On the Viability of TeVeS through Hamiltonian Analysis

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LCDM, Modified Gravity or new Dark Matter models?
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Tensor–Vector–Scalar gravity (TeVeS)

Bekenstein (2004)

• Relativistic gravitation theory for the Modified Newtonian Dinamics (MOND) paradigm

Milgrom (1983)

• Motivated by galactic dynamics, for which it accounts without hypothetical dark matter.
• Recently MONDian dynamics connected to entropic gravity on de Sitter space.

Verlinde (2016)

• For phenomenological consequences, see the talk by Costas Skordis.

Modified Gravitational Theories

- phenomenologically motivated (dark matter, dark energy)
- have to be checked for:

- renormalizability (power-counting)
- internal consistency (assessed by canonical analysis)

What is the situation in General Relativity?

- not renormalizable (at higher loops there appear terms which are at least quadratic in the Riemann curvature tensor)
- canonically consistent

Arnowitt, Deser, Misner (1959–1962)

- Typically, (modified) gravitational theories are gauge field theories, i.e. field theories with constraints
- The Hamiltonian analysis is essential to determine their consistency mainly in terms of absence of negative-norm states (ghosts)
- Example: Higher-derivative gravitational theories

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} + \alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2\right)$$

- renormalizable due to the fourth-order space-time derivatives in the Lagrangian, such that the graviton propagator behaves as $1/k^4$ for $k \to \infty$
- inconsistence due to the presence of ghosts:

$$\frac{1}{k^2 + k^4/m_\alpha^2} = \frac{1}{k^2} - \frac{1}{k^2 + m_\alpha^2},$$

where the second term with wrong sign is a ghost of mass squared $m_\alpha^2 \sim 1/2\kappa\alpha$

What are the ghosts?

• Ghosts are fields with the wrong sign for the *kinetic term*, e.g.

$$S = \int d^4x(-(\partial\phi)^2 + (\partial\xi)^2 - V(\phi,\xi))$$

(with metric of signature $(-,+,+,+)$)
• Ghosts are not unstable by themselves - they have reasonable equations of motion
• Hamiltonian is unbounded from below, leading to inconsistencies when the ghost *couples* with ordinary fields
• With vanishing total energy, ghosts and ordinary fields can exchange arbitrary amounts of energy, already at *classical level*
• In quantum theories, ghost modes render the vacuum unstable (the system is catastrophically unstable, as the lifetime of vacuum vanishes!)

Hamiltonian analysis of systems with constraints

The Hamiltonian formalism is at the basis of quantization:
- canonical quantization

$$[\hat{q}(t), \hat{p}(t)] = i\hbar$$

- path integral quantization

$$\langle q, t | q_0, t_0 \rangle = \int \mathcal{D}q\mathcal{D}p \; e^{i \int_{t_0}^{t} d\tau [p(\tau)\dot{q}(\tau) - H(p(\tau), q(\tau))]}$$

(For Hamiltonians quadratic in momenta the Gaussian integrals can be exactly performed and the transition amplitude written in terms of a path integral over generalized coordinates only)

• The Hamiltonian analysis of systems with constrains was initiated by Dirac

Dirac (1950,1951,1958)
see also Dirac (1964)
Itzykson and Zuber (1980)
Chaichian and Nelipa (1984)
Gitman and Tyutin (1990)
Henneaux and Teitelboim (1994)

• Consider a system with N degrees of freedom described by the Hamiltonian

$$H(p_i, q_i), \quad i = 1, \ldots, N$$

and subject to a number of *primary constraints*

$$\phi_a(p_i, q_i) = 0, \quad a = 1, \ldots, m'$$

- the constraints reduce the number of independent degrees of freedom.

• Simplest solution: solve the constraint equations, but this is often complicated.
• Elegant solution: include the constraints in the Hamiltonian, by the use of Lagrange multipliers:

$$H'(p_i, q_i) = H(p_i, q_i) + \sum_{a=1}^{m} \lambda_a \phi_a(p_i, q_i)$$

• The evolution of an arbitrary dynamical variable $f(p_i, q_i)$ is given by the Poisson bracket with the new Hamiltonian:

$$\dot{f}(p_i, q_i) = \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = \{H, f\} + \sum_{a=1}^{m} \lambda_a \{\phi_a, f\},$$

where

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right).$$

• Consistency conditions: the constraints have to be time-independent, i.e.

$$\dot{\phi}_a(p_i, q_i) = \{H, \phi_a\} + \sum_{b=1}^{m} \lambda_b \{\phi_b, \phi_a\} = 0, \quad a = 1, \ldots, m'.$$

This is a *weak equality*, in the sense introduced by Dirac, i.e. we take $\phi_a(p_i, q_i) = 0$ after calculating Poisson's brackets.

