Modern effective interactions

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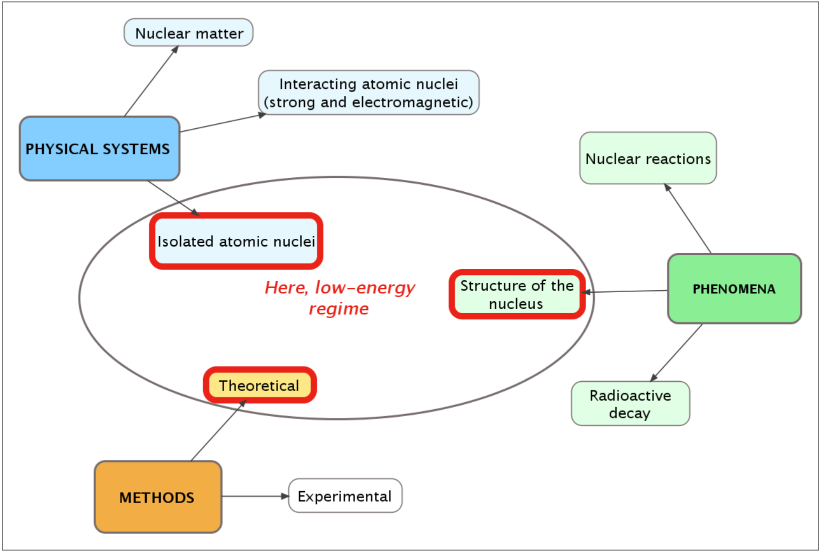
Nuclear matter
Interacting atomic nuclei (strong and electromagnetic)
PHYSICAL SYSTEMS
Nuclear reactions
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Structure of the nucleus
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Radioactive decay
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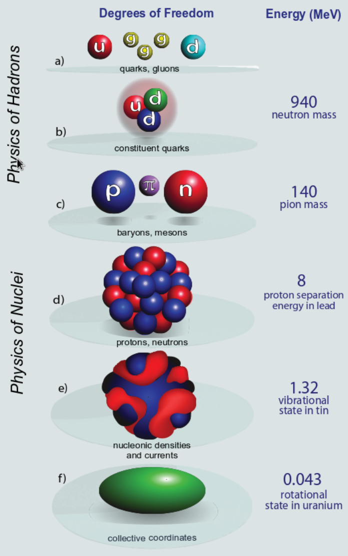
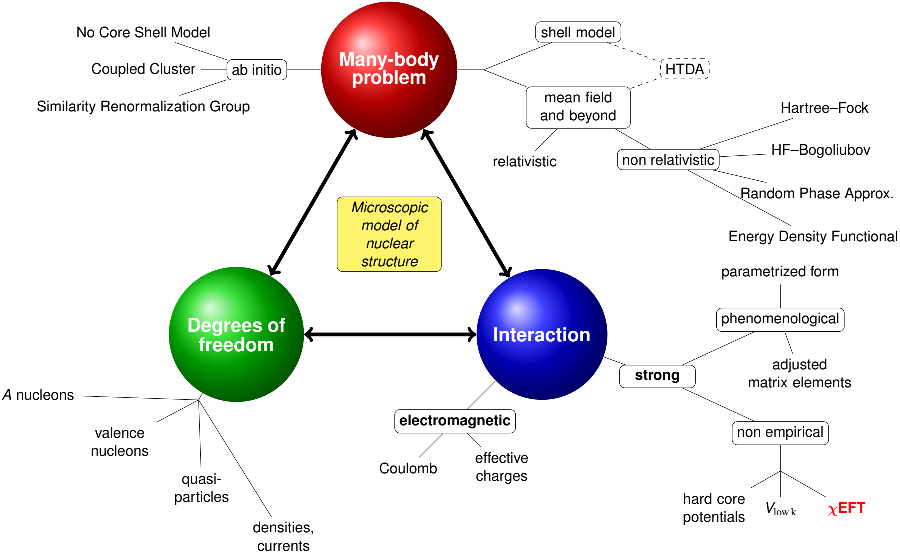


DOE/NSF NSAC, Long-Range Plan 2007



Many-body problem
No Core Shell Model
Coupled Cluster
ab initio
Similarity Renormalization Group
shell model
HTDA
mean field and beyond
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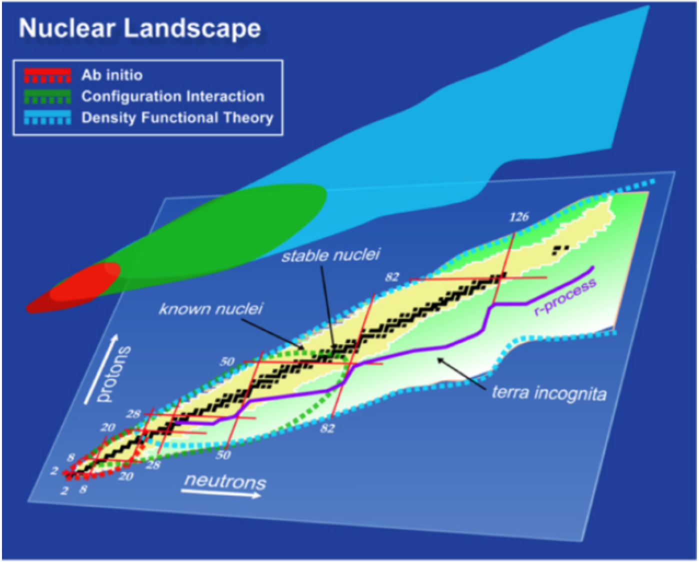
A nucleons
valence nucleons
quasi-particles
densities, currents
electromagnetic
Coulomb
effective charges
non empirical
hard core potentials
$V_{\text{low k}}$
χEFT



R.J. Furnstahl, NPB (Suppl.) **228** (2012).

OUTLINE

GOALS OF THE LECTURE

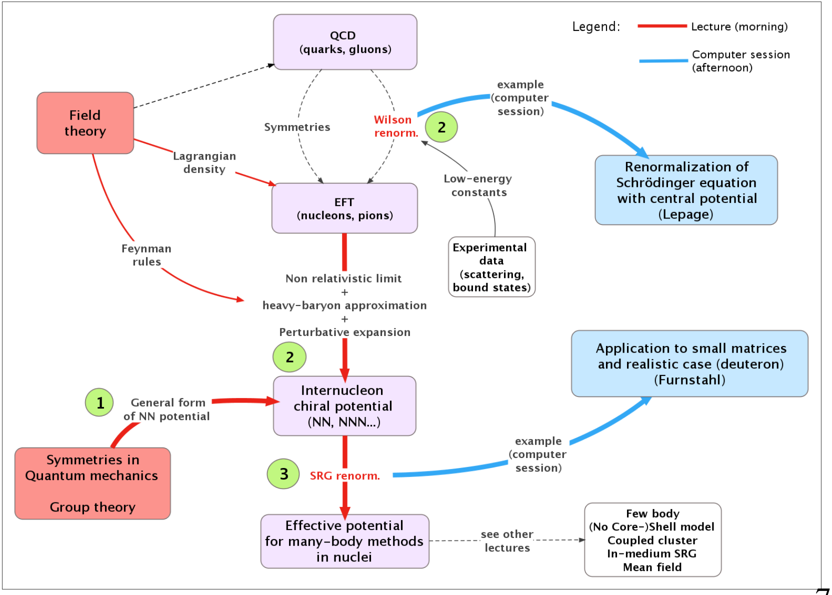
- General form of a two-body potential
- Notion of renormalization (Wilson and by similarity tranformation SRG)
- Introduction to chiral potentials

THEORETICAL AND MATHEMATICAL TOOLS

- Quantum mechanics (including symmetries)
- Group representation theory
- Field theory

