Affine generalized diffeomorphisms

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San Carlos de Bariloche, January 2018

Outline

- Linear system of supergravity in two dimensions
- Generalized diffeomorpshims
- Generalized Scherk–Schwarz Ansatz

[G. B, M. Cederwall, A. Kleinschmidt, J. Palmkvist, H. Samtleben, 1708.08936]

Motivations

Exceptional geometries for Kac–Moody groups $E_9 \subset E_{10} \subset E_{11}$.

- $\hookrightarrow$ Relation to the E_{10} cosmological billard
- $\hookrightarrow$ Exceptional cosmology?

E_9 is the first infinite dimensional and nonetheless tractable

- $\hookrightarrow$ General gauged supergravity in two dimensions

Integrable system of equations

- $\hookrightarrow$ Integrable structures in eleven dimensions?
- $\hookrightarrow$ Integrable structures in gauged supergravity?

String theory effective action on $\mathbb{R}^{1,1} \times T^8$

- $\hookrightarrow$ Loop amplitudes — Affine Eisenstein series.

Affine symmetry in supergravity

In three dimensions, the bosonic Lagrangian of (ungauged) $\mathcal{N} = 16$ supergravity is

$$\mathcal{L} = \sqrt{-g}R - \frac{1}{30}\mathrm{tr}\, P \star P$$

where $\mathcal{V} \in E_{8(8)}/(Spin(16)/\mathbb{Z}_2)$ defines the Maurer–Cartan form

$$\mathcal{V}^{-1}d\mathcal{V} = B + P$$

with $B \in \mathfrak{so}(16)$ and P is in the **128** complement.

Affine symmetry in supergravity

After reduction on a circle with the metric ansatz

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \rho^2 d\varphi^2$$

one gets the Lagrangian

$$\mathcal{L} = \rho\Big(\sqrt{-g}R - \frac{1}{30}\mathrm{tr}\, P \star P\Big)$$

with equations of motion

$$d{\star}d\rho = 0\,, \quad d(\rho{\star}P){+}[B,\rho{\star}P] = 0\,, \quad R_{\mu\nu} = \nabla_\mu\partial_\nu\rho{+}\frac{1}{30}\mathrm{tr}\, P_\mu P_\nu\,.$$

Affine symmetry in supergravity

$$d\star d\rho = 0\ , \quad d(\rho\star P) + [B,\rho\star P]\ , \quad R_{\mu\nu} = \nabla_\mu\partial_\nu\rho + \frac{1}{30}\mathrm{tr}\, P_\mu P_\nu\ .$$

Then $\star d\rho = d\tilde{\rho}$, and there exists a holomorphic function $\mathcal{V}(\gamma)$ such that [Breitenlohner, Maison]

$$\mathcal{V}(\gamma)^{-1}d\mathcal{V}(\gamma) = B + \frac{1+\gamma^2}{1-\gamma^2}P + \frac{2\gamma}{1-\gamma^2}\star P$$

provided (with $\gamma_+\gamma_- = 1$)

$$\gamma = \gamma_\pm(\rho,\tilde{\rho}) = \frac{w-\tilde{\rho}\pm\sqrt{(\tilde{\rho}-w)^2-\rho^2}}{\rho}\ .$$

and one checks that $M(w)$ is a meromorphic function in $E_{8(8)}$

$$d(\mathcal{V}(\gamma)\mathcal{V}^\dagger(\tfrac{1}{\gamma})) = 0 \quad\Rightarrow\quad \mathcal{V}(\gamma)\mathcal{V}^\dagger(\tfrac{1}{\gamma}) = M(w)\ .$$

Affine symmetry in supergravity

$$d \star d\rho = 0 \,, \quad d(\rho \star P) + [B, \rho \star P] \,, \quad R_{\mu\nu} = \nabla_\mu \partial_\nu \rho + \frac{1}{30} \mathrm{tr}\, P_\mu P_\nu \;.$$

Then $\star d\rho = d\tilde{\rho}$, and there exists a holomorphic function $\mathcal{V}(\gamma)$ such that [Breitenlohner, Maison]

$$\mathcal{V}(\gamma)^{-1} d\mathcal{V}(\gamma) = B + P + 2\gamma \sum_{n=0}^{\infty} \gamma^{2n} (\star P + \gamma P)$$

provided (with $\gamma_+ \gamma_- = 1$)

$$\gamma = \gamma_\pm(\rho, \tilde{\rho}) = \frac{w - \tilde{\rho} \pm \sqrt{(\tilde{\rho} - w)^2 - \rho^2}}{\rho} \;.$$

and one checks that $M(w)$ is a meromorphic function in $E_{8(8)}$

$$d(\mathcal{V}(\gamma)\mathcal{V}^\dagger(\tfrac{1}{\gamma})) = 0 \quad \Rightarrow \quad \mathcal{V}(\gamma)\mathcal{V}^\dagger(\tfrac{1}{\gamma}) = M(w) \;.$$

