrak{M}, \omega, Q = \{\Theta, \cdot\})$ (sympl. Lie 2-algebroids) $\overset{1-1}{\longleftrightarrow}$ CAs.
  Roytenberg '02
  $$\{\Theta, \Theta\} = 0 \ .$$

- Bruce, Grabowski '16 proved: Vaisman's pre-CA $\overset{1-1}{\longleftrightarrow}$ symplectic almost Lie 2-algebroids.
  $$\{\{\Theta, \Theta\}, f\} = 0 \ .$$
  Interpreted as the strong constraint in Deser, Stasheff '14

- Interestingly, Bruce, Grabowski '16 also define symplectic nearly Lie 2-algebroids, and show that the Jacobi identity and homomorphism of a CA are obstructed by
  $$\{\{\{\{\Theta, \Theta\}, A\}, B\}, C\} \quad \text{and} \quad \{\{\{\{\Theta, \Theta\}, f\}, A\}, B\} \ .$$
  Writing them in local coord's, exactly identified with $SC_1(A, B, C)$ and $SC_2(A, B)$!

$\rightsquigarrow$ supermanifold description of the DFT algebroid: symplectic nearly Lie 2-algebroid.

Epilogue

Take-home messages

- The geometric structure of DFT is in between two Courant algebroids
- A DFT algebroid defined — corresponds to symplectic nearly Lie 2-algebroids
- There is a pre-CA structure between a DFTA and a CA

Other comments and questions

- Nonassociative R-flux violates strong constraint of DFT

  Blumenhagen, Fuchs, Hassler, Lüst, Sun '13; see also Hohm, Kupriyanov, Lüst, Traube '17
- Approach is systematic, seems applicable to higher structures

  in this context: Grützmann '10; Ikeda, Uchino '10; or: Hull '07; Pacheco, Waldram '08; & c.