References:
[1] J. Dobaczewski, “Interactions, symmetry breaking and effective fields”, lecture at the Ecole Joliot-Curie de Physique Nucléaire (2002). ▸ Link to EJC 02
[2] E. Epelbaum, “Nuclear Forces from chiral effective field theory”, lecture at the 2009 Joliot-Curie School of Nuclear Physics. ▸ Link to EJC 09
[3] R. Machleidt and D. R. Entem, Phys. Rep. **503** (2011).
[4] R.J. Furnstahl, “Renormalization Group in Nuclear Physics”, Nucl. Phys. B (Suppl.) 228 (2012).
[5] G. P. Lepage, “How to renormalize the Schrödinger equation”, arXiv:nucl-th/9706029v1 (1997). ▸ Link to arXiv

Outline



QCD
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Legend:
Lecture (morning)
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(afternoon)
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Schrödinger equation
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rules
Non relativistic limit
+
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+
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Few body
(No Core-)Shell model
Coupled cluster
In-medium SRG
Mean field

PART 1: General form of a two-nucleon potential

1. Operator form in momentum space
2. Symmetries
3. Spin-isospin operator basis
4. Momentum structure functions
5. Final expression and Henley–Miller classification

Definitions

- Individual momenta before $\boldsymbol{p}_i = \hbar \boldsymbol{k}_i$ and after $\boldsymbol{p}'_i = \hbar \boldsymbol{k}'_i$ interaction.
- The partial matrix element $\langle \boldsymbol{k}'_1 \boldsymbol{k}'_2 | \hat{V}_{NN} | \boldsymbol{k}_1 \boldsymbol{k}_2 \rangle$ is at the same time a function of momenta and an operator in spin and isospin spaces

$$\langle \boldsymbol{k}'_1 \boldsymbol{k}'_2 | \hat{V}_{NN} | \boldsymbol{k}_1 \boldsymbol{k}_2 \rangle = \sum \mathcal{F}(\boldsymbol{k}'_i, \boldsymbol{k}_j)\, \hat{O}_s(\hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2) \otimes \hat{O}_t(\hat{\boldsymbol{\tau}}_1, \hat{\boldsymbol{\tau}}_2) \quad (1)$$

where $\hat{\boldsymbol{\sigma}}_i$ and $\hat{\boldsymbol{\tau}}_i$ are the Pauli spin-$1/2$ and isospin-$1/2$ matrices.

Momentum variables

- Two-nucleon system isolated $\Rightarrow$ two-body problem reduces to a one-body problem in the center-of-mass frame.
- Instead of individual momenta $\rightsquigarrow$ Jacobi momenta (here, relative and total momenta)

$$\boldsymbol{k} = \frac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2)\,, \quad \boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2 \quad \text{(before interaction)} \tag{2a}$$

$$\boldsymbol{k}' = \frac{1}{2}(\boldsymbol{k}_1' - \boldsymbol{k}_2')\,, \quad \boldsymbol{K}' = \boldsymbol{k}_1' + \boldsymbol{k}_2' \quad \text{(after interaction)} \tag{2b}$$

$\Rightarrow$ the momentum structure is a priori a function of $\boldsymbol{k}$, $\boldsymbol{k}'$, $\boldsymbol{K}$, $\boldsymbol{K}'$, and one can write

$$\langle \boldsymbol{k}_1' \boldsymbol{k}_2' | \hat{V}_{NN} | \boldsymbol{k}_1 \boldsymbol{k}_2 \rangle = \langle \boldsymbol{k}' \boldsymbol{K}' | \hat{V}_{NN} | \boldsymbol{k} \boldsymbol{K} \rangle\,.$$

Invariance properties and conservation laws

1. invariance by translation in time $\Rightarrow$ conservation of energy and $\hat{V}_{NN}$ is Hermitean
2. invariance by translation in space $\Rightarrow$ conservation of total momentum
3. invariance by a change of Galilean frame
4. invariance by rotation $\Rightarrow$ conservation of total angular momentum and the spin-space part of $\hat{V}_{NN}$ is a scalar
5. invariance by space reflection $\Rightarrow$ conservation of parity
6. invariance by time reversal
7. invariance by permutation
8. $\hat{V}_{NN}$ commutes with $\hat{T}_z = \frac{1}{2}\left(\hat{\tau}_{1,z} + \hat{\tau}_{2,z}\right) \Rightarrow$ conservation of neutron and proton numbers

Transformation properties of spin and isospin Pauli matrices

1. Hermiticity (spin and isospin)

$$\hat{\boldsymbol{\sigma}}_i^\dagger = \hat{\boldsymbol{\sigma}}_i \qquad \hat{\boldsymbol{\tau}}_i^\dagger = \hat{\boldsymbol{\tau}}_i \tag{3}$$

2. translation in space (spin and isospin): invariant
3. change of Galilean frame (spin and isospin): invariant
4. rotation (spin only): $\hat{\boldsymbol{\sigma}}_i$ transforms as a vector
5. space reflection (spin only)

$$\hat{\Pi}\hat{\boldsymbol{\sigma}}_i\hat{\Pi}^{-1} = \hat{\boldsymbol{\sigma}}_i \tag{4}$$

6. time reversal (spin and isospin)

$$\hat{\mathcal{T}}\hat{\boldsymbol{\sigma}}_i\hat{\mathcal{T}}^{-1} = -\hat{\boldsymbol{\sigma}}_i \tag{5}$$

$$\hat{\mathcal{T}}\hat{\boldsymbol{\tau}}_{i,x/z}\hat{\mathcal{T}}^{-1} = \hat{\boldsymbol{\tau}}_{i,x/z} \tag{6}$$

$$\hat{\mathcal{T}}\hat{\boldsymbol{\tau}}_{i,y}\hat{\mathcal{T}}^{-1} = -\hat{\boldsymbol{\tau}}_{i,y} \tag{7}$$

7. permutation (spin and isospin): indices $1 \leftrightarrow 2$
8. commutation relations for isospin operators:

$$[\hat{\tau}_x, \hat{\tau}_y] = 2\, i\, \hat{\tau}_z \quad \text{(+ circular permutations)} \tag{8}$$

Consequences for the momentum structure

1. Hermiticity:
$$\mathcal{F}(\boldsymbol{k}',\boldsymbol{K}';\boldsymbol{k},\boldsymbol{K}) = \mathcal{F}(\boldsymbol{k},\boldsymbol{K};\boldsymbol{k}',\boldsymbol{K}')^* \tag{9}$$
2. invariance by translation in space
$$\mathcal{F}(\boldsymbol{k}',\boldsymbol{K}';\boldsymbol{k},\boldsymbol{K}) = \delta(\boldsymbol{K}'-\boldsymbol{K})\,\mathcal{F}(\boldsymbol{k}',\boldsymbol{k},\boldsymbol{K}) \tag{10}$$
3. invariance by a change of Galilean frame
$$\mathcal{F}(\boldsymbol{k}',\boldsymbol{k},\boldsymbol{K}) = \mathcal{F}(\boldsymbol{k}',\boldsymbol{k}) \tag{11}$$
4. invariance by rotation: $\mathcal{F}(\boldsymbol{k}',\boldsymbol{k})$ and $\hat{O}_s(\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)$ are **spherical tensors of the same rank**, fully contracted to form a scalar Spherical tensors
5. invariance by space reflection:
$$\hat{\Pi}\boldsymbol{k}\hat{\Pi}^{-1} = -\boldsymbol{k} \;\Rightarrow\; \mathcal{F}(\boldsymbol{k}',\boldsymbol{k}) = \mathcal{F}(-\boldsymbol{k}',-\boldsymbol{k}) \tag{12}$$

$\Rightarrow \mathcal{F}$ involves products of an **even number** of momentum vectors