Affine symmetry in supergravity

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$$d(\mathcal{V}(\gamma)\mathcal{V}^\dagger(\tfrac{1}{\gamma})) = 0 \quad \Rightarrow \quad \mathcal{V}(\gamma)\mathcal{V}^\dagger(\tfrac{1}{\gamma}) = M(w) \; .$$

$\hookrightarrow$ $M(w)$ in the centrally extended loop group over $E_{8(8)}$

The central element acts as a Weyl rescaling of $g_{\mu\nu}$

For $a > 0$ and $b \in \mathbb{R}$ [Julia, Nicolai]

$$\gamma_\pm(aw + b, a\rho, a\tilde\rho + b) = \gamma_\pm(w, \rho, \tilde\rho) \qquad \star\, d(a\rho) = d(a\tilde\rho + b)$$

$\hookrightarrow$ $\mathbb{R}_+ \ltimes \mathbb{R} \subset SL(2,\mathbb{R}) \subset \text{Vir}$

Affine symmetry in supergravity

$$\mathfrak{e}_9 = \left\langle T^A_m \,:\, A = 1, \ldots, 248, \quad m \in \mathbb{Z} \right\rangle \oplus \mathbb{R}\mathsf{K} \oplus \mathbb{R}\mathsf{d}\,.$$

With $T^A_m \rightsquigarrow \mathcal{V}(t)$, $\mathsf{K} \rightsquigarrow \sqrt{-g} = e^{2\sigma}$, $\mathsf{d} \sim L_0 \rightsquigarrow \rho$, whereas the additional generator $L_{-1} \rightsquigarrow \tilde{\rho}$.

The algebra is

$$\begin{aligned} \left[T^A_m, T^B_n\right] &= f^{AB}{}_C\, T^C_{m+n} + \eta^{AB} m\, \delta_{m+n,0}\, \mathsf{K}\,, \\ \left[\mathsf{d}, T^A_m\right] &= -m T^A_m\,, \end{aligned}$$

with $f^{AC}{}_D f^{BD}{}_C = 60\, \eta^{AB}$.

Complete Kac–Moody groups

The minimal Kac–Moody group is defined as the group generated from all the one-dimensional subgroups associated to real roots.

$$g(x_\alpha|_I) = \prod_{\alpha \in I \subset \Delta_{\mathrm{Re}}} \exp(x_\alpha E_\alpha)$$

The complete Kac–Moody group is obtained by a certain completion associated to a Borel subgroup, such that I can include infinitely many positive real roots in Δ_{Re}.

The minimal affine group is defined from the loop group of finite Laurent series in w.

The complete affine group is defined from the loop group of meromorphic functions of w on $\mathbb{C}$. By E_9 we mean the complete affine group.

Ex: $\mathcal{M}(w) = 1 + \frac{2w\mathscr{C} + 2\mathscr{C}^2}{w^2 - c^2} + \frac{2}{3}\frac{\left(\mathscr{C}^2 - c^2\right)\left(2w\mathscr{C} + \mathscr{C}^2\right)}{(w^2 - c^2)^2}$ with $\mathscr{C}^5 = 5c^2\mathscr{C}^3 - 4c^4\mathscr{C}$.

Section constraint for Kac–Moody groups

The quadratic Casimir,

$$C_2 = \frac{1}{2}\eta_{AB} : T^A T^B : = \sum_{\alpha\in\Delta_+} E_{-\alpha}E_\alpha + \frac{1}{2}(H,H) + (\varrho, H) ,$$

such that for any $|v\rangle \in R(\lambda)$ highest weight module

$$C_2|v\rangle = \frac{1}{2}(\lambda, \lambda + 2\varrho)|v\rangle .$$

Then for the highest weight vector $|\lambda\rangle \in R(\lambda)$

$$\Big(\eta_{AB} T^A \otimes T^B - (\lambda,\lambda)\Big)|\lambda\rangle \otimes |\lambda\rangle = 0 .$$