• If the consistency condition is not automatically fulfilled, it has to be imposed as a *secondary constraint*:

$$\Phi_a(p_i, q_i) = \dot{\phi}_a(p_i, q_i) = 0$$

• The consistency of the secondary constraints has to be checked again, which may lead to ternary etc. constraints.
• The procedure is repeated for each new constraint until all are compatible with the equations of motion.
• Assume that one finds altogether *m* constraints (primary, secondary, ternary etc.) $\Phi_a(p_i, q_i)$, all satisfying the time-independence condition,

$$\dot{\Phi}_a(p_i, q_i) = \{H, \Phi_a\} + \sum_{b=1}^{m} \lambda_b \{\Phi_b, \Phi_a\} = 0, \quad a = 1, \ldots, m.$$

• If $\det |\{\Phi_a, \Phi_b\}| \neq 0$ the above consistency equations can be solved for λ_as, which are thus fixed.

- The constraints satisfying $\det|\{\Phi_a, \Phi_b\}| \neq 0$ are called *second-class constraints*.
- Define *Dirac brackets*,

$$\{f, g\}_{DB} = \{f, g\}_{PB} - \sum_{a,b} \{f, \Phi_a\}_{PB} |\{\Phi_a, \Phi_b\}|^{-1} \{\Phi_b, g\}_{PB},$$

which automatically take into account the constraints, since for any function on the phase space one can confirm

$$\{\Phi_a, f\}_{DB} = 0, \quad a = 1, \ldots, m$$

- In the canonical quantization, it is the *Dirac bracket* of dynamical variables which is promoted to commutator.

First-class constraints: What happens if $\det|\{\Phi_a, \Phi_b\}| = 0$?

- Some constraints have vanishing Poisson brackets with all others.
- The arbitrary Lagrange multipliers λ_a cannot be fixed $\Rightarrow$ the dynamical variables $f(p_i, q_i; t)$ at any moment are not uniquely determined by their initial values and equations of motion:

$$f(p_i, q_i; t + \delta t) = f(p_i, q_i; t) + \delta t \dot{f}(p_i, q_i; t)$$

i.e.,

$$f(p_i, q_i; t+\delta t) = f(p_i, q_i; t) + \delta t\{H, f(p_i, q_i; t)\} + \delta t \sum_{a=1}^{m} \lambda_a \{\Phi_a, f(p_i, q_i; t)\}.$$

- Take now some other values λ'_a for the Lagrange multipliers:

$$\Delta f(p_i, q_i; t + \delta t) = \delta t \sum_{a=1}^{m} \left(\lambda_a - \lambda'_a\right) \{\Phi_a, f(p_i, q_i; t)\},$$

which is a *gauge transformation* of f generated by the constraints Φ_a.

- All the values of $f(p_i, q_i; t + \delta t)$ for different λ_as describe the *same* physical state

Gauge fixing: all first-class constraints generate gauge transformations

• To fix a value of λ_a, one needs *subsidiary constraints*, also called *gauge fixing conditions*:

$$\chi_b(p_i, q_i) = 0, \quad b = 1, \ldots, m,$$

with the requirements:
- time-independence

$$\dot{\chi}_b = \{H, \chi_b\} + \sum_{a=1}^{m} \lambda_a \{\phi_a, \chi_b\} = 0$$

- number of subsidiary constraints equals number of first-class constraints;
- integrability condition for the system of equations:

$$\det |\{\phi_a, \chi_b\}| \neq 0, \quad a = b = 1, 2, \ldots, m,$$

- for convenience, one chooses $\{\chi_a, \chi_b\} = 0$.

• With this choice, the Lagrange multipliers can be completely fixed and the gauge freedom removed.

• Number of physical degrees of freedom, *n*, assuming:
2N-dimensional phase space
M first-class constraints
M' second-class constraints

$$n = \frac{2N - M - 2M'}{2}$$

• The (gauge-fixed) path integral for a system with first-class constraints:

$$\langle q_1'', \ldots, q_N''; t'' | q_1', \ldots, q_N'; t' \rangle = \int \prod_t \prod_{i=1}^{N} \frac{\mathcal{D}p_i(t)\mathcal{D}q_i(t)}{(2\pi)^{N-M}} \prod_{a,c} \delta(\chi_a)\delta(\phi_c) \det|\{\chi_a, \phi_c\}| \times \exp\left\{ \mathrm{i} \int_{t'}^{t''} dt \left[\sum_i p_i \dot{q}_i - H(p_i, q_i) \right] \right\}$$

• In gauge field theories, the determinant analogous to $\det|\{\chi_a, \phi_c\}|$, usually dependent on gauge fields in the non-Abelian case, is exponentiated using Grassmann variables, leading to the appearance of *Faddeev–Popov ghosts*.