Consequences for the momentum structure

6. invariance by time reversal + Hermiticity

$$\mathcal{F}(\boldsymbol{k}', \boldsymbol{k}) = \mathcal{F}(-\boldsymbol{k}, -\boldsymbol{k}') \tag{13}$$

7. invariance by permutation: same constraint as from space reflection

$$\hat{P}_{12}\boldsymbol{k}\hat{P}_{12}^{-1} = -\boldsymbol{k} \;\Rightarrow\; \mathcal{F}(\boldsymbol{k}', \boldsymbol{k}) = \mathcal{F}(-\boldsymbol{k}', -\boldsymbol{k}) \tag{14}$$

   $\Rightarrow$ no additional constraint on $\mathcal{F}$

8. commutation with $\hat{T}_z$: any $\mathcal{F}(\boldsymbol{k}', \boldsymbol{k})$ commutes with isospin operators
   $\Rightarrow$ no additional constraint on $\mathcal{F}$

Group representation method[1]

Notations:

- $\mathcal{E}_s$ = Hilbert space of spin states of 1 nucleon ($s = 1/2$)
  $\Rightarrow \dim \mathcal{E}_s = 2s + 1$
- $\mathcal{E}_t$ = Hilbert space of isospin states of 1 nucleon ($t = 1/2$)
  $\Rightarrow \dim \mathcal{E}_t = 2t + 1$
- $\mathcal{E} = \mathcal{E}_s \otimes \mathcal{E}_t$ = Hilbert space of spin and isospin states of 1 nucleon
- $\mathcal{E}^{\otimes N}$ = Hilbert space of spin and isospin states of N nucleons
- $E = (\mathcal{E}^{\otimes N})^* \otimes \mathcal{E}^{\otimes N}$ = vector space of linear operators acting on $\mathcal{E}^{\otimes N}$ (two-nucleon spin-isospin operators)

Remark: F^* is the dual space of the vector space F, that is the vector space of linear applications from F to $\mathbb{R}$ (such as the scalar product)

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[1] Method due to Phillips and Schat, Phys. Rev. C **88**, 034002 (2013).

Group representation method

Main ideas:

1. Build irreducible representations (irreps) D of the group SU(4) in the vector space E
2. Decompose the restriction of D to the subgroup $\mathrm{SU}(2)_s \times \mathrm{SU}(2)_t \subset \mathrm{SU}(4)$ into irreps of $\mathrm{SU}(2)_s \times \mathrm{SU}(2)_t$ in the vector space E (using so-called branching rules[2]) $\Rightarrow$ each such irrep is labeled by spin and isospin quantum numbers which are the ranks of the corresponding spin and isospin spherical-tensor operators

[2] See, e.g, Hecht and Pang, J. Math. Phys. **10**, 1571 (1969).

3) Spin-isospin operator basis

Application to two-nucleon spin-isospin operators

- Irrep of SU(4) in the spin-isospin space of 1 nucleon: fundamental representation $R_1 = \overset{4}{\square}$ (of dimension $(2s+1)(2t+1) = 4$)
- Irreducible decomposition of the tensor product $R_1^{\otimes 2} = \overset{4}{\square} \otimes \overset{4}{\square}$ into irreps R_2 of SU(4) in the spin-isospin space of 2 nucleons:

$$\overset{4}{\square} \otimes \overset{4}{\square} = \overset{6}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus \overset{10}{\square\square} \tag{15}$$

- Branching rules: irreducible decomposition of $\overset{6}{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ and $\overset{10}{\square\square}$ into irreps of $\mathrm{SU}(2)_s \times \mathrm{SU}(2)_t$ denoted by (D_S, D_T) [3]

$$\overset{6}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = (D_0, D_1) \oplus (D_1, D_0) \qquad \overset{10}{\square\square} = (D_0, D_0) \oplus (D_1, D_1) \tag{16}$$

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[3] D_J is the irrep of SU(2) in the vector space of angular-momentum states $|JM\rangle$ (of dimension $2J+1$).

Application to two-nucleon spin-isospin operator basis

- Using the property

$$(D_{S_1}, D_{T_1}) \otimes (D_{S_2}, D_{T_2}) = (D_{S_1} \otimes D_{S_2}, D_{T_1} \otimes D_{T_2}) \tag{17}$$

and the Clebsch–Gordan series

$$D_{j_1} \otimes D_{j_2} = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} D_J \tag{18}$$

decompose $R_1^{\otimes 2} \otimes R_1^{\otimes 2}$ into irreps of $\mathrm{SU}(2)_s \times \mathrm{SU}(2)_t$

$$\begin{aligned} R_1^{\otimes 2} \otimes R_1^{\otimes 2} &= 4\,(D_0, D_0) \oplus 6\,(D_1, D_0) \oplus 2\,(D_2, D_0) \qquad \text{(isoscalar, } T = 0) \\ &\oplus 6\,(D_0, D_1) \oplus 9\,(D_1, D_1) \oplus 3\,(D_2, D_1) \qquad \text{(isovector, } T = 1) \\ &\oplus 2\,(D_0, D_2) \oplus 3\,(D_1, D_2) \oplus (D_2, D_2) \qquad \text{(isotensor, } T = 2) \end{aligned} \tag{19}$$

- Spin-isospin operator basis contains 4 spin-scalar–isospin-scalar operators, 6 spin-vector–isospin-scalar operators, ... and 1 rank-2-spin–rank-2-isospin operator

Explicit form

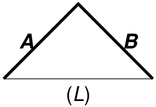
Spin operators

Rank S	Operators	Number
0	$\mathbb{1}$, $\hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2$	2
1	$\hat{\boldsymbol{\sigma}}_1 \pm \hat{\boldsymbol{\sigma}}_2$, $\hat{\boldsymbol{\sigma}}_1 \times \hat{\boldsymbol{\sigma}}_2$	3
2	$\{\hat{\boldsymbol{\sigma}}_1 \otimes \hat{\boldsymbol{\sigma}}_2\}_2$	1

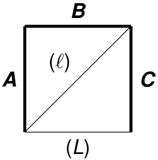
Isospin operators

Rank T	Operators	Number
0	$\mathbb{1}$, $\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2$	2
1	$\hat{\boldsymbol{\tau}}_1 \pm \hat{\boldsymbol{\tau}}_2$, $\hat{\boldsymbol{\tau}}_1 \times \hat{\boldsymbol{\tau}}_2$	3
2	$\{\hat{\boldsymbol{\tau}}_1 \otimes \hat{\boldsymbol{\tau}}_2\}_2$	1

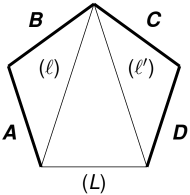
Tensor products of vectors



$$= \{\boldsymbol{A} \otimes \boldsymbol{B}\}_L \tag{20a}$$



$$= \{\{\boldsymbol{A} \otimes \boldsymbol{B}\}_\ell \otimes \boldsymbol{C}\}_L \tag{20b}$$



$$= \{\{\boldsymbol{A} \otimes \boldsymbol{B}\}_\ell \otimes \{\boldsymbol{C} \otimes \boldsymbol{D}\}_{\ell'}\}_L \tag{20c}$$

Elementary tensor-product structures

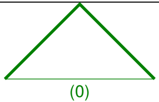
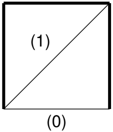
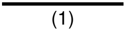
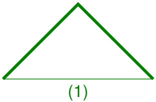
Problem: determine all the independent (non redundant) tensor structures of fixed rank L from a given set of momentum vectors with repetitions allowed, called elementary structures

Example for 2-nucleon case: scalar structures from momentum vectors at hand $\boldsymbol{k}$ and $\boldsymbol{k}'$