$\hookrightarrow$ General section constraint

$$\Big(\eta_{AB} T^A \otimes T^B + 1 - (\lambda,\lambda)\Big)|p\rangle \otimes |q\rangle = |q\rangle \otimes |p\rangle .$$

Generalized diffeomorphism for Kac–Moody groups

$$\mathcal{L}_\xi |V\rangle = \langle \partial_V|\xi\rangle|V\rangle - \eta_{AB}\langle\partial_\xi|T^A|\xi\rangle T^B|V\rangle + \big((\lambda,\lambda)-1\big)\langle\partial_\xi|\xi\rangle|V\rangle \ ,$$

i.e. in index notation

$$\mathcal{L}_\xi V^M = \xi^N\partial_N V^M - \Big(\eta_{AB}(T^A)^M{}_N(T^B)^P{}_Q + \big(1-(\lambda,\lambda)\big)\delta^M_N\delta^P_Q\Big) V^N\partial_P\xi^Q \ ,$$

that reduces to standard diffeomorphisms upon using the section constraint

$$\Big(\eta_{AB}(T^A)^M{}_N(T^B)^P{}_Q + \big(1-(\lambda,\lambda)\big)\delta^M_N\delta^P_Q\Big) V^N\partial_P\xi^Q = V^N\partial_N\xi^M \ .$$

Generally does not close!

Generalized diffeomorphism for E_9

For E_9 the generalized vector $|V\rangle$ and coordinates $|Y\rangle$ lie in the fundamental module that is defined by the CFT Hilbert space of eight chiral bosons on the E_8 torus $\mathbb{R}^8/E_8$, with character

$$\begin{aligned}\chi_{R(\Lambda_9)}(q) = \mathrm{Tr}_{R(\Lambda_9)} q^{L_0} &= \frac{\sum_{Q\in E_8} q^{Q^2/2}}{\prod_{n>0}(1-q^n)^8} \\ &= q^{\frac{1}{3}} \frac{E_4(\tau)}{\eta(\tau)^8} \ .\end{aligned}$$

Sugawara defines the Virasoro generators from the current T^A_m

$$L^{(k)}_m = \frac{1}{2(k+30)} \sum_{n\in\mathbb{Z}} \eta_{AB} : T^A_n T^B_{m-n} :$$

Generalized diffeomorphism for E_9

$$L_m^{(k)} = \frac{1}{2(k+30)} \sum_{n\in\mathbb{Z}} \eta_{AB} : T_n^A T_{m-n}^B :$$

So the operator appearing in the section constraint

$$\begin{aligned} C_n \equiv \eta_{(n)\mathcal{A}\mathcal{B}}\, T^{\mathcal{A}} \otimes T^{\mathcal{B}} &\equiv \sum_{p\in\mathbb{Z}} \eta_{AB} T_p^A \otimes T_{n-p}^B - \mathbb{1} \otimes L_n - L_n \otimes \mathbb{1} \\ &= 32(\mathbb{1} \otimes L_n + L_n \otimes \mathbb{1} - L_n^{(2)}) \end{aligned}$$

where $\frac{1}{32} C_n$ acts on $R(\Lambda_9) \otimes R(\Lambda_9)$ as the Virasoro algebra on the Ising CFT module.

$$R(\Lambda_9) \otimes R(\Lambda_9) = \mathrm{Vir}^3_{1,1} \otimes R(2\Lambda_9)_0 \oplus \mathrm{Vir}^3_{2,1} \otimes R(\Lambda_1)_{-3/2} \oplus \mathrm{Vir}^3_{2,2} \otimes R(\Lambda_8)_{-15/16}\,.$$

Generalized diffeomorphism for E_9

One gets the generalized diffeomorphism

$$\begin{aligned}
\mathcal{L}_\xi |V\rangle &= \langle \partial_V|\xi\rangle|V\rangle + \langle\partial_\xi|(C_0-1)|\xi\rangle \otimes |V\rangle \\
&= \langle \partial_V|\xi\rangle|V\rangle + \eta_{AB}\sum_{n\in\mathbb{Z}} \langle\partial_\xi|T^A_n|\xi\rangle\, T^B_{-n}|V\rangle \\
&\quad -\langle\partial_\xi|L_0|\xi\rangle|V\rangle - \langle\partial_\xi|\xi\rangle(L_0+1)|V\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\left([\mathcal{L}_\xi,\mathcal{L}_\eta] - \tfrac{1}{2}\mathcal{L}_{\mathcal{L}_\xi\eta-\mathcal{L}_\eta\xi}\right)|V\rangle \\
=\;& \frac{1}{4}\,\langle\partial_\eta|C_{-1}\big(\langle\partial_\eta|C_1|\eta\rangle\otimes|\xi\rangle - \langle\partial_\eta|C_1|\xi\rangle\otimes|\eta\rangle\big)\otimes|V\rangle \\
&-\frac{1}{4}\,\langle\partial_\xi|C_{-1}\big(\langle\partial_\xi|C_1|\xi\rangle\otimes|\eta\rangle - \langle\partial_\xi|C_1|\eta\rangle\otimes|\xi\rangle\big)\otimes|V\rangle\,.
\end{aligned}$$