• In the course of the Hamiltonian analysis, it may appear that some degrees of freedom contribute to the Hamiltonian with kinetic terms of wrong sign (ghosts):

$$H = A_1 p_1^2 + A_2 p_2^2 \ldots - A_k p_k^2 + \ldots, \qquad A_i \geq 0$$

• This happens inevitably in nondegenerate higher derivative theories (Ostrogradsky instabilities)

Ostrogradsky (1850)

• Ostrogradsky's theorem can be evaded if the theory is *degenerate*, i.e. gauge field theories.
• The only way to render such instabilities harmless is to remove them by constraints which prevent the physical phase space to extend in the unstable directions.

Hamiltonian analysis of TeVeS

• Bimetric theory of gravity with extra gravitational degrees of freedom carried by one vector and one scalar field.

Bekenstein (2004)

• Bekenstein frame: for gravitational fields, with metric $\tilde{g}_{\mu\nu}$ and connection $\tilde{\nabla}_\mu$.
• Physical frame: for the matter fields, with metric $g_{\mu\nu}$ and connection ∇_μ, such that

$$g_{\mu\nu} = e^{-2\phi}\tilde{g}_{\mu\nu} - 2\sinh(2\phi)A_\mu A_\nu$$

• Vector field timelike and normalized with respect to the Bekenstein metric,

$$A_\mu A^\mu = -1$$

• The scalar and gravitational fields are coupled nonminimally to the spacetime metric, leading to an involved local dynamics.
• Stability of the theory investigated by Hamiltonian analysis

Chaichian, Klusoň, Oksanen, Tureanu (2014)

- Action of TeVeS:

$$S = S_{\tilde{g}} + S_A + S_\phi + S_m$$

- Action for the tensor field is of Einstein–Hilbert type:

$$S_{\tilde{g}} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\tilde{R}$$

- Action for vector field:

$$S_A = -\frac{1}{32\pi G}\int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}[\kappa F_{\mu\nu}F^{\mu\nu} - 2\lambda(A_\mu A^\mu + 1)]$$

with $F_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ and λ - Lagrange multiplier.

- Action for scalar field:

$$S_\phi = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\left[\mu(\tilde{g}^{\mu\nu} - A^\mu A^\nu)\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + V(\mu)\right],$$

where μ is a non-dynamical dimensionless scalar field and $V(\mu)$ is an arbitrary function that typically depends on a scale.

• All matter fields, denoted generically by χ^A, couple to the physical metric $g_{\mu\nu}$:

$$S_m = \int_{\mathcal{M}} d^4x \sqrt{-g} \mathcal{L}[g, \chi^A, \nabla\chi^A].$$

• For simplicity, we consider a scalar matter field χ with the action

$$S_m = -\int_{\mathcal{M}} d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \mathcal{V}(\chi) \right].$$

• Relation between the determinants of the two metrics:

$$g = e^{-4\phi} \left[1 - (1 - e^{-4\phi})(A_\mu A^\mu + 1) \right] \tilde{g}.$$

• For the analysis of the canonical structure, use Arnowitt–Deser–Misner (ADM) decomposition of the gravitational field.

Arnowitt, Deser, Misner (1959)

- Space-time is assumed to admit a foliation into a union of nonintersecting spacelike hypersurfaces Σ_t, with normal vector n_μ, which are parameterized by the time t.
- The Bekenstein metric $\tilde{g}_{\mu\nu}$ induces a metric $h_{\mu\nu}$ on Σ_t, which is defined as

$$h_{\mu\nu} = \tilde{g}_{\mu\nu} + n_\mu n_\nu$$

- Introduce a timelike vector t_μ, such that $t_\mu \tilde{\nabla} t = 1$, and decompose it along the tangent and normal to Σ_t:

$$t^\mu = N n^\mu + N^\mu,$$

where $N = -n_\mu t^\mu$ is the lapse and $N^\mu = h^\mu{}_\nu t^\nu$ is the shift vector on Σ_t.
- ADM variables, describing the foliation: N, N^μ, $h_{\mu\nu}$

• Decomposition of the metric $\tilde{g}_{\mu\nu}$ in terms of the ADM variables:

$$\tilde{g}_{00} = -N^2 + N^i h_{ij} N^j,\ \tilde{g}_{0i} = h_{ij} N^j,\ \tilde{g}_{ij} = h_{ij}.$$

• Decomposition of the vector field A_μ into components tangent and normal to Σ_t:

$$_\perp A_\mu = h_\mu^{\ \nu} A_\nu,\ A_{\mathbf{n}} = n^\mu A_\mu,$$

respectively, where $h_\mu^{\ \nu} = h_{\mu\rho}\tilde{g}^{\rho\nu} = \delta_\mu^{\ \nu} + n_\mu n^\nu$ is the projection operator onto Σ_t. (The components of the vector field are expressed as $A_0 = NA_{\mathbf{n}} + N^i A_i$ and $A_i = {}_\perp A_i$.)