- using 2 vectors in the product: $\boldsymbol{k} \cdot \boldsymbol{k}$, $\boldsymbol{k} \cdot \boldsymbol{k}'$, $\boldsymbol{k}' \cdot \boldsymbol{k}'$
- using 3 vectors in the product: no non-vanishing structures (for example $(\boldsymbol{k} \times \boldsymbol{k}') \cdot \boldsymbol{k} = 0$)
- using 4 vectors: no structures independent of those with fewer vectors (for example $(\boldsymbol{k} \times \boldsymbol{k}') \times (\boldsymbol{k} \times \boldsymbol{k}') = \boldsymbol{k} \cdot \boldsymbol{k} - \boldsymbol{k}' \cdot \boldsymbol{k}'$, so it is not elementary)

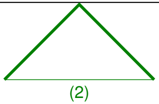
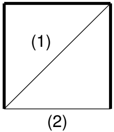
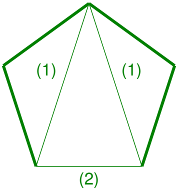
4) MOMENTUM STRUCTURE FUNCTIONS

Independent elementary tensor structures (green: allowed by parity)

Rank L	Elementary tensor products		Hermitean momentum structures
0		(scalar product)	k, k', $\boldsymbol{k}' \cdot \boldsymbol{k}$ (or q, p, $\boldsymbol{q} \cdot \boldsymbol{p}$)
		(scalar triple product)	**0**
1		(“initial” vector)	$i\boldsymbol{q}$, $\boldsymbol{p}$
		(vector product)	$i\boldsymbol{k}' \times \boldsymbol{k} = i\boldsymbol{q} \times \boldsymbol{p}$

with $k = \sqrt{\boldsymbol{k} \cdot \boldsymbol{k}}$, $\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$ and $\boldsymbol{p} = \frac{1}{2}(\boldsymbol{k}' + \boldsymbol{k})$

Independent elementary tensor structures (green: allowed by parity)

Rank L	Elementary tensor products	Hermitean momentum structures
2		$\{\boldsymbol{q} \otimes \boldsymbol{q}\}_2, \{\boldsymbol{p} \otimes \boldsymbol{p}\}_2, i\,\{\boldsymbol{q} \otimes \boldsymbol{p}\}_2$
		$\{(\boldsymbol{q} \times \boldsymbol{p}) \otimes \boldsymbol{q}\}_2, i\,\{(\boldsymbol{q} \times \boldsymbol{p}) \otimes \boldsymbol{p}\}_2$
		$\{(\boldsymbol{q} \times \boldsymbol{p}) \otimes (\boldsymbol{q} \times \boldsymbol{p})\}_2$

5) Final expression

Henley–Miller classification

$$\langle \boldsymbol{k}_1' \boldsymbol{k}_2' | \hat{V}_{NN} | \boldsymbol{k}_1 \boldsymbol{k}_2 \rangle = \delta(\boldsymbol{K}' - \boldsymbol{K}) \left(\langle \boldsymbol{k}' | \hat{v}^{(\mathrm{I})} | \boldsymbol{k} \rangle + \langle \boldsymbol{k}' | \hat{v}^{(\mathrm{II})} | \boldsymbol{k} \rangle + \langle \boldsymbol{k}' | \hat{v}^{(\mathrm{III})} | \boldsymbol{k} \rangle + \langle \boldsymbol{k}' | \hat{v}^{(\mathrm{IV})} | \boldsymbol{k} \rangle \right) \tag{21}$$

- Class I: isospin invariant (isoscalar)
- Class II: charge symmetric $V_{nn} = V_{pp} \neq V_{np}$ (isotensor)
- Class III: charge symmetry breaking but commutes with $\hat{\boldsymbol{T}}^2$ (isovector $\propto \hat{T}_z$)
- Class IV: full isospin symmetry breaking (remaining isovector)

5) Final expression

Class I (isospin invariant)

$$
\begin{aligned}
\langle \boldsymbol{k}' | \hat{v}^{(\mathrm{I})} | \boldsymbol{k} \rangle = {} & (V_C^{(\mathrm{I})} + W_C^{(\mathrm{I})}\, \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2)\, \mathbb{1}_s + (V_S^{(\mathrm{I})} + W_S^{(\mathrm{I})}\, \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2)\, \hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2 \\
& + (V_{LS}^{(\mathrm{I})} + W_{LS}^{(\mathrm{I})}\, \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2)\, i\, (\boldsymbol{k}' \times \boldsymbol{k}) \cdot (\hat{\boldsymbol{\sigma}}_1 + \hat{\boldsymbol{\sigma}}_2) \\
& + \Big[(V_T^{(\mathrm{I})} + W_T^{(\mathrm{I})}\, \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2)\, \{\boldsymbol{q} \otimes \boldsymbol{q}\}_2 + (V_T'^{(\mathrm{I})} + W_T'^{(\mathrm{I})}\, \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2)\, \{\boldsymbol{p} \otimes \boldsymbol{p}\}_2 \\
& \quad + (V_{L\sigma}^{(\mathrm{I})} + W_{L\sigma}^{(\mathrm{I})}\, \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2)\, \{(\boldsymbol{k}' \times \boldsymbol{k}) \otimes (\boldsymbol{k}' \times \boldsymbol{k})\}_2 \Big] \cdot \{\hat{\boldsymbol{\sigma}} \otimes \hat{\boldsymbol{\sigma}}\}_2
\end{aligned}
\tag{22}
$$

where all form factors V and W are real scalar functions of k, k' and $\boldsymbol{k}' \cdot \boldsymbol{k}$, symmetric under the exchange of $\boldsymbol{k}$ and $\boldsymbol{k}'$, and $\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$, $\boldsymbol{p} = \frac{1}{2}(\boldsymbol{k}' + \boldsymbol{k})$

Classes II, III and IV

$$\langle \boldsymbol{k}'|\hat{v}^{(\mathrm{II})}|\boldsymbol{k}\rangle = \Big[V_C^{(\mathrm{II})} + V_S^{(\mathrm{II})}\,\hat{\boldsymbol{\sigma}}_1\cdot\hat{\boldsymbol{\sigma}}_2 + V_{LS}^{(\mathrm{II})}\,i\,(\boldsymbol{k}'\times\boldsymbol{k})\cdot(\hat{\boldsymbol{\sigma}}_1+\hat{\boldsymbol{\sigma}}_2) + \Big(V_T^{(\mathrm{II})}\,\{\boldsymbol{q}\otimes\boldsymbol{q}\}_2 + V_T'^{(\mathrm{II})}\,\{\boldsymbol{p}\otimes\boldsymbol{p}\}_2 + V_{L\sigma}^{(\mathrm{II})}\,\{(\boldsymbol{k}'\times\boldsymbol{k})\otimes(\boldsymbol{k}'\times\boldsymbol{k})\}_2\Big)\cdot\{\hat{\boldsymbol{\sigma}}\otimes\hat{\boldsymbol{\sigma}}\}_2\Big]\,\{\hat{\boldsymbol{\tau}}\otimes\hat{\boldsymbol{\tau}}\}_{20} \tag{23}$$

$$\langle \boldsymbol{k}'|\hat{v}^{(\mathrm{III})}|\boldsymbol{k}\rangle = \Big[V_C^{(\mathrm{III})} + V_S^{(\mathrm{III})}\,\hat{\boldsymbol{\sigma}}_1\cdot\hat{\boldsymbol{\sigma}}_2 + V_{LS}^{(\mathrm{III})}\,i\,(\boldsymbol{k}'\times\boldsymbol{k})\cdot(\hat{\boldsymbol{\sigma}}_1+\hat{\boldsymbol{\sigma}}_2) + \Big(V_T^{(\mathrm{III})}\,\{\boldsymbol{q}\otimes\boldsymbol{q}\}_2 + V_T'^{(\mathrm{III})}\,\{\boldsymbol{p}\otimes\boldsymbol{p}\}_2 + V_{L\sigma}^{(\mathrm{III})}\,\{(\boldsymbol{k}'\times\boldsymbol{k})\otimes(\boldsymbol{k}'\times\boldsymbol{k})\}_2\Big)\cdot\{\hat{\boldsymbol{\sigma}}\otimes\hat{\boldsymbol{\sigma}}\}_2\Big]\,\underbrace{(\hat{\boldsymbol{\tau}}_1+\hat{\boldsymbol{\tau}}_2)_0}_{2\,\hat{T}_z} \tag{24}$$

