The Nuclear Shell Model

N. A. Smirnova

CEN Bordeaux-Gradignan, France

Bridging Methods in Nuclear Theory
IPHC, Strasbourg, June 26–30, 2017

The Nuclear Shell Model

Table of contents

Lecture of A. P. Zuker "Beyond the Shell Model" on Friday afternoon

Structure of complex nuclei

- Nuclear charge (matter) density distribution $\rho_{ch}(\vec{r})$ ($\rho_m(\vec{r})$) with sharp radius $R \simeq r_0 A^{1/3}$.
- **Empirical evidence on the existence of an average potential and the corresponding shell structure**

  (From masses, nucleon separation energies, low-energy spectra, etc.)
- **Independent-particle motion near Fermi-level.**

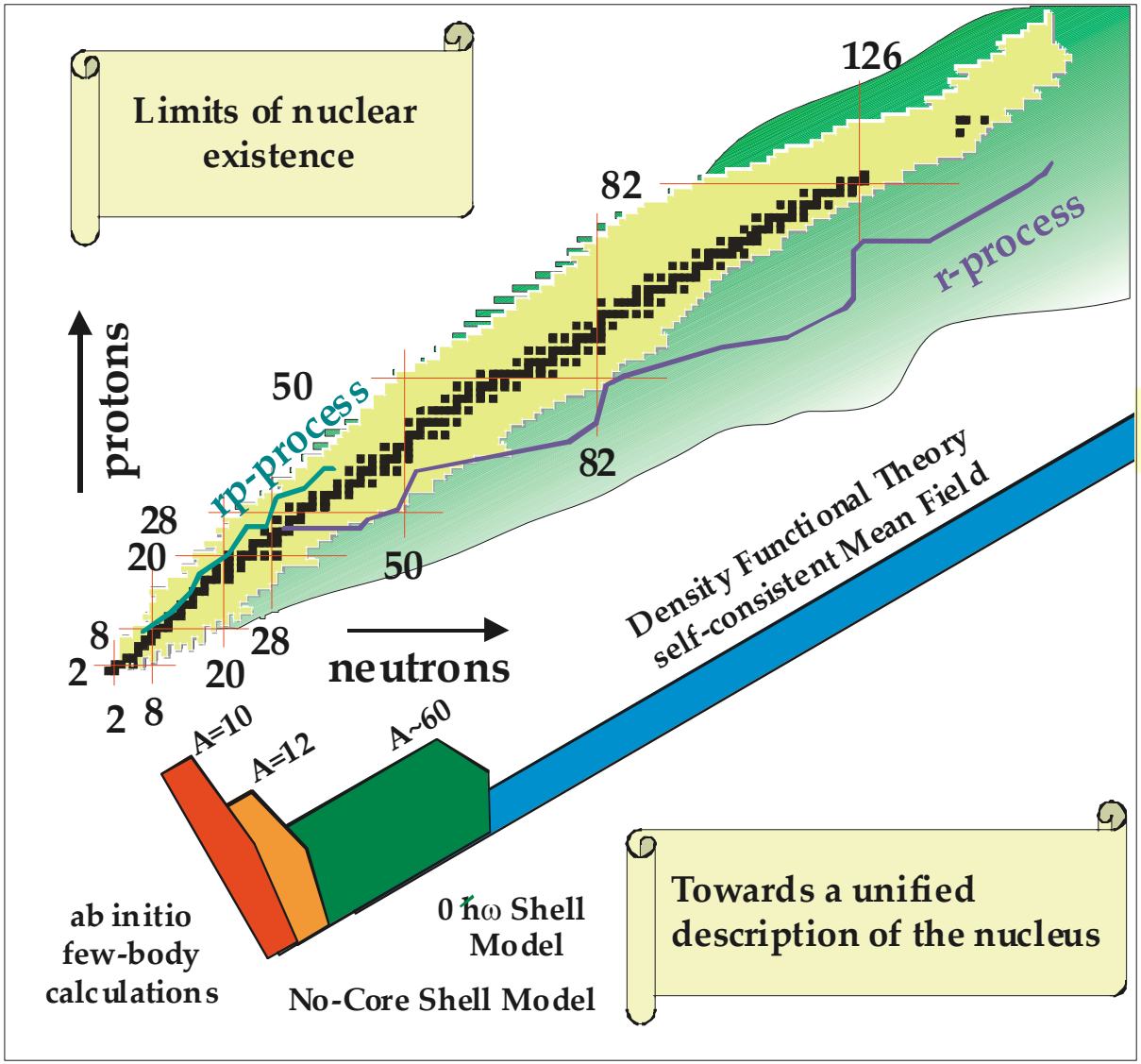
  (From nucleon transfer reactions: nuclear mean-field with strong spin-orbit splitting and large shell gaps)
- **Pairing (superfluid behavior) at low excitation energy.**