11D supergravity on $\mathbb{R}^{1,10-d} \times T^d$

The metric field

$$g_{\mu\nu} \ , \quad A^I_\mu \ , \qquad M^{IJ} \ .$$

The 3-form

$$C_{\mu\nu\rho} \ , \quad B_{\mu\nu I} \ , \qquad A_{\mu IJ} \ , \qquad a_{IJK} \ .$$

The dual 6-form

$$\ldots \ , \quad B_{\mu\nu IJKL} \ , \quad A_{\mu I_1 I_2 I_3 I_4 I_5} \ , \qquad a_{I_1\ldots I_6} \ .$$

The dual graviton

$$\ldots \ , \quad B_{\mu\nu I_1\ldots I_6,J} \ , \quad A_{\mu I_1\ldots I_7,J} \ , \qquad a_{I_1\ldots I_8,J} \ .$$

11D supergravity on $\mathbb{R}^{1,10-d} \times T^d$

The scalars combine in the adjoint

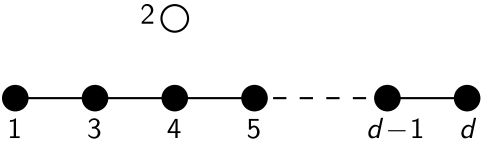
$$\mathcal{M}^{MN} = (M^{IJ}, a_{IJK}, a_{I_1 \dots I_6}, a_{I_1 \dots I_8, J}) \ ,$$

the 1-forms combine in $R(\Lambda_d)$

$$A_{\mu M} = (A^I_\mu, A_{\mu IJ}, A_{\mu I_1 \dots I_5}, A_{\mu I_1 \dots I_7, J}) \ ,$$

the 2-forms combine in $R(\Lambda_1)$

$$B_{\mu\nu\alpha} = (B_{\mu\nu I}, B_{\mu\nu I_1 \dots I_4}, B_{\mu\nu I_1 \dots I_6, J}) \ ,$$



Gradient dual fields

One has also fields of the type $A_{9,3}$ dual to the 3-form [Boulanger, Sundell, West]

$$\partial^b F_{a_1 a_2 a_3 a_4} = \frac{40}{10!}\,\varepsilon^b{}_{c_1\cdots c_{10}}\,\partial^{c_1}\partial_{[a_1} A^{c_2\ldots c_{10},}{}_{a_2 a_3 a_3]}\,,$$

or at first order

$$\partial^b A_{a_1 a_2 a_3} = \frac{1}{9!}\,\varepsilon^b{}_{c_1\cdots c_{10}}\,\partial^{c_1} A^{c_2\ldots c_{10},}{}_{a_1 a_2 a_3} + 3\partial_{[a_1} X^b{}_{a_2 a_3]}\,,$$

Repeating the procedure one gets $A_{9,3}$, $A_{9,9,3}$, $A_{9,9,9,3}$, etc... that we shall write $A_{9^n,3}$ for short.

11D supergravity on $\mathbb{R}^{1,10-d} \times T^d$

The metric field

$$g_{\mu\nu} \ , \quad A_\mu^I \ , \qquad M^{IJ} \ .$$

The $(9^n, 3)$-form

$$\ldots \ , \quad C_{\mu\nu\rho 9^n} \ , \quad B_{\mu\nu 9^n, I} \ , \qquad A_{\mu 9^n, IJ} \ , \qquad a_{9^n, IJK} \ .$$

The dual $(9^n, 6)$-form

$$\ldots \ , \quad B_{\mu\nu 9^n, IJKL} \ , \quad A_{\mu 9^n, I_1 I_2 I_3 I_4 I_5} \ , \qquad a_{9^n, I_1 \ldots I_6} \ .$$