$$\langle \boldsymbol{k}'|\hat{v}^{(\mathrm{IV})}|\boldsymbol{k}\rangle = i\,(\boldsymbol{k}'\times\boldsymbol{k})\cdot\Big(V_1^{(\mathrm{IV})}\,(\hat{\boldsymbol{\sigma}}_1-\hat{\boldsymbol{\sigma}}_2)\,(\hat{\boldsymbol{\tau}}_1-\hat{\boldsymbol{\tau}}_2)_0 + V_2^{(\mathrm{IV})}\,(\hat{\boldsymbol{\sigma}}_1\times\hat{\boldsymbol{\sigma}}_2)\,(\hat{\boldsymbol{\tau}}_1\times\hat{\boldsymbol{\tau}}_2)_0\Big) \tag{25}$$

Remark: the Coulomb potential is a combination of classes I, II and III

PART 2: Introduction to chiral potentials

1. From QCD to chiral EFT Lagrangian
2. Derivation of the internucleon potential
3. Regularization and Wilson renormalization

Fundamental interactions

- Strong interaction: quantum chromodynamics (QCD) by Politzer, Wilczek et Gross (Nobel prize in 2004); responsible for nuclear binding
- Electromagnetic and weak interactions: electroweak theory by Glashow, Salam and Weinberg (Nobel prize 1979); weak interaction responsible for β decay of nuclei; electrostatic (Coulomb) interaction responsible for limit of stability (fission of heavy nuclei)
- QCD non usable at the energy scale of atomic nuclei because relevant degrees of freedom are nucleons and pions, not quarks and gluons

$\Rightarrow$ need for building effective interactions betweens degrees of freedom adapated to nuclear-structure scales

Chiral symmetry of QCD and pions

‣ Introduction to field theory

- Decoupling of light-quark (u, d, s) and heavy-quark (c, b, t) sectors
- $m_u c^2 \approx 2.5$ MeV, $m_d c^2 \approx 5$ MeV and $m_s c^2 \approx 101$ MeV
  $\Rightarrow$ u et d quarks only (2 flavors)
- $m_u, m_d \ll m_{\text{hadrons}} \Rightarrow$ limit of vanishing mass in $\mathscr{L}_{\text{QCD}}$: chiral symmetry ($\psi_{\text{quarks}} \to \gamma_5 \psi_{\text{quarks}}$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$) and isospin symmetry (mixing of u and d fields; true if $m_u = m_d$)
- Spontaneous chiral-symmetry breaking (solution does not have symmetry of Lagrangian)
  $\Rightarrow$ massless Goldstone bosons associated = pions (mesons $\pi^0 \to u\bar{u}/d\bar{d}$, $\pi^+ \to u\bar{d}$, $\pi^+ \to u\bar{d}$)
- In fact $m_\pi \neq 0$ but small because chiral symmetry is approximate ($m_{u,d} \neq 0$ but small)
  $\Rightarrow$ pions reflect at the same time spontaneous and explicit breaking of chiral symmetry

Chiral effective-field theory

- Most general Lagrangian respecting all symmetries of underlying theory (QCD), especially chiral symmetry, using nucleon and pions field

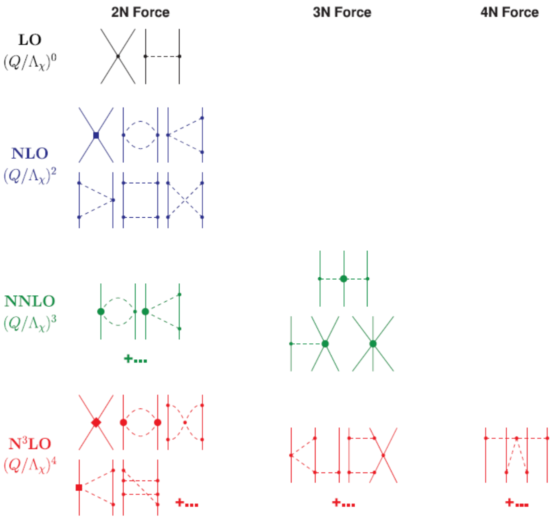
$$\mathscr{L}_{\text{eff}} = \mathscr{L}_N + \mathscr{L}_\pi + \mathscr{L}_{\pi N} \tag{26}$$

- Chiral perturbation theory:
  - $\mathscr{L}_{\text{eff}}$ contains an infinite number of terms $\Rightarrow$ need to order these terms according to decreasing importance
  - Truncation of $\mathscr{L}_{\text{eff}}$ as a function of $(Q/\Lambda_\chi)^\nu$ where Q is a momentum transfer, $\Lambda_\chi \sim 1$ GeV the chiral-symmetry breaking scale, and ν an integer which depends on the number of interacting nucleons and the number of exchanged mesons
  - At a given truncation order ν, $\mathscr{L}_{\text{eff}}^{(\nu)}$ contains a finite number of terms

1) From QCD to chiral EFT

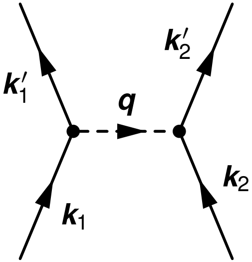
Hierarchy of chiral forces

R. Machleidt, D.R. Entem / Physics Reports 503 (2011) 1–75



Definition

$V = i\,\mathcal{M}$ where $\mathcal{M}$ is the scattering amplitude of the process (example of one-pion exchange two-nucleon potential)



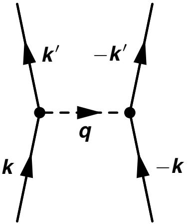
In a potential description of interactions, the propagation time of the exchanged pions is neglected ⇒ instantaneous interaction

2) Derivation of the internucleon potential

Calculation by the Feynman rules

In the heavy-baryon approximation ($m_N \gg m_\pi$) and non relativistic limit, the dominant long-range part of the pion–nucleon Lagrangian is

$$\mathscr{L}_{\pi N}^{(AV)} = -\frac{g_A}{2F_\pi}\overline{N}\Big(\boldsymbol{\tau}\cdot\big[(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\pi}\big]\Big)N \tag{27}$$



$$V_{NN}^{(AV)} = i\underbrace{\left(-\frac{g_A}{2F_\pi}\right)(\boldsymbol{\sigma}_1\cdot\boldsymbol{q})\,\tau_1^a}_{\text{left vertex}}\times\underbrace{\frac{i\delta_{ab}}{-\boldsymbol{q}^2-m_\pi^2}}_{\text{pion propagator}}\times\underbrace{\left(-\frac{g_A}{2F_\pi}\right)(\boldsymbol{\sigma}_2\cdot\boldsymbol{q})\,\tau_2^b}_{\text{right vertex}} \tag{28}$$

$$= -\left(\frac{g_A}{2F_\pi}\right)^2\frac{(\boldsymbol{\sigma}_1\cdot\boldsymbol{q})(\boldsymbol{\sigma}_2\cdot\boldsymbol{q})}{q^2+m_\pi^2}\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 \text{ (one-pion exchange potential)} \tag{29}$$

Design of effective theories[4]

