L_∞ algebras and CFT

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Introduction: L_∞

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- L_∞ algebras first appeared in higher spin gauge theories with field dependent gauge parameters (Berends, Burgers, van Dam, 1985)

$$[\delta_{\lambda_1}, \delta_{\lambda_2}]\Phi = \delta_{C(\lambda_1,\lambda_2,\Phi)}\Phi\,.$$

- They also appeared as "generalized" gauge symmetries of closed string field theory. Higher products from

$$\delta_\lambda \Phi = \sum_n \ell_n(\lambda, \Phi^{n-1})\,, \qquad \mathcal{F}(\phi) = \sum_n \ell_n(\Phi^n)$$

plus relations.(Zwiebach, 1993)

- It was shown that they are also realized in effective theories, like Yang-Mills etc. (Hohm, Zwiebach, 2017)
- Studied in the mathematics literature as strong homotopy algebras. (Lada, Stasheff, 1993)

Introduction: $\mathcal{W}$ algebras

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- $\mathcal{W}$ algebras appear as extended (global) symmetry algebras of 2D conformal field theories (CFTs), i.e. extensions of the Virasoro algebra (Zamolodchikov, 1985)
- More recently, it was realized that they are the holographic duals of higher spin theories in AdS_3. (Henneaux, Rey, 2010),(Campoleoni, Fredenhagen, Pfenninger, Theisen, 2010), (Gaberdiel, Gopakumar, 2010)
- Since higher spin gauge theories were the first framework, where L_∞ appeared, also $\mathcal{W}$ algebras should show an L_∞ structure
- Since $\mathcal{W}$ algebras are well understood and highly non-trivial closed and associative algebras, they could provide a huge class of non-trivial L_∞ algebras

Aim: Make this explicit.

L_∞ algebras

L_∞ algebras

Definition of L_∞ algebra:

- Graded vector space: $X = \bigoplus_n X_n$
- Multi-linear products: $\ell_n(x_1, \ldots, x_n)$ of degree $\deg(\ell_n) = n - 2$
- These are graded commutative, i.e.

$$\ell_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (-1)^\sigma \epsilon(\sigma; x)\, \ell_n(x_1, \ldots, x_n)\,,$$

where $(-1)^\sigma$ is sign of the permutation and the Koszul sign $\epsilon(\sigma; x)$ is defined by considering a graded commutative algebra $\Lambda(x_1, x_2, \ldots)$ with $x_i \wedge x_j = (-1)^{x_i x_j} x_j \wedge x_i$ and reading the sign from

$$x_1 \wedge \ldots \wedge x_k = \epsilon(\sigma; x)\, x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(n)}\,.$$

L_∞ algebras

L_∞ algebras

- Relations of L_∞ algebra

$$\mathcal{J}_n(x_1, \dots, x_n) := \sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{\sigma} \epsilon(\sigma; x)$$

$$\ell_j\Big(\ell_i(x_{\sigma(1)} \, , \dots , x_{\sigma(i)}) \, , x_{\sigma(i+1)}, \dots , x_{\sigma(n)}\Big) = 0 \, .$$

where the permutations are restricted to the ones with

$$\sigma(1) < \cdots < \sigma(i), \qquad \sigma(i+1) < \cdots < \sigma(n) \, .$$

Note: the L_∞ algebras of interest here are concentrated in

- X_0: symmetry parameters ε
- X_{-1}: the basic fields W

L_∞ algebras

L_∞ algebras

In this case, only the n-products

$$\ell_n(\varepsilon, W^{n-1})\,, \qquad \ell_n(\varepsilon_1, \varepsilon_2, W^{n-2})$$

can be non-trivial.

The schematic form of the first relations is

$$\mathcal{J}_2 = \ell_1\ell_2 - \ell_2\ell_1\,,$$

$$\mathcal{J}_3 = \ell_1\ell_3 + \ell_2\ell_2 + \ell_3\ell_1\,,$$

$$\mathcal{J}_4 = \ell_1\ell_4 - \ell_2\ell_3 + \ell_3\ell_2 - \ell_4\ell_1\,,$$

$$\mathcal{J}_5 = \ell_1\ell_5 + \ell_2\ell_4 + \ell_3\ell_3 + \ell_4\ell_2 + \ell_5\ell_1\,,$$

where the sign reflects the factor $(-1)^{i(j-1)}$ and only $\mathcal{J}_n$ with two or three symmetry parameters are non-trivial.

