Algebraic Structures in Double and Exceptional Field Theory

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- O.H, Zwiebach, 1701.08824
- O.H., Kupriyanov, Lüst, Traube, 1709.10004
- O.H., Samtleben, 1707.06693 & to appear
- O.H., work in progress

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Road Map

- Duality covariant formulation in 1) gauged supergravity (‘embedding tensor formalim’) and 2) double/exceptional field theory requires redundant or unphysical objects $\Rightarrow$ ‘higher equivalences’

- analogous features in algebraic topology and homotopy theory, where ‘∞-algebras’ allow one “to live with slightly false algebraic identities in a new world where they become effectively true.” [D. Sullivan]

- Features of physical theories usually taken for granted [e.g.: “continuous symmetries $\equiv$ Lie algebras”] hold only ‘up to homotopy’, which quite likely provides deep pointers for (so far) elusive underlying mathematical structure of DFT/ExFT

Overview

- Strongly Homotopy (sh) or ∞-Algebras

- Field Theories and L_∞ Algebras $\rightarrow$ weakly constrained DFT?

- Leibniz (or Loday) Algebras and their Chern-Simons Gauge Theory

- Topological Phase of $E_{8(8)}$ ExFT as Leibniz Chern-Simons Theory

- General Remarks and Outlook

Strongly Homotopy Lie or L$_\infty$ Algebras

An L$_\infty$ algebras is a graded vector space [Zwiebach (1993), Lada & Stasheff (1993)]

$$X = \bigoplus_{n\in\mathbb{Z}} X_n ,$$

equipped with *multilinear and graded antisymmetric* brackets or maps

$$x_1, \dots, x_n \mapsto \ell_n(x_1, \dots, x_n) \in X_{n-2+\sum_i |x_i|} ,$$

satisfying, for each $n = 1, 2, 3, \dots$, the *generalized Jacobi identities*

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_\sigma (-1)^\sigma \epsilon(\sigma; x)\, \ell_j\big(\ell_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}\big) = 0$$

with the sum over all permutations of n objects with partially ordered arguments (‘unshuffles’), $\sigma(1) \leqslant \cdots \leqslant \sigma(i)$, $\sigma(i+1) \leqslant \cdots \leqslant \sigma(n)$,

and *Koszul sign* $\epsilon(\sigma; x)$, determined for any graded algebra with $x_i x_j = (-1)^{x_i x_j} x_j x_i$ by $x_1 \cdots x_k = \epsilon(\sigma; x)\, x_{\sigma(1)} \cdots x_{\sigma(k)}$

Explicit L_∞-relations

For $n = 1$ we learn that $\ell_1 \equiv Q$ is nil-potent:

$$\ell_1(\ell_1(x)) \; = \; 0$$

For $n = 2$ we learn that ℓ_1 is a derivation of $\ell_2 \equiv [\cdot, \cdot]$:

$$\ell_1(\ell_2(x_1, x_2)) \; = \; \ell_2(\ell_1(x_1), x_2) + (-1)^{x_1} \ell_2(x_1, \ell_1(x_2))$$

For $n = 3$ we learn that $\ell_2 \equiv [\cdot, \cdot]$ satisfies Jacobi only ‘up to homotopy’

$$\begin{aligned} 0 \; = \; & \ell_2(\ell_2(x_1, x_2), x_3) + 2 \text{ terms} \\ & + \ell_1(\ell_3(x_1, x_2, x_3)) \\ & + \ell_3(\ell_1(x_1), x_2, x_3) + 2 \text{ terms} \end{aligned}$$

For $n = 4$ we learn that $\ell_2 \ell_3 + \ell_3 \ell_2$ is zero ‘up to homotopy’, i.e., up to the the failure of ℓ_1 to act as a derivation on ℓ_4

plus infinitely more relations

Constructing L_∞ Algebras

Given a bilinear antisymmetric 2-bracket $[\cdot,\cdot]$, is there L_∞ with $\ell_2 = [\cdot,\cdot]$?