  (S_n versus S_{2n}; two-nucleon spectra, comparison of spectra of even-even and even-odd, odd-even nuclei, etc.)
- **Low-lying multipole (quadrupole) modes, vibrational or rotational energy structures.**

  (Coulomb excitation; scattering of charge particles; heavy-ion fusion-evaporation reactions, etc)

The aim of the microscopic theory is to describe these motions starting from a NN force.

Nuclear many-body problem



Nuclear many-body problem

Non-relativistic Hamiltonian for A nucleons

$$\hat{H} = \sum_{i=1}^{A} \frac{\vec{p}_i^2}{2m} + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j)$$

$$\hat{H} = \underbrace{\sum_{i=1}^{A} \left[\frac{\vec{p}_i^2}{2m} + U(\vec{r}_i) \right]}_{\hat{H}^{(0)}} + \underbrace{\sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j) - \sum_{i=1}^{A} U(\vec{r}_i)}_{\hat{V}},$$

Mean-field theories

Search for the most optimum mean-field potential starting from a given two-body interaction + correlations

Shell model

Schematic average potential + residual interaction

Variational approach to the nuclear many-body problem

Starting point: antisymmetric product wave function

$$\Psi(1,2,\ldots,A) = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_{\alpha_1}(1) & \phi_{\alpha_1}(2) & \ldots & \phi_{\alpha_1}(A) \\ \phi_{\alpha_2}(1) & \phi_{\alpha_2}(2) & \ldots & \phi_{\alpha_2}(A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{\alpha_A}(1) & \phi_{\alpha_A}(2) & \ldots & \phi_{\alpha_A}(A) \end{vmatrix}$$

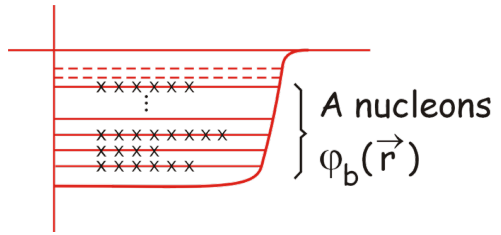
The best wave function is determined via a variational principle:

$$\delta\langle\Psi|H|\Psi\rangle = \langle\delta\Psi|H|\Psi\rangle = 0 \quad \text{with} \int |\phi_{\alpha_i}(\vec{r})|^2 d\vec{r} = 1.$$

Self-consistent mean-field potential

Hartree-Fock equations

$$\left(-\frac{\hbar^2}{2m}\Delta + U_H(\vec{r})\right)\phi_i(\vec{r}) + \int U_F(\vec{r},\vec{r'})\phi_i(\vec{r'})d\vec{r'} = \varepsilon_i\phi_i(\vec{r})$$



Direct (Hartree) term

$$U_H(\vec{r}) = \sum_{b\in F}\int \phi_b^*(\vec{r'})W(\vec{r},\vec{r'})\phi_b(\vec{r'})d\vec{r'}$$

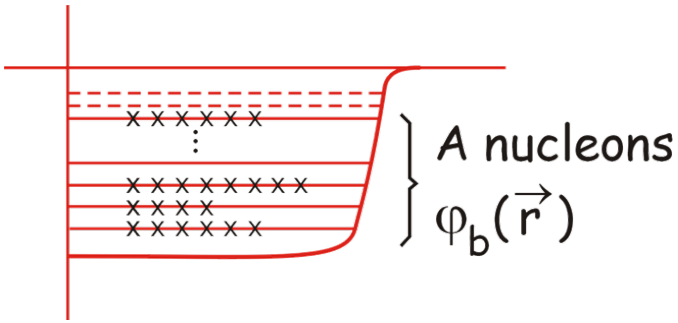
Exchange (Fock) term

$$U_F(\vec{r},\vec{r'}) = \sum_{b\in F}\phi_b^*(\vec{r})W(\vec{r},\vec{r'})\phi_b(\vec{r'})$$

Iterative solution of Hartree-Fock equations:

$$\begin{array}{ccccccc} \phi_i^{(0)}(\vec{r}) & & \phi_i^{(1)}(\vec{r}),\varepsilon_i^{(1)} & & \dots & & \phi_i^{(HF)}(\vec{r}),\varepsilon_i^{(HF)} \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ U_H^{(0)}(\vec{r}), U_F^{(0)}(\vec{r},\vec{r'}) & & U_H^{(1)}(\vec{r}), U_F^{(1)}(\vec{r},\vec{r'}) & & \dots & & U^{(HF)}(\vec{r}) \end{array}$$