Basics of $\mathcal{W}$ algebras

Basics of $\mathcal{W}$ algebras

$\mathcal{W}$ algebras: extended (chiral) symmetry algebras in 2D CFT

Particular feature: contain products of the generators

For concreteness, consider the $\mathcal{W}_3$ algebra which has two generators $\mathbf{W} = \{T, W\}$ of conformal dimension 2 and 3.

By Laurant expansion, these contain infinitly many generators for global symmetries

$$T(z) = \sum_{n \in \mathbb{Z}} L_n \, z^{-n-2} \,, \qquad W(z) = \sum_{n \in \mathbb{Z}} W_n \, z^{-n-3}$$

The algebraic structure can either be written as a commutator algebra for the modes are as a operator product expansion for the fields.

Basics of $\mathcal{W}$ algebras

Basics of $\mathcal{W}$ algebras

$$\frac{1}{\hbar} T(z) \circ T(w) = \frac{c/2}{(z-w)^4} + 2\left(\frac{T(w)}{(z-w)^2} + \frac{1}{2}\frac{\partial T(w)}{(z-w)}\right) ,$$

$$\frac{1}{\hbar} T(z) \circ W(w) = 3\left(\frac{W(w)}{(z-w)^2} + \frac{1}{3}\frac{\partial W(w)}{(z-w)}\right) ,$$

$$\frac{1}{\hbar} W(z) \circ W(w) = \frac{c/3}{(z-w)^6}$$

$$+ \alpha\left(\frac{T(w)}{(z-w)^4} + \frac{1}{2}\frac{\partial T(w)}{(z-w)^3} + \frac{3}{20}\frac{\partial^2 T(w)}{(z-w)^2} + \frac{1}{30}\frac{\partial^3 T(w)}{(z-w)}\right)$$

$$+\beta\left(\frac{\Lambda^{\mathrm{qu}}(w)}{(z-w)^2} + \frac{1}{2}\frac{\partial \Lambda^{\mathrm{qu}}(w)}{(z-w)}\right) .$$

Basics of $\mathcal{W}$ algebras

Basics of $\mathcal{W}$ algebras

- $\Lambda^{\rm qu}$ denotes the normal ordered product

$$\Lambda^{\rm qu} = N(TT) - \hbar\frac{3}{10}\partial^2 T$$

- Modes satisfy the Jacobi-identity for

$$\alpha = 2\,, \qquad \beta = \frac{32}{5c+22\hbar}\,.$$

- Classical limit: $\hbar = 0$, i.e. commutator $\rightarrow$ Poisson bracket

$$\{\cdot,\cdot\}_{\rm PB} = \lim_{\hbar\to 0}\frac{1}{i\hbar}\,[\cdot,\cdot]\,.$$

and $\Lambda^{\rm cl} = T(z)\cdot T(z)$.

$\mathcal{W}$ as $\mathbf{L}_\infty$ algebras

$\mathcal{W}$ as L_∞ algebras

From the higher spin AdS_3-CFT_2 duality we expect that these two structures are related.

Show next: Classical $\mathcal{W}$ algebras that describe infinitely many global symmetries are L_∞ algebras $\rightarrow$ thus one has a set of highly non-trivial solutions to the L_∞ relations.