Yes: extend the space V by a second copy V^*

$$X_1 = V^* \xrightarrow{\ \ell_1\ } X_0 = V$$

$$\ell_1(v^*) \;=\; v \;, \qquad \ell_3(u,v,w) \;=\; -\mathsf{Jac}(u,v,w)^*$$

But: 'trivial' because $v \sim w$ iff $v - w = \ell_1(\cdot)$, no further extension

Non-trivial if Jacobiator lives in proper subspace or, more generally,

in image of linear map $\mathcal{D} : U \to V$: $\mathsf{Jac}(\cdot,\cdot,\cdot) = \mathcal{D}f(\cdot,\cdot,\cdot)$.

<u>Theorem:</u>

$$X_2 \cong \mathsf{Ker}(\mathcal{D}) \xrightarrow{\ \ell_1 = \iota\ } X_1 = U \xrightarrow{\ \ell_1 = \mathcal{D}\ } X_0$$

carries L_∞ structure, provided $[\mathsf{Im}(\mathcal{D}), V] \subset \mathsf{Im}(\mathcal{D})$, with

$\ell_3(\cdot,\cdot,\cdot) = -f(\cdot,\cdot,\cdot)$ and generally *non-trivial* $\ell_4(\cdot,\cdot,\cdot,\cdot)$

Field Theories & Weakly Constrained DFT

Dictionary L_∞ algebra $\longleftrightarrow$ field theory:

$$\cdots \to X_1 \xrightarrow{\ell_1} X_0 \xrightarrow{\ell_1} X_{-1} \xrightarrow{\ell_1} X_{-2} \to \cdots$$
$$\chi \qquad\quad \xi \qquad\quad \Psi \qquad\quad \mathrm{EOM}$$

Gauge transformations and field equations:

$$\delta_\xi \Psi \;=\; \ell_1(\xi) + \ell_2(\xi,\Psi) - \tfrac{1}{2}\,\ell_3(\xi,\Psi,\Psi) + \cdots$$
$$0 \;=\; \ell_1(\Psi) - \tfrac{1}{2}\,\ell_2(\Psi,\Psi) - \tfrac{1}{3!}\,\ell_3(\Psi,\Psi,\Psi) + \cdots$$

gauge algebra closes 'up to homotopy': trivial parameters $\xi = \ell_1(\chi)$

Example: Courant algebroid/gauge structure of DFT, with $\ell_2 = [\cdot,\cdot]_c$,
defines L_∞ algebra with $\ell_4 = 0$ [Roytenberg & Weinstein (1998)]
→ generalization to weakly constrained? Indeed, in general L_∞ non-trivial

$$\ell_2(\chi_1,\chi_2) \;=\; \langle \mathcal{D}\chi_1, \mathcal{D}\chi_2\rangle \; (\;=\; \partial^M \chi_1\, \partial_M \chi_2 \;=\; 0\,)$$

→ still *very non-trivial* (non-local projected product needed)
[A. Sen (2016)]

Leibniz Algebras and their Chern-Simons Theory

Leibniz (or Loday) algebra: vector space with product $\circ$, satisfying

$$x \circ (y \circ z) \; = \; (x \circ y) \circ z + y \circ (x \circ z)$$

If $\circ$ antisymmetric $\Rightarrow$ Lie algebra

Defines symmetry variations: $\delta_x y \; = \; \mathcal{L}_x y \; \equiv \; x \circ y$ that close:

$$\begin{aligned}
[\mathcal{L}_x, \mathcal{L}_y] z \; &\equiv \; \mathcal{L}_x(\mathcal{L}_y z) - \mathcal{L}_y(\mathcal{L}_x z) \; = \; x \circ (y \circ z) - y \circ (x \circ z) \\
&= \; (x \circ y) \circ z \; = \; \mathcal{L}_{x \circ y} z
\end{aligned}$$