Self-consistent mean-field potential



Direct (Hartree) term

$$U_H(\vec{r}) = \sum_{b\in F} \int \phi_b^*(\vec{r'}) W(\vec{r},\vec{r'}) \phi_b(\vec{r'}) d\vec{r'}$$

$$U_H(\vec{r}) = \int \rho(\vec{r'}) W(\vec{r},\vec{r'}) d\vec{r'}$$

$$\rho(\vec{r}) = \sum_{b\in F} |\phi_b(\vec{r})|^2$$

Example: a delta-force

$$W(\vec{r},\vec{r'}) \propto \delta(\vec{r}-\vec{r'})$$

$$U_H(\vec{r}) \propto \rho(\vec{r})$$

Hartree-Fock ground state properties

From the product HF wave function, a number of ground-state properties can be calculated:

$$\Psi_{HF}(1,2,\ldots,A) = \frac{1}{\sqrt{A!}}\mathcal{A}\prod_{i=1}^{A}\phi_{\alpha_i}^{HF}(i)$$

$$\begin{aligned}
\langle\Psi_{HF}|\hat{H}|\Psi_{HF}\rangle &= E_0 \\
\langle\Psi_{HF}|\sum_{i=1}^{A}\hat{r}_i^2|\Psi_{HF}\rangle &= \langle r^2\rangle \\
\langle\Psi_{HF}|\sum_{i=1}^{A}\hat{\rho}(\vec{r}_i)|\Psi_{HF}\rangle &= \rho(\vec{r})
\end{aligned}$$

Application to open-shell nuclei and calculation excitation spectra require inclusion of correlations (pairing, etc) $\rightarrow$ beyond mean-field techniques (lecture of Ph. Quentin).

Shell model: energy matrix diagonalization

Non-relativistic Hamiltonian for A nucleons

$$\hat{H} = \sum_{i=1}^{A} \underbrace{\left[\frac{\vec{p}_i^2}{2m} + U(\vec{r}_i)\right]}_{\hat{h}_i} + \underbrace{\sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j) - \sum_{i=1}^{A} U(\vec{r}_i)}_{\hat{V}}$$

Construction of a basis from single-particle states

$$\hat{h}\phi_\alpha(\vec{r}) = \varepsilon_\alpha \phi_\alpha(\vec{r}) \quad \rightarrow \quad \{\varepsilon_\alpha,\, \phi_\alpha(\vec{r})\}$$

Spherical potential $U(\vec{r}) = U(r)$: $\alpha = \{n_\alpha, l_\alpha, j_\alpha, m_\alpha\}$

$$\phi_{nljm}(\vec{r}) = \frac{R_{nlj}(r)}{r} \underbrace{\left[Y_l(\theta,\varphi) \times \chi_{\frac{1}{2}}\right]_m^{(j)}}_{\sum_{m_l m_s} (l m_l \frac{1}{2} m_s | jm) Y_{l m_l}(\theta,\varphi) \chi_{\frac{1}{2} m_s}}$$

Single-particle wave functions

Radial differential equation

$$-\frac{\hbar^2}{2m}R''(r)+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}R(r)+\left[U(r)R(r)+a_{ls}f_{ls}(r)\right]R(r)=\varepsilon R(r)$$

Normalization condition

$$\int|\phi_{nlsjm}(\vec{r})|^2d\vec{r}=\sum_{m_l m_s m_l' m_s'}(lm_l\frac{1}{2}m_s|jm)(lm_l'\frac{1}{2}m_s'|jm)\int Y_{lm_l}(\theta,\phi)Y_{lm_l'}(\theta,\phi)d\Omega$$

$$\times\langle\chi_{\frac{1}{2}m_s}|\chi_{\frac{1}{2}m_s'}\rangle\int_0^\infty|R_{nl}(r)|^2dr=\int_0^\infty|R_{nl}(r)|^2dr=1$$

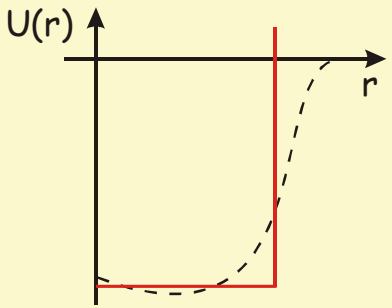
Parity

$$\hat{P}\phi_{nlsjm}(\vec{r})=\hat{P}\phi_{nlsjm}(-\vec{r})=(-1)^l\phi_{nlsjm}(\vec{r})$$