The dual graviton $(9^n, 8, 1)$-form

$$\ldots \ , \quad B_{\mu\nu 9^n, I_1 \ldots I_6, J} \ , \quad A_{\mu 9^n, I_1 \ldots I_7, J} \ , \qquad a_{9^n, I_1 \ldots I_8, J} \ .$$

Additional shift gauge invariance

The scalars combine in the adjoint, Σ in $R(\Lambda_d) \otimes \overline{R}(\Lambda_9)$
[Hohm, Samtleben]

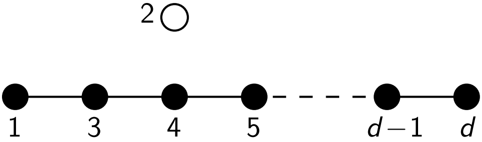
$$\mathcal{M}^{MN} = (M^{IJ}, a_{IJK}, a_{I_1 \dots I_6}, a_{I_1 \dots I_8, J} + \Sigma_{I_1 \dots I_8, J}, a_{9,3} + \Sigma_{9,3}, \dots) \ ,$$

the 1-forms combine in $R(\Lambda_d)$, Σ in $R(\Lambda_d) \otimes \overline{R}(\Lambda_8)$

$$A_{\mu M} = (A^I_\mu, A_{\mu IJ}, A_{\mu I_1 \dots I_5}, A_{\mu I_1 \dots I_7, J} + \Sigma_{\mu I_1 \dots I_7, J}, \dots) \ ,$$

the 2-forms combine in $R(\Lambda_1)$, Σ in $R(\Lambda_d) \otimes \overline{R}(\Lambda_7)$

$$B_{\mu\nu\alpha} = (B_{\mu\nu 9^n, I}, B_{\mu\nu 9^n, I_1 \dots I_4}, B_{\mu\nu 9^n, I_1 \dots I_6, J} + \Sigma_{\mu\nu 9^n, I_1 \dots I_6, J}, \dots) \ ,$$



Additional shift gauge invariance

The scalars combine in the adjoint, Σ in $R(\Lambda_d) \otimes \overline{R}(\Lambda_9)$

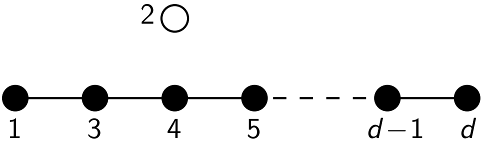
$$\mathcal{M}^{MN} + \Sigma_M{}^{N_1 \dots N_{d-8}} \ ,$$

the 1-forms combine in $R(\Lambda_d)$, Σ in $R(\Lambda_d) \otimes \overline{R}(\Lambda_8)$

$$A_{\mu M} + \Sigma_{\mu M}{}^{N_1 \dots N_{d-7}} \ ,$$

the 2-forms combine in $R(\Lambda_1)$, Σ in $R(\Lambda_d) \otimes \overline{R}(\Lambda_7)$

$$B_{\mu\nu MN} + \Sigma_{\mu\nu M}{}^{N_1 \dots N_{d-6}} \ ,$$



Additional shift gauge invariance

The scalars combine in the adjoint, Σ in $R(\Lambda_9) \otimes \overline{R}(\Lambda_9)$

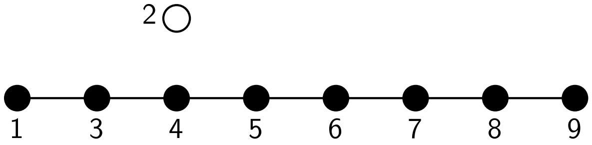
$$\mathcal{M}^{MN} + \Sigma_{M}{}^{N} \ ,$$

the 1-forms combine in $R(\Lambda_9)$, Σ in $R(\Lambda_9) \otimes \overline{R}(\Lambda_8)$

$$A_{\mu M} + \Sigma_{\mu M}{}^{\alpha} \ ,$$

the 2-forms combine in $R(\Lambda_1)$, Σ in $R(\Lambda_9) \otimes \overline{R}(\Lambda_7)$

$$B_{\mu\nu\alpha} + \Sigma_{\mu\nu M}{}^{\dot{\alpha}} \ ,$$



Generalized diffeomorphism for E_9

One gets the generalized diffeomorphism

$$\mathcal{L}_{\xi,\Sigma}|V\rangle = \langle\partial_V|\xi\rangle|V\rangle + \langle\partial_\xi|(C_0 - 1)|\xi\rangle \otimes |V\rangle + \langle\pi_\Sigma|C_{-1}|\Sigma\rangle \otimes |V\rangle$$