(Anti-)symmetrizing in x, y:

$$[\mathcal{L}_x, \mathcal{L}_y] z \; = \; \mathcal{L}_{[x,y]} z \;, \qquad \mathcal{L}_{\{x,y\}} z \; = \; 0$$

Thus, $\{,\}$ defines 'trivial vector'. Jacobiator is trivial:

$$\sum_{\text{antisym}} 3[[x_1, x_2], x_3] - \{x_1 \circ x_2, x_3\} \; = \; 0$$

'Trivial space' forms ideal of bracket: $[\cdot, \{\;,\;\}] = \{\cdot, \cdot\}$. Thus:

<u>Theorem:</u> Any Leibniz algebra defines L_∞ algebra with $\ell_2 = [\cdot, \cdot]$

Leibniz-valued Gauge Fields and Chern-Simons Action

Leibniz-valued one-form with gauge transformations

$$\delta_\lambda A_\mu \;=\; D_\mu \lambda \;\equiv\; \partial_\mu \lambda - A_\mu \circ \lambda$$

This closes up to 'higher gauge transformations' (c.f. trivial parameters).

Generalized Chern-Simons action

$$S_{\mathrm{CS}} \;\equiv\; \int \mathrm{d}^3 x \, \epsilon^{\mu\nu\rho} \big\langle A_\mu \, , \, \partial_\nu A_\rho - \tfrac{1}{3} A_\nu \circ A_\rho \big\rangle$$

is gauge invariant provided the inner product $\langle \, , \rangle$ is invariant and

$$\langle x, \{\cdot, \cdot\} \rangle \;=\; 0 \quad \forall x$$

$\Rightarrow$ situation in 3D gauged SUGRA in embedding tensor formalism

[de Wit, Nicolai & Samtleben (2001–2002)]

$\Rightarrow$ any Leibniz algebra with $\langle \, , \rangle$ as above defines Chern-Simons theory

$\Rightarrow$ general dimensions: tensor hierarchy (& corresponding L_∞ algebra)

‘Unbroken Phase’ of $E_{8(8)}$ ExFT

Fields: $e_\mu{}^a,\ A_\mu{}^M,\ B_{\mu M},\ \mathcal{M}_{MN}$, coordinates: (x^μ, Y^M), $M = 1,\ldots,248$

Action:

$$S \;=\; \int \mathrm{d}^3x\,\mathrm{d}^{248}Y e\left(\widehat{R} + \frac{1}{240}D^\mu\mathcal{M}^{MN}D_\mu\mathcal{M}_{MN} - V(\mathcal{M}) + \mathcal{L}_{\text{top}}(A,B)\right)$$

Consider subsector that truncates $\mathcal{M}_{MN}$, say by setting:

$$\mathcal{M}_{MN} \;=\; \delta_{MN} \;:\qquad \mathrm{E}_{8(8)} \;\rightarrow\; \mathrm{SO}(16)$$

However, we want *unbroken phase*, so we set (illegally):

$$\mathcal{M}_{MN} \;=\; 0$$

Perhaps justification in suitably reformulated/enlarged theory?

c.f. unconstrained ‘doubled ’metric $\rightarrow \alpha'$ corrections

[O.H., Siegel & Zwiebach (2013)]

first-order formulation with degenerate frame field? [E. Witten (1988)]

Big Chern-Simons Theory

Leibniz algebra unifies 3D Poincaré and (doubled) generalized diffeos:

$$\Xi \;=\; \big(\xi^a\,,\;\lambda_a\,;\;\Lambda^M\,,\;\Sigma_M\big)$$

$$\Xi_1\circ\Xi_2 \;=\; \big(\xi^a_{12}\,,\;\lambda_{12a}\,;\;\Lambda^M_{12}\,,\;\Sigma_{12M}\big)$$

where [$R^M \equiv f^{MN}{}_K\,\partial_N\Lambda^K + \Sigma^M$, constraint: $\Sigma^M\otimes\partial_M = 0$, etc.]