Examples of spherically-symmetric potentials

- Square-well potential + strong spin-orbit term

→ $J_{l+1/2}(kr)$



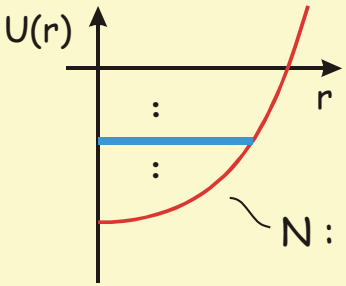
ex. N = 4

s ——— $l=0$ (100)

d ——— $l=2$ (93)

g ——— $l=4$ (75.5)

- Harmonic oscillator potential + orbital + spin-orbit term



→ $(\nu r)^l e^{\frac{-\nu^2 r^2}{2}} L_{n-1}^{l+1/2}(\nu^2 r^2)$

$(\nu = \sqrt{\frac{m\omega}{\hbar}})$

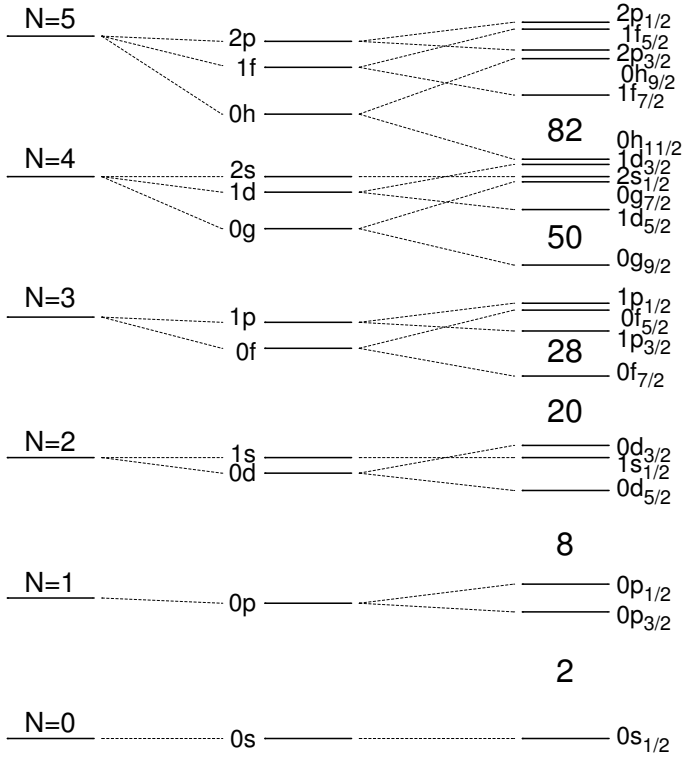
Harmonic oscillator potential

$$U(r) = \frac{m\omega^2 r^2}{2} + \alpha \vec{l}\cdot\vec{l} + \beta \vec{l}\cdot\vec{s}$$

$$\varepsilon_N = \hbar\omega\left(2n + l + \frac{3}{2}\right) = \hbar\omega\left(N + \frac{3}{2}\right) ,$$

$$N = 0, 1, 2, \ldots ,$$
$$l = N, N-2, \ldots, 1 \text{ or } 0$$
$$n = (N-l)/2 .$$

Harmonic oscillator potential possesses many **symmetry properties** which make it a preferable choice as a basis.



M. Mayer (1949)

O. Haxel, H. Jensen, H.E. Suess (1949)

Isospin

The idea of W.Heisenberg: proton and neutron are considered as two states of a nucleon.

$$v_{\pi\pi} \approx v_{\nu\nu} \approx v_{\pi\nu} \qquad \pi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \nu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Isospin operators (in analogy with the Pauli matrices):

$$\vec{t} = \frac{1}{2}\vec{\tau}, \quad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Single-particle wave functions

$$\phi_\nu(r) = \phi(\vec{r}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \phi(\vec{r})\, \theta_{t=1/2, m_t=1/2}\,,$$
$$\phi_\pi(r) = \phi(\vec{r}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \phi(\vec{r})\, \theta_{t=1/2, m_t=-1/2}\,.$$

Isospin and classification of nuclear states

$$\hat{T} = \sum_{i=1}^{A} \hat{t}_i, \quad \hat{T}_z = \sum_{i=1}^{A} \hat{t}_{zi} \, .$$