or

$$\mathcal{L}_{\xi,\Sigma}V^M = \xi^N\partial_N V^M + C_0^M{}_N{}^P{}_Q\partial_P\xi^Q V^N - \partial_N\xi^N V^M + C_{-1}^M{}_N{}^P{}_Q\Sigma_P{}^Q V^N$$

and so

$$\begin{aligned}\mathcal{L}_{\xi,\Sigma}|V\rangle = {} & \langle\partial_V|\xi\rangle|V\rangle + \eta_{AB}\sum_{n\in\mathbb{Z}}\langle\partial_\xi|T_n^A|\xi\rangle T_{-n}^B|V\rangle \\ & -\langle\partial_\xi|L_0|\xi\rangle|V\rangle - \langle\partial_\xi|\xi\rangle(L_0+1)|V\rangle \\ & +\eta_{AB}\sum_{n\in\mathbb{Z}}\mathrm{Tr}\left[T_{n-1}^A\Sigma\right]T_{-n}^B|V\rangle \\ & -\mathrm{Tr}\left[L_{-1}\Sigma\right]|V\rangle - \mathrm{Tr}\left[\Sigma\right]L_{-1}|V\rangle\end{aligned}$$

Decomposition in E_8

$\langle\partial|$ solving the section constraint can be rotated using E_9 to the vacuum state $\langle 0|$ (highest weight vector).

$$\langle\partial_1| = \langle 0|g_{\mathbb{R}} \;, \qquad \langle p_1| = \langle 0|n_1 g_{\mathbb{Z}} \;.$$

Then $\langle\partial_1|\otimes\langle\partial_2|(C_0-1)+\langle\partial_2|\otimes\langle\partial_1| = 0$ implies that

$$\langle\partial_2| = \langle 0|(\partial_0 + T_1^A\partial_A)g_{\mathbb{R}} \;, \qquad \langle p_2| = \langle 0|(n_2 + T_1^A Q_A)g_{\mathbb{Z}} \;.$$

Using the form $\langle\partial_i| = \langle 0|(\partial_{i0} + T_1^A\partial_{iA})$ one gets

$$\begin{aligned} 0 &= \langle\partial_1|\otimes\langle\partial_2|(C_0-1)+\langle\partial_2|\otimes\langle\partial_1| \\ &= \langle 0|\otimes\langle 0|\,\partial_{1A}\partial_{2B}\left(\Pi^{AB}{}_{CD}\,T_1^C\otimes T_1^D - T_1^A T_1^B\otimes\mathbb{1} - \mathbb{1}\otimes T_1^B T_1^A\right) \;, \end{aligned}$$

where $\Pi^{AB}{}_{CD} \equiv 2\,\delta_C^{(A}\delta_D^{B)} - f^A{}_{CE}f^{EB}{}_D$.

Decomposition in E_8

Take the expansion

$$\Sigma = \left(\sigma_1 + \sigma_{2A}\, T^A_{-1} + \sigma_{3AB}\, T^A_{-1}\, T^B_{-1} + \dots\right) |0\rangle\langle 0| \\ - \left(\Sigma_{0A} + \Sigma_{1A,B}\, T^B_{-1} + \Sigma_{2A,BC}\, T^B_{-1}\, T^C_{-1} + \dots\right) |0\rangle\langle 0|\, T^A_1 \,,$$

$$|\xi\rangle = \left(\xi^0 + \eta_{AB}\xi^A_1\, T^B_{-1} + \xi_{2AB}\, T^A_{-1}\, T^B_{-1} + \dots\right) |0\rangle \;,$$

and $\langle\partial_i| = \langle 0|(\partial_{i0} + T^A_1\partial_{iA})$, one gets

$$\mathcal{L}_{\xi,\Sigma} = \xi^0\partial_0 + \xi^A_1\partial_A - \partial_A\xi^0\, T^A_1 + \partial_0\xi^0(L_0 - 1) + \partial_A\xi^A_1 L_0 + \left(f^B{}_{CA}\partial_B\xi^C_1 + \Sigma_{0A}\right) T^A_0 \\ +\sigma_1 L_{-1} - \left(\partial_0\xi^A_1 + \Pi^{AB,CD}\partial_B\xi_{2CD} - f^{ABC}\Sigma_{1B,C}\right)\eta_{AE}\, T^E_{-1} + \sum_{n>1}\omega_{nA}\, T^A_{-n} \;.$$