- Low-energy phenomena can be sensitive to short-distance physics, but not its details
- Freedom to redesign the short-distance interaction (Lagrangian, potential...) $\Rightarrow$ effective theories describing any low-energy data with arbitrary precision
    1. Incorporate in the interaction the correct long-range behavior in the potential (supposed to be known from underlying theory, including parameters)
    2. Introduce a cutoff to exclude explicit high-momentum contributions and make interactions regular at $r = 0$
    3. Add counterterms to the interaction to mimic the short-distance/high-momentum effects and remove the cutoff dependence

[4] See G. P. Lepage lecture notes, "How to renormalize the Schrödinger equation", arXiv:nucl-th/9706029v1 (1997). ▸ Link to arXiv

Application to leading-order chiral potential

Renormalization applied to chiral effective Lagrangian yields at LO

$$\hat{V}_{NN}^{(LO)} = \hat{v}_{1\pi} + \hat{v}_{ct}^{(0)} \tag{30}$$

where

long range: $\langle \boldsymbol{k}' | \hat{v}_{1\pi} | \boldsymbol{k} \rangle = -\left(\frac{g_A}{2f_\pi}\right)^2 \frac{(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{q})(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{q})}{q^2 + m_\pi^2} \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2 \, f_\Lambda(k', k)$ (31a)

short range: $\langle \boldsymbol{k}' | \hat{v}_{ct}^{(0)} | \boldsymbol{k} \rangle = \Big(C_S(\Lambda) + C_T(\Lambda) \, \hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2 \Big) \, f_\Lambda(k', k)$ (31b)

typical cutoff function: $f_\Lambda(k', k) = e^{-(k'^6 + k^6)/\Lambda^6}$ (31c)

$C_S(\Lambda)$ and $C_T(\Lambda)$ constants are to be fitted to some low-energy data (typically scattering data at specific energies) for a given cutoff [5]

[5] E. Epelbaum et al., Nucl. Phys. A**747** (2005)

Renormalization of Schrödinger equation

Renormalization applied to the Schrödinger equation of a spinless particle in a local, central potential

$$\Big[-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big)+\frac{1}{2\mu r^2}\mathbf{L}^2\Big]\psi(\mathbf{r})+V(r)\psi(\mathbf{r})=E\psi(\mathbf{r})\,. \qquad (32)$$

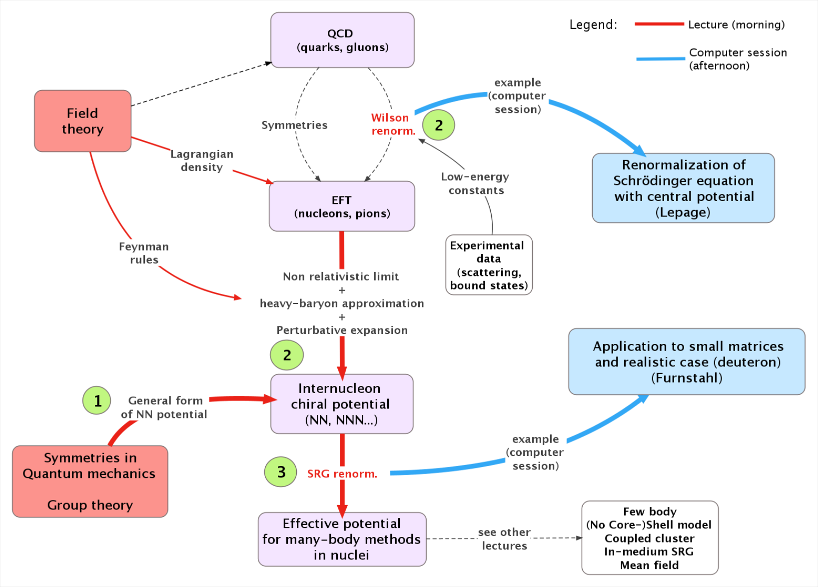
Setting $\psi(\mathbf{r})=\frac{u_\ell(r)}{r}\;Y_\ell^m(\hat{r})$ we get

$$-\frac{\hbar^2}{2m}\Big(-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}\Big)\,u_\ell(r)+V(r)\,u_\ell(r)=E\,u_\ell(r)\,. \qquad (33)$$

Work to do

- Compute eigenvalues E for a given “bare” potential $V(r)$ whose long-range behavior is known
- Replace $V(r)$ with an effective potential $V_{\mathrm{eff}}(r)$ having the same long-range form and counterterms with 2 constants
- Fit the 2 parameters to the least-bound state energy
- Plot relative error on remaining eigenvalues for bound states as a function of energy (Lepage plot)

Summary



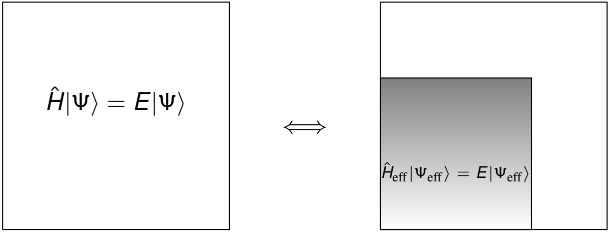
Legend:
Lecture (morning)
Computer session (afternoon)
QCD (quarks, gluons)
Field theory
Symmetries
Wilson renorm.
2
example (computer session)
Renormalization of Schrödinger equation with central potential (Lepage)
Lagrangian density
Low-energy constants
EFT (nucleons, pions)
Experimental data (scattering, bound states)
Feynman rules
Non relativistic limit
+
heavy-baryon approximation
+
Perturbative expansion
2
Application to small matrices and realistic case (deuteron) (Furnstahl)
1
General form of NN potential
Internucleon chiral potential (NN, NNN...)
example (computer session)
Symmetries in Quantum mechanics
Group theory
3
SRG renorm.
Effective potential for many-body methods in nuclei
see other lectures
Few body
(No Core-)Shell model
Coupled cluster
In-medium SRG
Mean field

PART 3: SRG transformation of $\hat{V}_{NN}$ for nuclear-structure calculations

1. Renormalization by similarity transformation
2. Application to the potential
3. Evolution of other operators

Effective potentials for nuclear-structure calculations

- Non observable character of a potential $\hat{V}$ and a Hamiltonian $\hat{H}$: only the eigenvalues of $\hat{H}$ are observables (can be measured), neither its matrix elements nor its eigenvectors
- Reduction to a restricted Hilbert space for practical reasons: need for a transformation preserving the spectrum of $\hat{H}$



SRG method[6]

- Idea: succession of infinitesimal, unitary transformations $\hat{U}_s$ of a Hamiltonian $\hat{H}$ to bring it into a simpler form for subsequent nuclear-structure calculations (Wegner 1994, Glazek et Wilson 1993)
- "Flow equation" for the transformed Hamiltonian $\hat{H}_s = \hat{U}_s \hat{H} \hat{U}_s^\dagger$, $\hat{U}_{s=0} = \mathbb{1}$

$$\hat{U}_s \text{ unitarity: } \hat{U}_s^{-1} = \hat{U}_s^\dagger \Rightarrow \hat{U}_s \frac{d\hat{U}_s^\dagger}{ds} = -\frac{d\hat{U}_s}{ds}\hat{U}_s^\dagger \tag{34}$$

$$\begin{aligned}
\frac{d\hat{H}_s}{ds} &= \frac{d\hat{U}_s}{ds}\hat{H}\hat{U}_s^\dagger + \hat{U}_s\hat{H}\frac{d\hat{U}_s^\dagger}{ds} \\
&= -\hat{U}_s\frac{d\hat{U}_s^\dagger}{ds}\underbrace{\hat{U}_s\hat{H}\hat{U}_s^\dagger}_{\hat{H}_s} + \underbrace{\hat{U}_s\hat{H}}_{\hat{H}_s\hat{U}_s}\frac{d\hat{U}_s^\dagger}{ds} \\
&= \left[\hat{\eta}_s, \hat{H}_s\right] \quad \text{where } \hat{\eta}_s = -\hat{U}_s\frac{d\hat{U}_s^\dagger}{ds} = \frac{d\hat{U}_s}{ds}\hat{U}_s^\dagger
\end{aligned} \tag{35}$$

- Generator of the transformation: Hermitean operator $\hat{G}_s$ defined by

$$\hat{\eta}_s \equiv \left[\hat{G}_s, \hat{H}_s\right] \tag{36}$$

[6] R. J. Furnstahl, Nucl. Phys. B (Suppl.) **228** (2012).