Charge independent Hamiltonian: $\left[\hat{H}, \hat{T}\right] = 0 \, .$

$$M_T = \frac{1}{2}(N - Z), \quad \frac{1}{2}(N - Z) \leq T \leq \frac{A}{2} \, .$$

Realistic situation

$$m_p \approx m_n; \quad \hat{V}_{Coulomb} = \sum_{i<j=1}^{Z} \frac{e^2}{|\vec{r}_i - \vec{r}_j|}$$

$$\hat{V}_{Coulomb} = \hat{V}^{(T=0)} + \hat{V}^{(T=1)} + \hat{V}^{(T=2)} \, .$$

$$E(T, M_T) = a(T) + b(T)M_T + c(T)M_T^2 \, ,$$

Two-particle wave function for identical fermions

Angular-momentum coupled state ($j_a \neq j_b$)

$$\Phi_{j_a(1)j_b(2);JM}(\vec{r}_1,\vec{r}_2) = \sum_{m_a m_b} (j_a\, m_a\, j_b\, m_b | J\, M)\phi_{j_a m_a}(\vec{r}_1)\phi_{j_b m_b}(\vec{r}_2) = \left[\phi_{j_a}(\vec{r}_1) \times \phi_{j_b}(\vec{r}_2)\right]^{(J)}_M$$

$j_a \equiv (n_a l_a j_a)$, $J = |j_a - j_b|, |j_a - j_b| + 1, \ldots, j_a + j_b$, while $M = -J, -J+1, \ldots, J-1, J$.

Not antisymmetric with respect to permutation of two identical fermions !

Normalized and antisymmetric state

$$\Phi_{j_a j_b;JM}(\vec{r}_1,\vec{r}_2) = \frac{1}{\mathcal{N}} \sum_{m_a m_b} (j_a\, m_a\, j_b\, m_b | J\, M) \left[\phi_{j_a m_a}(\vec{r}_1)\phi_{j_b m_b}(\vec{r}_2) - \phi_{j_b m_b}(\vec{r}_1)\phi_{j_a m_a}(\vec{r}_2)\right]$$

Since the Clebsch-Gordan coefficients have the following property:

$$(j_a\, m_a\, j_b\, m_b\, | J\, M) = (-1)^{j_a + j_b - J}(j_b\, m_b\, j_a\, m_a | J\, M)$$

$$\Phi_{j_a j_b;JM}(\vec{r}_1,\vec{r}_2) = \frac{1}{\mathcal{N}} \sum_{m_a m_b} \Big[(j_a\, m_a\, j_b\, m_b | J\, M)\phi_{j_a m_a}(\vec{r}_1)\phi_{j_b m_b}(\vec{r}_2)$$
$$-(-1)^{j_a + j_b - J}(j_b\, m_b\, j_a\, m_a | J\, M)\phi_{j_b m_b}(\vec{r}_1)\phi_{j_a m_a}(\vec{r}_2)\Big]$$

Two-particle wave function for identical fermions

Normalized and antisymmetric state

$j_a \neq j_b$

$$\Phi_{j_a j_b;JM}(\vec{r}_1,\vec{r}_2) = \frac{1}{\sqrt{2}}\left\{\left[\phi_{j_a}(\vec{r}_1)\times\phi_{j_b}(\vec{r}_2)\right]^{(J)}_M - (-1)^{j_a+j_b+J}\left[\phi_{j_b}(\vec{r}_1)\times\phi_{j_a}(\vec{r}_2)\right]^{(J)}_M\right\}$$

$j_a \equiv (n_a l_a j_a)$, $J = |j_a - j_b|, |j_a - j_b|+1, \ldots, j_a + j_b$, while $M = -J, -J+1, \ldots, J-1, J$.

$j_a = j_b = j$

$$\Phi_{j^2;JM}(\vec{r}_1,\vec{r}_2) = \frac{1+(-1)^J}{2}\left[\phi_j(\vec{r}_1)\times\phi_j(\vec{r}_2)\right]^{(J)}_M$$