$\mathcal{W}$ as $\mathbf{L}_\infty$ algebras

$\mathcal{W}$ as L_∞ algebras

Two graded vector-spaces: $X_0 = \oplus \varepsilon_i(z)$ and $X_{-1} = \oplus W_i(z)$

The infinitesimal variation of the chiral fields under the symmetries generated by W_i can be determined using

$$\delta_{\varepsilon_i} W_j(z) = \frac{1}{2\pi i} \oint dw \, \varepsilon_i(w) \, W_i(w) \circ W_j(z) \,,$$

The right hand side can be generically evaluated using the form of the OPE between quasi-primary fields

$$W_i(w) \, W_j(z) = \sum_k C_{ij}{}^k \, \frac{a^n_{ijk}}{n!} \, \frac{\partial^n \phi_k(z)}{(w-z)^{h_i+h_j-h_k-n}}$$

with

$$a^n_{ijk} = \binom{2h_k + n - 1}{n}^{-1} \binom{h_k + h_i - h_j + n - 1}{n} \,.$$

$\mathcal{W}$ as $\mathbf{L}_\infty$ algebras

$\mathcal{W}$ as $\mathbf{L}_\infty$ algebras

In a $\mathcal{W}$ algebra the fields $\phi_k(z)$ can themselves be products of the primary generators $W_i(z)$. The ${C_{ij}}^k$ are structure constants, some of them have to be bootstrapped.

Thus, for the variation of the generators we obtain

$$\delta_{\varepsilon_i} W_j(z) = \sum_{\substack{m,n \in \mathbb{Z}_0^+ \\ m+n = h_{ijk}-1}} {C_{ij}}^k \frac{a_{ijk}^n}{m!\,n!}\,\partial^m \varepsilon_i(z)\,\partial^n \phi_k(z)$$

with $h_{ijk} = h_i + h_j - h_k$.

Read off

$$\delta_{\varepsilon_i} W_j = \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\,\ell_{n+1}^{W_j}(\varepsilon_i, \mathbf{W}^n)$$

$\mathcal{W}$ as $\mathbf{L}_\infty$ algebras

$\mathcal{W}$ as L_∞ algebras

Now that we have access to the $\ell_n(\varepsilon, W^{n-1})$ how to get the $\ell_n(\varepsilon_1, \varepsilon_2, W^{n-2})$?

The closure requirement

$$[\delta_{\varepsilon_i}, \delta_{\varepsilon_j}]W_k = \delta_{-\mathbf{C}(\varepsilon_i,\varepsilon_j,\mathbf{W})}W_k$$

is equivalent to the L_∞ relations with two symmetry parameters with

$$\mathbf{C}(\varepsilon_i, \varepsilon_j, \mathbf{W}) = \sum_l \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} \ \ell^{\varepsilon_l}_{n+2}(\varepsilon_i, \varepsilon_j, \mathbf{W}^n)\,.$$

Note, in general one gets field dependent symmetry parameters.

$\mathcal{W}$ as $\mathbf{L_\infty}$ algebras

$\mathcal{W}$ as L_∞ algebras

Using CFT techniques one can finally read off

$$\mathbf{C}(\varepsilon_i, \varepsilon_j, \mathbf{W}) = \sum_l \sum_k C_{ij}{}^k \, \mathcal{P}_{ijk}(\varepsilon_i, \varepsilon_j) \, \partial_l \phi_k(\mathbf{W}) \in X_0 \, .$$

with universal

$$\mathcal{P}_{ijk}(\varepsilon_i, \varepsilon_j) = \sum_{\substack{r,s \in \mathbb{Z}_0^+ \\ r+s = h_{ijk}-1}} \kappa_{ijk}^{rs} \, \partial^r \varepsilon_i \, \partial^s \varepsilon_j$$

with

$$\kappa_{ijk}^{rs} = \frac{(-1)^r (2h_k - 1)!}{r! s! (h_i + h_j + h_k - 2)!} \prod_{t=0}^{s-1} (2h_i - 2 - r - t) \prod_{u=0}^{r-1} (2h_j - 2 - s - u) \, .$$

$\mathcal{W}$ as $\mathbf{L}_\infty$ algebras

$\mathcal{W}$ as L_∞ algebras

L_∞ relations with three symmetry parameters are equivalent to a vanishing Jacobi-identity

$$\sum_{\text{cycl}} \left[\delta_{\varepsilon_i}, [\delta_{\varepsilon_j}, \delta_{\varepsilon_k}]\right] = 0\,.$$