Choice of the generator

Flow equation of $\hat{H}_s$ in terms of $\hat{G}_s$

$$\frac{d\hat{H}_s}{ds} = \left[\left[\hat{G}_s, \hat{H}_s\right], \hat{H}_s\right] \tag{37}$$

With an appropriate choice of the generator $\hat{G}_s$ defined by $\hat{\eta}_s \equiv \left[\hat{G}_s, \hat{H}_s\right]$, one can tailor the final form of the Hamiltonian $\hat{H}_\infty = \lim_{s\to\infty} \hat{H}_s$

- $\hat{G}_s = \hat{T}$ (relative kinetic energy): the Hamiltonian is driven to a diagonal form (see computer session)
- $\hat{G}_s = \begin{pmatrix} \hat{P}\hat{H}_s\hat{P} & 0 \\ 0 & \hat{Q}\hat{H}_s\hat{Q} \end{pmatrix}$, where $\hat{P}$ and $\hat{Q}$ are projectors such that $\hat{P} + \hat{Q} = 1$ and $\hat{P}\hat{Q} = \hat{Q}\hat{P} = 0$: the Hamiltonian is driven to a block-diagonal form (useful to decouple low- and high-momentum states)

Evolution of the potential: simple numerical example

$H = T + V$ with $T = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}$ and $V = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$ (spectrum of H : 7 and 17)

Transformation of hamiltonian matrix : $H(s) = U(s)\, H\, U(s)^\dagger$

SRG flow equation for the transformed potential matrix $V(s) \equiv H(s) - T$

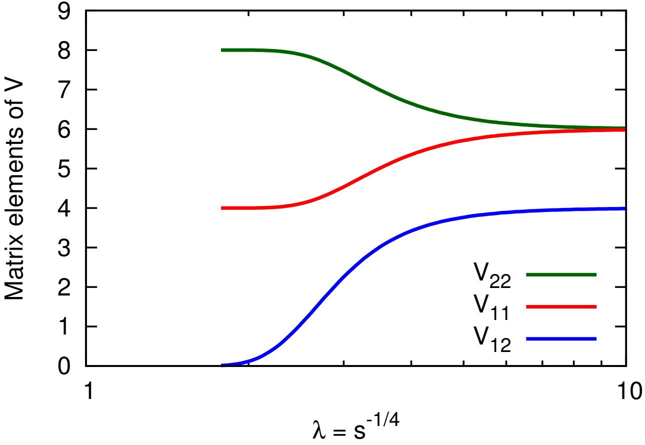
$$\frac{dV(s)}{ds} = [[T, V(s)]\,, T + V(s)]$$

with $V(s) = \begin{pmatrix} V_{11}(s) & V_{12}(s) \\ V_{12}(s) & V_{22}(s) \end{pmatrix}$, hence the nonlinear order-1 differential system

$$\begin{cases} \dfrac{dV_{11}}{ds} = -12\, V_{12}^2(s) \\ \dfrac{dV_{22}}{ds} = 12\, V_{12}^2(s) \\ \dfrac{dV_{12}}{ds} = -6\, V_{12}(s)\left(6 + V_{22}(s) - V_{11}(s)\right) \end{cases} \quad \text{with} \quad \begin{cases} V_{11}(0) = 6 \\ V_{22}(0) = 6 \\ V_{12}(0) = 4 \end{cases}$$

Evolution of the potential: simple numerical example

Numerical solution using the Runge–Kutta method of order 4



$$\lim_{s\to\infty} H(s) = \begin{pmatrix} 7 & 0 \\ 0 & 17 \end{pmatrix}$$

Induced interactions

- Flow equation in operator form
- Practical calculations require a choice of basis
- In basis of physical states of A nucleons (Slater determinants for example, see lecture by Ph. Quentin), SRG evolution of N-body interactions with $N < A$ induce $N+1$-interactions, $N+2$-interactions...

$$\frac{dV_s}{ds} = \Big[\big[\underbrace{\sum a^\dagger a}_{\text{1-body } \hat{G}_s}, \underbrace{\sum a^\dagger a^\dagger aa}_{\text{2-body } \hat{H}_s}\big], \underbrace{\sum a^\dagger a^\dagger aa}_{\text{2-body } \hat{H}_s}\Big]$$

$$= \cdots + \underbrace{\sum a^\dagger a^\dagger a^\dagger aaa}_{\text{3-body}} + \cdots \qquad (38)$$

- In practical calculations, truncation of normal ordered right-hand side[7]

[7] See, e.g., P. Roth et al., Phys. Rev. Lett. **109**, 052501 (2009).

Determination of the unitary transformation $\hat{U}_s$

- By flow equation: according to the relation between $\hat{\eta}_s$ and $\hat{U}_s$

$$\eta_s = \frac{d\hat{U}_s}{ds}\,\hat{U}_s^\dagger$$

and the definition of the generator

$$\hat{\eta}_s \equiv \left[\hat{G}_s, \hat{H}_s\right] \ ,$$

and using unitarity of $\hat{U}_s$, one can deduce the flow equation

$$\frac{d\hat{U}_s}{ds} = \left[\hat{G}_s, \hat{H}_s\right]\,\hat{U}_s \qquad (39)$$

$\Rightarrow$ $\hat{U}_s$ evolved at the same time as $\hat{H}_s$

- By diagonalization of $\hat{H}_s$: eigenstates $|\Psi_i(s)\rangle$

$$\hat{U}_s = \sum_i |\Psi_i(s)\rangle\langle\Psi_i(0)| \qquad (40)$$

Transformed operators

Let $\hat{O}$ be a Hermitean operator (observable). After SRG evolution up to s, the transformed operator is given by

$$\hat{O}_s = \hat{U}_s \hat{O} \hat{U}_s^\dagger \,. \tag{41}$$

It can be calculated directly by matrix multiplication once $\hat{U}_s$ is calculated, or evolved along with the Hamiltonian according to a similar flow equation

$$\frac{d\hat{O}_s}{ds} = \left[\left[\hat{G}_s, \hat{H}_s\right], \hat{O}_s\right] \tag{42}$$

SRG transformation of a matrix

Let $H = T + V$ be a symmetric real matrix of order n, where T is diagonal.

- Reproduce the above numerical example.
- Compute the matrix $P(0)$ of eigenvectors of the initial matrix H.
- Calculate the unitary transformation matrix $U(s)$ for an arbitrary value of s.
- Compute the matrix $P(s)$ of eigenvectors of evolved Hamiltonian matrix $H(s)$.
- Establish the relation between $U(s)$, $P(s)$ and $P(0)$ and check it numerically.