Decomposition in E_8

Decomposing the vector $|V\rangle$ accordingly,

$$|V\rangle = \Big(V^0 + \eta_{AB} V_1^A T_{-1}^B + V_{2AB} T_{-1}^A T_{-1}^B + \dots \Big) |0\rangle ,$$

one obtains for the action on its lowest components

$$\begin{aligned}
\mathcal{L}_{\xi,\Sigma} V^0 &= \xi^0 \partial_0 V^0 - V^0 \partial_0 \xi^0 + \xi_1^A \partial_A V^0 - V_1^A \partial_A \xi^0 , \\
\mathcal{L}_{\xi,\Sigma} V_1^A &= \xi^0 \partial_0 V_1^A - V^0 \partial_0 \xi_1^A \\
&\quad + \xi_1^B \partial_B V_1^A + V_1^A \partial_B \xi_1^B - \big(f^{EA}{}_B f^C{}_{DE} \partial_C \xi_1^D + f^{CA}{}_B \Sigma_{0C} \big) V_1^B \\
&\quad - \Pi^{BA,CD} V_{2CD} \partial_B \xi^0 - \Pi^{AB,CD} V^0 \partial_B \xi_{2CD} + f^{ABC} \Sigma_{1B,C} V^0 ,
\end{aligned}$$

Decomposition in E_8

Decomposing the vector $|V\rangle$ accordingly,

$$|V\rangle = \Big(V^0 + \eta_{AB} V_1^A T_{-1}^B + V_{2AB} T_{-1}^A T_{-1}^B + \ldots \Big) |0\rangle \ ,$$

one obtains for the action on its lowest components

$$\begin{aligned}
\mathcal{L}_{\xi,\Sigma} V^0 &= \xi^0 \partial_0 V^0 - V^0 \partial_0 \xi^0 + \xi_1^A \partial_A V^0 - V_1^A \partial_A \xi^0 \ , \\
\mathcal{L}_{\xi,\Sigma} V_1^A &= \xi^0 \, \partial_0 V_1^A - V^0 \, \partial_0 \xi_1^A \\
&\quad + \xi_1^B \partial_B V_1^A + V_1^A \partial_B \xi_1^B - \left(f^{EA}{}_B f^C{}_{DE} \, \partial_C \xi_1^D + f^{CA}{}_B \Sigma_{0C} \right) V_1^B \\
&\quad - \Pi^{BA,CD} V_{2CD} \, \partial_B \xi^0 - \Pi^{AB,CD} V^0 \, \partial_B \xi_{2CD} + f^{ABC} \Sigma_{1B,C} V^0 \ ,
\end{aligned}$$

Affine Scherk–Schwarz Ansatz

The loop group twist matrix defines the scalar fields and the scaling factor $-g = e^{2\sigma}$ as

$$e^{-2\sigma(x,Y)}\mathcal{M}(x,Y) = U_{\text{loop}}(Y)^T e^{-2\sigma(x)} M(x) U_{\text{loop}}(Y) \ ,$$

The Virasoro parabolic subgroup twist matrix ρ and $\tilde{\rho}$ as

$$U_{\text{Vir}}^T(Y) = e^{\varsigma(Y)L_{-1}} e^{\upsilon(Y)L_0} \ \Rightarrow \ \rho(x,Y) = e^{-\upsilon(Y)}\rho(x) \ , \quad \tilde{\rho}(x,Y) = e^{-\upsilon(Y)}(\tilde{\rho}(x) - \varsigma(Y)) \ .$$

They combine into

$$U(Y) = U_{\text{Vir}}(Y) U_{\text{loop}}(Y) \ .$$

Affine Scherk–Schwarz Ansatz

The derivative of the twist matrix decomposes in currents

$$U^T(Y) \otimes \langle \overleftarrow{\partial}_Y | = U^T \underline{L}_0 \otimes \langle \partial \upsilon | + U^T \underline{L}_{-1} \otimes e^{-\upsilon} \langle \partial \varsigma | + \sum_n U^T \underline{T}^A_n \otimes \langle j_{nA} | + U^T \otimes \langle j_c | \ .$$

for

$$(U^T)^{-1} \partial_M U^T(Y) = j_{M\alpha} T^\alpha \quad \Rightarrow \quad \partial_M U^T(Y) = j_{M\alpha} U^T T^\alpha$$

and we have the Scherk–Schwarz Ansatz for the generalized diffeomomorphisms

$$|V\rangle = U^{-T} |\underline{V}\rangle \ ,$$
$$|\xi\rangle = U^{-T} |\underline{\xi}\rangle \ ,$$

$$\Sigma = e^{-\upsilon} U^{-T} \Big(\sum_n \underline{T}^A_{1+n} |\underline{\xi}\rangle \langle j_{nA}| + \underline{L}_1 |\underline{\xi}\rangle \langle \partial \upsilon | + \underline{L}_0 |\underline{\xi}\rangle e^{-\upsilon} \langle \partial \varsigma | \Big) + e^{-\upsilon} \varsigma |\xi\rangle \langle \partial_\xi | \ ,$$

with underlined fields independent of Y.