Important consequence: if $j_a = j_b = j$, then $J = 0, 2, 4, \ldots, 2j-1$

$$(\nu 0d_{5/2})^2: \quad J = 0, 2, 4$$

Two-particle wave function for protons and neutrons

$j_a \neq j_b$

$$\Phi_{j_a j_b;JMTM_T}(\vec{r}_1,\vec{r}_2) = \left\{ \left[\phi_{j_a}(\vec{r}_1)\times\phi_{j_b}(\vec{r}_2)\right]^{(J)}_M + (-1)^{j_a+j_b+J+T} \left[\phi_{j_b}(\vec{r}_1)\times\phi_{j_a}(\vec{r}_2)\right]^{(J)}_M \right\} \frac{\Theta_{TM_T}}{\sqrt{2}}$$

$$\begin{aligned}
\Theta_{1,1} &= \theta_{1/2,1/2}(1)\theta_{1/2,1/2}(2)\,, \\
\Theta_{1,-1} &= \theta_{1/2,-1/2}(1)\theta_{1/2,-1/2}(2)\,, \\
\Theta_{1,0} &= \left[\theta_{1/2,1/2}(1)\theta_{1/2,-1/2}(2) + \theta_{1/2,-1/2}(1)\theta_{1/2,1/2}(2)\right]/\sqrt{2}\,, \\
\Theta_{0,0} &= \left[\theta_{1/2,1/2}(1)\theta_{1/2,-1/2}(2) - \theta_{1/2,-1/2}(1)\theta_{1/2,1/2}(2)\right]/\sqrt{2}\,,
\end{aligned}$$

$j_a = j_b = j$

$$\Phi_{j^2;JMTM_T}(\vec{r}_1,\vec{r}_2) = \frac{1-(-1)^{J+T}}{2} \left[\phi_j(\vec{r}_1)\times\phi_j(\vec{r}_2)\right]^{(J)}_M \Theta_{TM_T}$$

Remark: if $j_a = j_b = j$, then $(J+T)$ is odd!

$$\begin{aligned}
&(\nu 0d_{5/2})^2: \quad J = 0,2,4 \\
&(\nu 0d_{5/2}\ \pi 0d_{5/2}): \quad J = 0,2,4 (T=1); \quad J = 1,3,5\,(T=0)
\end{aligned}$$

Many-particle wave function: $J(T)$-coupled states.

J-coupled state

- Consider N identical fermions in a single-j shell. We construct a totally antisymmetric and coupled to good J N-nucleon wave function from a set of totally antisymmetric $(N-1)$-nucleon wave functions coupled to all possible J':

$$\Phi^{j(N)}_{\chi JM}(\vec{r}_1,\ldots,\vec{r}_N) = \sum_{\chi' J'} \left[j^{N-1}(\chi' J')j| \right\} j^N \chi J \Big] \; \Phi^{j(N-1)}_{\chi' J' M'}(\vec{r}_1,\ldots,\vec{r}_{N-1})\; \phi_{jm}(\vec{r}_N)\,,$$

  where

$$\left[j^{N-1}(\chi' J')j| \right\} j^N \chi J \Big]$$

  are one-particle *coefficients of fractional parentage (cfp's)*

- Repeat this procedure for N' particles in j' orbital and so on. Construct thus basis states by consecutive coupling of angular momenta and antisymmetrization.

Many-particle wave function: *m*-scheme basis

Slater determinants

$$\Phi_\alpha(1,2,\ldots,A) = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_{\alpha_1}(\vec{r}_1) & \phi_{\alpha_1}(\vec{r}_2) & \cdots & \phi_{\alpha_1}(\vec{r}_A) \\ \phi_{\alpha_2}(\vec{r}_1) & \phi_{\alpha_2}(\vec{r}_2) & \cdots & \phi_{\alpha_2}(\vec{r}_A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{\alpha_A}(\vec{r}_1) & \phi_{\alpha_A}(\vec{r}_2) & \cdots & \phi_{\alpha_A}(\vec{r}_A) \end{vmatrix}$$

where $\alpha_i = (n_i, l_i, j_i, m_i)$ and α stores a set of single-particle configurations $\{\alpha_1, \alpha_2, \ldots, \alpha_A\}$

$$M = \sum_{i=1}^{A} m_i$$

Projection on J can be performed.

Exercise 1

Basis construction

Consider 2 neutrons in $0f_{7/2}$ orbital.

- Write down the basis in J-coupled form.
- Write down the basis in m-scheme.
- How many different J-states exist in this model space?