Vectors as rank-1 tensors

- A triplet (v_1, v_2, v_3) is said to be a vector of $\mathbb{R}^3$ with respect to rotations if it transforms as follows under rotation in a fixed frame

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = R \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{43}$$

  where R is the rotation matrix

- Spherical components of a vector $\boldsymbol{v}$ of Cartesian components (v_x, v_y, v_z)

$$v_{\mp 1} = \pm \frac{1}{\sqrt{2}} (v_x \mp i\, v_y) \tag{44a}$$

$$v_0 = v_z \tag{44b}$$

Back to symmetries

[8] See lecture by H. Molique at this school (Friday June 30)

Vectors as rank-1 tensors

- Example: spherical harmonics $Y_1 = \left(Y_1^{-1}, Y_1^0, Y_1^1\right)$

$$\begin{pmatrix} Y_1^{-1}(\theta', \varphi') \\ Y_1^0(\theta', \varphi') \\ Y_1^1(\theta', \varphi') \end{pmatrix} = R \begin{pmatrix} Y_1^{-1}(\theta, \varphi) \\ Y_1^0(\theta, \varphi) \\ Y_1^1(\theta, \varphi) \end{pmatrix}$$

The three spherical components of Y_1 are Y_1^m with $-1 \leqslant m \leqslant 1$.

Back to symmetries

Rank-2 tensors

- Rank-2 tensor from two vectors: $T_2 \equiv \{\boldsymbol{u} \otimes \boldsymbol{v}\}_2$ has $2 \times 2 + 1 = 5$ spherical components $T_{2\mu}$

$$T_{2\mu} = \sum_{\mu_1,\mu_2} C^{2\mu}_{1\mu_1 1\mu_2} u_{\mu_1} v_{\mu_2} = \begin{cases} u_{\pm 1} v_{\pm 1} & \text{if } \mu = \pm 2 \\ \frac{1}{\sqrt{2}} \left(u_{\pm 1} v_0 + u_0 v_{\pm 1} \right) & \text{if } \mu = \pm 1 \\ \frac{1}{\sqrt{6}} \left(u_{+1} v_{-1} + u_{-1} v_{+1} + 2 u_0 v_0 \right) & \text{if } \mu = 0 \end{cases} \tag{45}$$

- Group-theoretical definition: set of 5 numbers that transform under a rotation $\mathcal{R}$ in the same way as the spherical harmonic $Y_2^m(\theta, \varphi)$, namely according to

$$Y_2^m(\theta', \varphi') = \sum_{m'=-2}^{2} \left[D^{(2)}_{mm'}(\mathcal{R}) \right]^* Y_2^{m'}(\theta, \varphi) \tag{46}$$

where $D^{(2)}_{mm'}(\mathcal{R})$ is the element (m, m') of the so-called Wigner rotation matrix, defined by $D^{(\ell)}_{mm'} = \langle \ell m | \hat{\mathcal{R}} | \ell m' \rangle$ where $\hat{\mathcal{R}}$ is the rotation operator

Back to symmetries

Lagrange formulation of classical Mechanics

1) One-particle system

- Degrees of freedom q_i (length or angle) and time derivatives $\dot{q}_i = \frac{dq_i}{dt}$ considered to be independent variables
- Lagrange function or Lagrangian L = difference between kinetic and potential energies

$$L(q_i, \dot{q}_i, t) \equiv T - V \tag{47}$$

- Action for fixed end-points

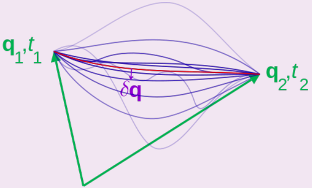
$$S = \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t)\, dt \tag{48}$$

Back to QCD

Lagrange formulation of classical Mechanics

1) ONE-PARTICLE SYSTEM

- Equations of motion result from variational principle: the action is stationary around the path in space-time corresponding to the solution



Credit: Wikipedia

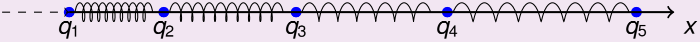
⇒ Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \tag{49}$$

Back to QCD

Lagrange formulation of classical Mechanics

2) MANY-PARTICLE SYSTEM: INFINITE CHAIN OF POINTLIKE MASSES



- Equilibrium solution: $q_n = n\,\Delta x$
- Lagrangian: $L(q_n, \dot{q}_n) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2}\, m\, \dot{q}_n^2 - \frac{1}{2}\, k\, (q_n - q_{n+1})^2\right]$
- Notation: $q_n = \varphi(n\Delta x, t)$ where the real, scalar function φ gives the abscissa on the x axis at time t (called a real, scalar field)

Back to QCD

Lagrange formulation of classical Mechanics

3) Continuum limit

- $\Delta x \to 0$
- Order 1 Taylor expansion

$$q_{n+1}(t) - q_n(t) = \varphi((n+1)\Delta x, t) - \varphi(n\Delta x, t) \approx \Delta x \left(\frac{\partial \varphi}{\partial x}\right)_{x=n\Delta x} \tag{50}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{2} k \left(q_n - q_{n+1}\right)^2 \approx \frac{1}{2} \rho c^2 \int_{-\infty}^{\infty} \left(\frac{\partial \varphi}{\partial x}\right)^2 dx \tag{51}$$

where $\rho = \dfrac{m}{\Delta x}$ and $c = \sqrt{\dfrac{k}{m}} \Delta x$. Similarly for the kinetic term

$$\sum_{n=-\infty}^{\infty} \frac{1}{2} k \dot{q}_n^2 \approx \frac{1}{2} \rho \int_{-\infty}^{\infty} \left(\frac{\partial \varphi}{\partial t}\right)^2 dx \tag{52}$$

Back to QCD

Lagrange formulation of classical Mechanics

3) Continuum limit

- Lagrangian becomes an integral over the coordinate variable x

$$L(t) = \int_{-\infty}^{\infty} \mathscr{L}(\varphi, \partial_x\varphi, \partial_t\varphi)\, dx \tag{53}$$

where $\mathscr{L}$ is a Lagrangian density (often improperly called Lagrangian)

$$\mathscr{L}(\varphi, \partial_x\varphi, \partial_t\varphi) = \frac{1}{2}\,\rho\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{1}{2}\,\rho\, c^2\left(\frac{\partial\varphi}{\partial x}\right)^2 \tag{54}$$

- Euler–Lagrange equation becomes, with implicit summation over repeated indices $\mu \in \{x, t\}$

$$\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathscr{L}}{\partial\varphi} = 0 \tag{55}$$

that is

$$\frac{1}{c^2}\,\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial^2\varphi}{\partial x^2} \quad \text{(longitudinal wave equation)} \tag{56}$$

Lagrangian for the Dirac equation

- Dirac equation as a relativistc equation of motion for a free spin-1/2 single particle

$$i\hbar\gamma^{\mu}\partial_{\mu}\Psi(\underline{x}) - mc\,\Psi(\underline{x}) = 0 \qquad (57)$$

where $\Psi(\underline{x})$ is the wavefunction of the particle (4-component spinor), $\underline{x} = (x^{\mu}, \mu = 0, ...3) = (ct, x, y, z)$ is a 4-vector, and γ^{μ} $(\mu = 0, 1, 2, 3)$ are the Dirac 4x4 matrices

$$\gamma^0 = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & I_2 \end{pmatrix} \quad \gamma^k = \begin{pmatrix} \mathbf{0} & \sigma^k \\ -\sigma^k & \mathbf{0} \end{pmatrix}$$

(I_2 =2x2 unit matrix, σ^k = Pauli matrix, $k = x, y, z$)

- Lagrangian density

$$\mathscr{L}(\Psi, \overline{\Psi}, \partial_{\mu}\Psi, \partial_{\mu}\overline{\Psi}) = \frac{i\hbar}{2}\left[\overline{\Psi}\gamma^{\mu}(\partial_{\mu}\Psi) - (\partial_{\mu}\overline{\Psi})\gamma^{\mu}\Psi\right] - mc\,\overline{\Psi}\Psi \qquad (58)$$

where $\overline{\Psi}$ is the conjugate spinor (field independent of Ψ)

$$\overline{\Psi} = \Psi^{\dagger}\gamma^0 \qquad (59)$$

Back to QCD

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