Affine Scherk–Schwarz Ansatz

From this one gets the gauge transformations

$$\delta_{\underline{\xi}}|\underline{V}\rangle \equiv U^T \mathcal{L}_{\xi,\Sigma}|V\rangle = \langle\underline{\theta}|\, \underline{C}_{-1}\,|\underline{\xi}\rangle \otimes |\underline{V}\rangle + \langle\underline{\vartheta}|\, \underline{C}_0\,|\underline{\xi}\rangle \otimes |\underline{V}\rangle$$
$$= \sum_{n\in\mathbb{Z}} \eta_{AB}\langle\underline{\theta}|\, T^A_{n-1}|\underline{\xi}\rangle\, T^B_{-n}|\underline{V}\rangle + \sum_{n\in\mathbb{Z}} \eta_{AB}\langle\underline{\vartheta}|\, T^A_{n}|\underline{\xi}\rangle\, T^B_{-n}|\underline{V}\rangle$$
$$- \langle\underline{\theta}|L_{-1}|\underline{\xi}\rangle|\underline{V}\rangle - \langle\underline{\theta}|\underline{\xi}\rangle L_{-1}|\underline{V}\rangle - \langle\underline{\vartheta}|L_0|\underline{\xi}\rangle|\underline{V}\rangle - \langle\underline{\vartheta}|\underline{\xi}\rangle L_0|\underline{V}\rangle$$

for constant parameters

$$\langle\underline{\theta}| \equiv \langle\underline{\partial}\upsilon|\underline{L}_1 + e^{-\upsilon}\langle\underline{\partial}\varsigma|(\underline{L}_0 - 1) + \sum_n \langle\underline{j}_{nA}|\underline{T}^A_{n+1} \;,$$
$$\langle\underline{\vartheta}| \equiv -\langle\underline{\partial}\upsilon|(\underline{L}_0 + 1) - e^{-\upsilon}\langle\underline{\partial}\varsigma|\underline{L}_{-1} - \sum_n \langle\underline{j}_{nA}|\underline{T}^A_{n} - \langle\underline{j}_c| \;,$$

Gauge algebra

The gauge algebra closes as ([Samtleben, Weidner] for $\langle\vartheta| = 0$)

$$[\delta_{\xi_1}, \delta_{\xi_2}]\, |\underline{V}\rangle \;=\; \delta_{\xi_{12}}\, |\underline{V}\rangle \; ,$$

with gauge parameter

$$|\xi_{12}\rangle \;\equiv\; \frac{1}{2}\big(\langle\underline{\theta}|\underline{C}_{-1} + \langle\underline{\vartheta}|\underline{C}_0\big)\big(|\underline{\xi}_1\rangle \otimes |\underline{\xi}_2\rangle - |\underline{\xi}_2\rangle \otimes |\underline{\xi}_1\rangle\big) \; ,$$

provided that the components of the embedding tensor satisfy the constraints

$$\begin{aligned} \langle\underline{\theta}| \otimes \langle\underline{\theta}|\underline{C}_{-1} + \langle\underline{\vartheta}| \otimes \langle\underline{\theta}|\,(\underline{C}_0 - 1) + \langle\underline{\theta}| \otimes \langle\underline{\vartheta}| \;&= 0 \; , \\ \langle\underline{\vartheta}| \otimes \langle\underline{\vartheta}|\underline{C}_0 + \langle\underline{\theta}| \otimes \langle\underline{\vartheta}|\underline{C}_{-1} &= 0 \; . \end{aligned}$$

Conclusion

⋆ Generalized diffeomorphisms for E_9 can be defined at the cost of additional shift gauge invariance (like for E_8)

⋆ The closure of the algebra only relies on Virasoro algebra and is identical for all affine groups

⋆ The Scherk–Schwarz Ansatz gives the expected gauge algebra with embedding tensor and 'trombone generalization'.

⋆ Loop amplitudes smoothly generalize from E_8 to E_9 (for Langlands Eisenstein).