• Canonical coordinates: h_{ij}, N, N^i, $A_{\mathbf{n}}$, A_i, λ, ϕ, μ and χ

• Canonical momenta: π^{ij}, π_N, π_i, $p_{\mathbf{n}}$, p^i, p_λ, p_ϕ, p_μ and p_χ

• Since the action is independent of the time derivatives of N, N^i, λ, $A_{\mathbf{n}}$ and μ, their canonically conjugated momenta are the primary constraints:

$$\pi_N = 0,\ \pi_i = 0,\ p_\lambda = 0,\ p_{\mathbf{n}} = 0,\ p_\mu = 0$$

• Total Hamiltonian:

$$H = \int_{\Sigma_t} d^3x (N\mathcal{H}_T + N^i\mathcal{H}_i + v_N\pi_N + v^i\pi_i + v_\lambda p_\lambda + v_{\mathbf{n}}p_{\mathbf{n}} + v_\mu p_\mu)$$

• Secondary constraints:

- "momentum constraint"

$$\mathcal{H}_i = -2h_{ij}D_k\pi^{jk} - A_i\partial_j p^j + (\partial_i A_j - \partial_j A_i)p^j + \partial_i\phi p_\phi + \partial_i\chi p_\chi = 0$$

- "Hamiltonian constraint"

$$\mathcal{H}_T = \mathcal{H}_T^{\mathrm{GR}} + \mathcal{H}_T^A + \mathcal{H}_T^\phi + \mathcal{H}_T^\chi = 0$$

- three more constraints

• The first-class constraints $\pi_N, \pi_i, \mathcal{H}_T, \mathcal{H}_i$ are associated with the invariance of the original theory under four-dimensional diffeomorphisms.

• Potential danger for stability: the matter contribution to the Hamiltonian constraint,

$$
\begin{aligned}
\mathcal{H}^{\chi}_{T} = {} & \frac{\sqrt{1-(1-e^{-4\phi})\mathcal{G}_{\lambda}}}{2\sqrt{h}(e^{-4\phi}-(1-e^{-4\phi})A_iA^i)} p_{\chi}^{2} \\
& - \frac{(1-e^{-4\phi})A_{\mathbf{n}}}{e^{-4\phi}-(1-e^{-4\phi})A_iA^i} A^i \partial_i \chi p_{\chi} \\
& + \sqrt{h\left(1-(1-e^{-4\phi})\mathcal{G}_{\lambda}\right)} \times \\
& \times \left[\frac{1-e^{-4\phi}}{2(e^{-4\phi}-(1-e^{-4\phi})A_iA^i)} (A^i\partial_i\chi)^2 + \frac{1}{2} h^{ij}\partial_i\chi\partial_j\chi + e^{-2\phi}\mathcal{V}(\chi) \right]
\end{aligned}
$$

• Kinetic term may be negative, depending on the sign of the denominator.

• Positive sign of

$$e^{-4\phi} - (1 - e^{-4\phi})A_i A^i$$

is essential for stability

• Note that

$$\det(g_{ij}) = e^{-2\phi}\left(e^{-4\phi} - (1 - e^{-4\phi})A_i A^i\right) h,$$

i.e. the same quantity determines the signature of the spatial part of the physical metric.

• Stability condition is equivalent to the requirement that Σ_t are spacelike with respect to the physical metric as well:

$$0 < \phi < \frac{1}{4}\ln\left(1 + \frac{1}{A_i A^i}\right).$$

(The first inequality was derived by Bekenstein (2004) as condition for no superluminal propagation of perturbations.)

Conclusions

• Any plausible modified theory of gravity has to be free of ghost instabilities.
• Canonical (Hamiltonian) analysis is an essential tool in determining the stability and consistency of gravitational theories.
• TeVeS does not exhibit ghosts or other singularities assuming $0 < \phi < \frac{1}{4} \ln \left(1 + \frac{1}{A_i A^i}\right)$ and the canonical structure is sound.