Example: $\mathcal{W}_3$ algebra

Example: $\mathcal{W}_3$ as $\mathbf{L}_\infty$

Example: $\mathcal{W}_3$ as L_∞

From the OPE one can read off

$$\delta_\varepsilon T = \underbrace{\frac{c}{12}\,\partial^3\varepsilon}_{\ell_1^T(\varepsilon)} + \underbrace{\left(2\,\partial\varepsilon\, T + \varepsilon\,\partial T\right)}_{\ell_2^T(\varepsilon, T)}$$

$$\delta_\varepsilon W = \underbrace{\left(3\,\partial\varepsilon\, W + \varepsilon\,\partial W\right)}_{\ell_2^W(\varepsilon, W)}$$

and

$$\delta_\eta W = \underbrace{\frac{c}{360}\,\partial^5\eta}_{\ell_1^W(\eta)} + \underbrace{\alpha\left(\frac{1}{6}\,\partial^3\eta\, T + \frac{1}{4}\,\partial^2\eta\,\partial T + \frac{3}{20}\,\partial\eta\,\partial^2 T + \frac{1}{30}\,\eta\,\partial^3 T\right)}_{\ell_2^W(\eta, T)}$$

$$+ \underbrace{\beta\left(\partial\eta\,(TT) + \frac{1}{2}\eta\,\partial(TT)\right)}_{-\frac{1}{2}\ell_3^W(\eta, T, T)},$$

Example: $\mathcal{W}_3$ as L_∞

Example: $\mathcal{W}_3$ as L_∞

From the commutator of two variations one can read off

$$\mathbf{C}(\varepsilon_1, \varepsilon_2, \mathbf{W}) = \varepsilon_1 \partial \varepsilon_2 - \partial \varepsilon_1\, \varepsilon_2 =: \ell_2^{\varepsilon}(\varepsilon_1, \varepsilon_2)\,,$$

$$\mathbf{C}(\,\varepsilon\,,\,\eta\,, \mathbf{W}) = \varepsilon\, \partial \eta - 2\, \partial \varepsilon\, \eta =: \ell_2^{\eta}(\varepsilon, \eta)$$

and

$$\mathbf{C}(\eta_1, \eta_2, \mathbf{W}) = \ell_2^{\varepsilon}(\eta_1, \eta_2) + \ell_3^{\varepsilon}(\eta_1, \eta_2, T)$$

with

$$\ell_2^{\varepsilon}(\eta_1, \eta_2) = \alpha \left(\frac{1}{30} \eta_1\, \partial^3 \eta_2 - \frac{1}{30} \partial^3 \eta_1\, \eta_2 + \frac{1}{20} \partial^2 \eta_1\, \partial \eta_2 - \frac{1}{20} \partial \eta_1\, \partial^2 \eta_2 \right)$$

and

$$\ell_3^{\varepsilon}(\eta_1, \eta_2, T) = \beta\, T\, (\eta_1 \partial \eta_2 - \partial \eta_1\, \eta_2)\ .$$

Example: $\mathcal{W}_3$ as $\mathbf{L}_\infty$

Example: $\mathcal{W}_3$ as L_∞

One has to check all non-trivial L_∞ relations. Consider e.g. the one $[\delta_\varepsilon, \delta_\eta]W = \delta_{C(\epsilon,\eta)}W$

$$\ell_1^W\big(\ell_2^\eta(\varepsilon,\eta)\big) = \ell_2^W\big(\ell_1^T(\varepsilon),\eta\big) + \ell_2^W\big(\varepsilon,\ell_1^W(\eta)\big)$$

holds only for $\alpha = 2$, and

$$\begin{aligned} 0 = \quad & \ell_2^W\big(\ell_2^\eta(\varepsilon,\eta),T\big) + \ell_2^W\big(\ell_2^W(\eta,T),\varepsilon\big) + \ell_2^W\big(\ell_2^T(T,\varepsilon),\eta\big) \\ & + \ell_3^W\big(\ell_1^T(\varepsilon),\eta,T\big) \end{aligned}$$

vanishing only for $16\alpha = 5c\beta$.
Finally, one also has the non-trivial $\mathcal{J}_4$ relation

$$0 = \ell_3^W\big(\ell_2^\eta(\varepsilon,\eta),T,T\big) - 2\,\ell_3^W\big(\ell_2^T(\varepsilon,T),\eta,T\big) - \ell_2^W\big(\varepsilon,\ell_3^W(\eta,T,T)\big)$$

that is zero for any choice of α and β.