Solution to exercise 1

Basis construction

- Basis in J-coupled scheme:

$$|(0f_{7/2})^2; J, T = 1\rangle \; J = 0, 2, 4, 6$$

Solution to exercise 1 (continued)

Basis construction

- Basis in m-scheme: $|m_1\, m_2; M\rangle$:

$$
\begin{array}{llll}
|\ \tfrac{7}{2},\ \tfrac{5}{2};\ 6\rangle & & & \\
|\ \tfrac{7}{2},\ \tfrac{3}{2};\ 5\rangle & & & \\
|\ \tfrac{7}{2},\ \tfrac{1}{2};\ 4\rangle & |\ \tfrac{5}{2},\ \tfrac{3}{2};\ 4\rangle & & \\
|\ \tfrac{7}{2},-\tfrac{1}{2};\ 3\rangle & |\ \tfrac{5}{2},\ \tfrac{1}{2};\ 3\rangle & & \\
|\ \tfrac{7}{2},-\tfrac{3}{2};\ 2\rangle & |\ \tfrac{5}{2},-\tfrac{1}{2};\ 2\rangle & |\ \tfrac{3}{2},\ \tfrac{1}{2};\ 2\rangle & \\
|\ \tfrac{7}{2},-\tfrac{5}{2};\ 1\rangle & |\ \tfrac{5}{2},-\tfrac{3}{2};\ 1\rangle & |\ \tfrac{3}{2},-\tfrac{1}{2};\ 1\rangle & \\
|\ \tfrac{7}{2},-\tfrac{7}{2};\ 0\rangle & |\ \tfrac{5}{2},-\tfrac{5}{2};\ 0\rangle & |\ \tfrac{3}{2},-\tfrac{3}{2};0\rangle & |\ \tfrac{1}{2},-\tfrac{1}{2};0\rangle \\
|-\tfrac{7}{2},\tfrac{5}{2};-1\rangle & |-\tfrac{5}{2},\ \tfrac{3}{2};-1\rangle & |-\tfrac{3}{2},\tfrac{1}{2};-1\rangle & \\
|-\tfrac{7}{2},\tfrac{3}{2};-2\rangle & |-\tfrac{5}{2},\ \tfrac{1}{2};-2\rangle & |-\tfrac{3}{2},-\tfrac{1}{2};-2\rangle & \\
|-\tfrac{7}{2},\tfrac{1}{2};-3\rangle & |-\tfrac{5}{2},\ \tfrac{1}{2};-3\rangle & & \\
|-\tfrac{7}{2},\tfrac{1}{2};-4\rangle & |-\tfrac{5}{2},\ \tfrac{3}{2};-4\rangle & & \\
|-\tfrac{7}{2},\tfrac{3}{2};-5\rangle & & & \\
|-\tfrac{7}{2},\tfrac{5}{2};-6\rangle & & &
\end{array}
$$

- There are 4 different J-states.

Solution of a many-body Schrödinger equation

Construct a basis in the valence space for each J

$$\Phi_k^{(J)} = \left\{ (j_a)_{J_a}^{n_a} (j_b)_{J_b}^{n_b} \dots \right\}_k^{(J)}, \quad \hat{H}^{(0)} \Phi_k = E_k^{(0)} \Phi_k$$

Expand unknown wave function in terms of basis functions

$$\Psi_p = \sum_{k=1}^{n} a_{kp} \Phi_k \quad \Rightarrow \quad \hat{H} \Psi_p = E_p \Psi_p \quad (\hat{H} = \hat{H}^{(0)} + \hat{V})$$

Multiplying by $\langle \Phi_l |$, we get a system of equations

$$\sum_{k=1}^{n} H_{lk} a_{kp} = E_p a_{lp}$$

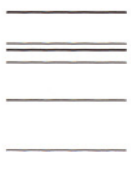
$\Rightarrow$ diagonalization of the matrix

$$H_{lk} = \langle \Phi_l | \hat{H} | \Phi_k \rangle = E_k^{(0)} \delta_{lk} + V_{lk}$$

Calculate Hamiltonian matrix $H_{ij}=\langle \phi_j | H | \phi_i \rangle$

— **Diagonalize to obtain eigenvalues**

$$\begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & & \\ \vdots & & \ddots & \\ H_{N1} & & \cdots & H_{NN} \end{pmatrix} \rightarrow$$