$$\lambda_{12a} \;=\; \epsilon_{abc}\,\lambda_1^b\,\lambda_2^c + 2\,\Lambda_{[1}{}^N\partial_N\lambda_{2]a}$$

$$\Sigma_{12M} \;=\; \mathbb{L}_1\Sigma_{2M} + \underline{\Lambda_2^N\partial_M R_{1N} - 2\,\alpha\,\xi^a_{[1}\partial_M\lambda_{2]a}} \quad \text{etc.}$$

For algebra element $\mathfrak{A}\equiv\big(e^a,\,\omega_a;\;A^M,\;B_M\big)$ inner product given by

$$\langle\mathfrak{A},\mathfrak{A}\rangle \;=\; \int \mathrm{d}^{248}Y\Big(2\,\alpha\,e^a\omega_a + 2\,A^M B_M - \underline{f^K{}_{MN}A^M\partial_K A^N}\Big)$$

CS action for $\mathfrak{A}_\mu$ precisely top. ExFT action!

Generalization of: pure 3D (super-)gravity $\equiv$ Chern-Simons theory

[Achucarro & Townsend (1986), Witten (1988)]

Consistent Kaluza-Klein to half-maximal $D = 3$ SUGRA

Duality: $O(d+1, d+1)$, coordinates $Y^{\mathcal{M}} \equiv Y^{[MN]}$, M, N fundamental

'Doubled vector' $\Upsilon \equiv (\Lambda^{MN}, \Sigma_{MN})$ satisfies Leibniz algebra w.r.t.

$$\Upsilon_1 \circ \Upsilon_2 \;\equiv\; \Big(\mathcal{L}_{\Upsilon_1}\Lambda_2^{MN}\,,\; \mathcal{L}_{\Upsilon_1}\Sigma_{2MN} + \frac{1}{4}\Lambda_2^{KL}\partial_{MN}K(\Upsilon_1)_{KL}\Big)$$

Generalized Scherk-Schwarz in terms of 'doubled' twist matrix:

$$\mathfrak{U}_{\bar{M}\bar{N}} \;\equiv\; \big(\rho^{-1}U^K{}_{[\bar{M}}U^L{}_{\bar{N}]}\,,\; -\frac{1}{4}\,\rho^{-1}(\partial_{KL}U^P{}_{\bar{M}})U_{P\bar{N}}\big)$$

reads for the gauge vectors:

$$\mathfrak{A}_\mu(x,Y) \;=\; \mathfrak{U}_{\bar{M}\bar{N}}(Y)\,A_\mu{}^{\bar{M}\bar{N}}(x)$$

and is consistent provided

$$\mathfrak{U}_{\bar{M}\bar{N}} \circ \mathfrak{U}_{\bar{K}\bar{L}} \;=\; -X_{\bar{M}\bar{N},\bar{K}\bar{L}}{}^{\bar{P}\bar{Q}}\,\mathfrak{U}_{\bar{P}\bar{Q}}$$

where X is the constant embedding tensor.

$\Rightarrow$ consistency of $D = 6$, $\mathcal{N} = (1,1)$ & $(2,0)$ SUGRA on AdS$_3 \times S^3$

Outlook & Remarks

- algebraic structures beyond Lie arise naturally in string/M-theory
- tensor hierarchy of gauged SUGRA & ExFT suggests ∞-algebra, difficult/unnatural in terms of Lie algebra
- unifying algebraic structure of M-theory? → Hermann and Martin's talks

  affine $E_{9(9)}$ works analogously to $E_{8(8)}$ → Guillaume's talk

  $\Rightarrow$ Lie algebra theory may be the "slightly wrong" framework
- novel products in HSZ theory [O.H., Siegel & Zwiebach (2013)]

  → interpretation as ∞-algebra?

  → more general story for (chiral) CFTs? → Ralph's talk
- global structure of doubled (extended) spaces? [O.H. & Zwiebach (2012)]