Remarks: $\mathcal{W}$ as $\mathbf{L}_\infty$

Remarks: $\mathcal{W}$ as L_∞

- All other L_∞ relations are also satisfied.
- Thus, one can also bootstrap the structure constants from the L_∞ relations.
- For higher $\mathcal{W}_N$ algebras more higher order products and relations are non-trivial.
- These can be considered as truncations of the enormous $\mathcal{W}_\infty[\mu]$ algebra, which is the asymptotic symmetry algebra of Vasiliev's $\mathrm{hs}[\mu]$ higher spin theory.

Classical $\mathcal{W}_N$ algebras admit a quantum generalizaton $\to$ what effects does that have in the L_∞ structure?

The quantum case

The quantum case

Recall: The point-wise product of functions generalizes to a normal ordered product between operator valued quantum fields, defined as

$$N(\phi\,\chi)(w) = \frac{1}{2\pi i}\oint_{\gamma(w)} dz\,\frac{\phi(z)\circ\chi(w)}{(z-w)}\,,$$

The symmetrized normal ordered product

$$A\star B = \frac{1}{2}\Big(N(AB)+N(BA)\Big)\,.$$

is commutative, but not associative.

The quantum case

The quantum case

Consider e.g.

$$(\varepsilon T)\star T - \varepsilon(T\star T) = \frac{1}{4\pi i}\oint dz\,\frac{\epsilon(z)\,\overbracket{T(z)\circ T}(w)}{(z-w)}$$

$$= \frac{c\hbar}{96}\partial^4\epsilon + \frac{\hbar}{2}\partial^2\epsilon\, T + \frac{\hbar}{2}\partial\epsilon\,\partial T\,,$$

where both sides depend on w.

- These corrections arise from the contraction of operators below the integral
- They are $\hbar$-suppressed relative to the leading order normal ordered products.
- Such corrections do appear throughout the quantum computation

Quantum L_∞

Quantum L_∞

Now, one can proceed as before by e.g. writing schematically

$$\delta_\varepsilon^{\rm qu}\,\Phi \sim \sum_n \varepsilon\ \underbrace{\Phi\star\cdots\star\Phi}_{n \text{ times}} = \sum_{n\geq 0}\frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\ L_{n+1}(\varepsilon,\underbrace{\Phi,\ldots,\Phi}_{n\text{ times}})\,.$$

However, the constactions lead to off-diagonal contributions in the quantum L_∞ relations

$$\begin{aligned}&\mathcal{J}^{\rm qu}_{m+n}(\epsilon_1,\ldots,\epsilon_m,x_1,\ldots,x_n)\\ &+\sum_{\substack{(y_1,\ldots,y_k)\\ \to(x_1,\ldots,x_n)}}\hbar^\xi\ \mathcal{J}^{\rm qu}_{m+k}(\epsilon_1,\ldots,\epsilon_m,\underbrace{y_1,\ldots,y_k}_{\to(x_1,\ldots,x_n)})=0\end{aligned}$$

which are model dependent.

Conclusions

Conclusions

- Classical $\mathcal{W}$ algebras are examples of L_∞ algebras with field dependent symmetry parameters.
- From the L_∞ point of view, these provide highly non-trivial solutions to the intricate L_∞ relations. Ultimative case is the $\mathcal{W}_\infty[\mu]$ algebra,
- It allowed to study a possible quantum deformation of the L_∞ structure that differs from the quantum loop-corrections in gauge theories.

- Would be nice not only to **rewrite** known theories in terms of L_∞ (see also (Hohm, Kupriyanov, Lüst, Traube, 2017)) but to use the latter for learning about new theory.