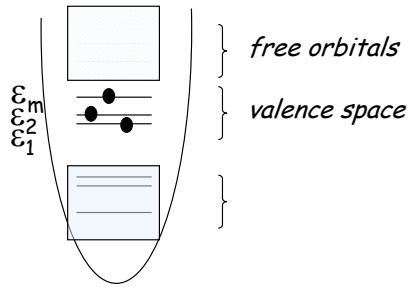


Basis dimension and choice of the model space

Basis dimension grows quickly

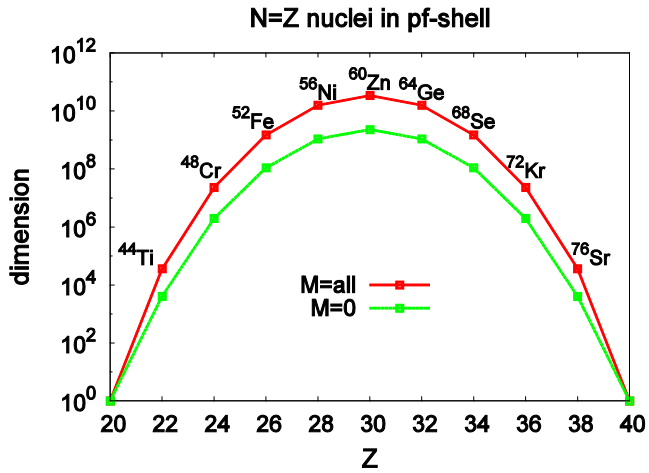
$$dim \approx \begin{pmatrix} \Omega_\pi \\ N_\pi \end{pmatrix} \begin{pmatrix} \Omega_\nu \\ N_\nu \end{pmatrix}$$

Model space: a few valence orbitals beyond the closed-shell core.



Example: $^{60}Zn_{30}$ in *pf*-shell

$$\begin{aligned} dim(^{60}\mathrm{Zn}) &= \begin{pmatrix} 20 \\ 10 \end{pmatrix} \begin{pmatrix} 20 \\ 10 \end{pmatrix} \\ = \frac{20!}{10!10!}\frac{20!}{10!10!} &\approx 3.4 \times 10^{10}. \end{aligned}$$



Practical shell-model for ^{18}O in *sd*-shell

$$\hat{H} = \hat{H}^{(0)} + \hat{V} = \hat{h}(1) + \hat{h}(2) + \hat{V}$$

Single-particle energies:

$$\varepsilon(0d_{5/2}) = E_B(^{17}_{8}\mathrm{O}_9\ ) - E_B(^{16}_{8}\mathrm{O}_8\ ) = -4.143\ \mathrm{MeV}$$

$$\varepsilon(1s_{1/2}) = \varepsilon(0d_{5/2}) + \mathrm{E_{ex}}(^{17}\mathrm{O};\ 1/2^+_1) = -3.273\ \mathrm{MeV}$$

$$\varepsilon(0d_{3/2}) = \varepsilon(0d_{5/2}) + \mathrm{E_{ex}}(^{17}\mathrm{O};\ 3/2^+_1) = 0.942\ \mathrm{MeV}$$

Basis of states for each (JT) denoted as $|j_a\, j_b\rangle_{JT}$:

$$0^+, T=1\ :\quad \begin{array}{l} |\Phi_1(0^+)\rangle \equiv |d^2_{5/2}\rangle_{01} \\ |\Phi_2(0^+)\rangle \equiv |s^2_{1/2}\rangle_{01} \\ |\Phi_3(0^+)\rangle \equiv |d^2_{3/2}\rangle_{01} \end{array} \quad \Rightarrow \quad \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$$

$$1^+, T=1\ :\quad \begin{array}{l} |\Phi_1(1^+)\rangle \equiv |d_{5/2}\, d_{3/2}\rangle_{11} \\ |\Phi_2(1^+)\rangle \equiv |s_{1/2}\, d_{3/2}\rangle_{11} \end{array} \quad \Rightarrow \quad \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

and so on for $J = 2, 3, 4$.

Exercise 2: Energies of 0^+ states in ^{18}O

Diagonal Two-body matrix elements (TBMEs)

$$H_{11} = 2\varepsilon(d_{5/2}) + \underbrace{\langle d^2_{5/2}|V|d^2_{5/2}\rangle_{01}}_{-2.82 \text{ MeV}}$$

$$H_{22} = 2\varepsilon(s_{1/2}) + \underbrace{\langle s^2_{1/2}|V|s^2_{1/2}\rangle_{01}}_{-2.12 \text{ MeV}}$$

$$H_{33} = 2\varepsilon(d_{3/2}) + \underbrace{\langle d^2_{3/2}|V|d^2_{3/2}\rangle_{01}}_{-2.18 \text{ MeV}}$$

Non-diagonal TBMEs

$$H_{12} = H_{21} = \underbrace{\langle d^2_{5/2}|V|s^2_{1/2}\rangle_{01}}_{-1.32 \text{ MeV}}$$

$$H_{23} = H_{32} = \underbrace{\langle s^2_{1/2}|V|d^2_{3/2}\rangle_{01}}_{-1.08 \text{ MeV}}$$

$$H_{13} = H_{31} = \underbrace{\langle d^2_{5/2}|V|d^2_{3/2}\rangle_{01}}_{-3.19 \text{ MeV}}$$

Practical shell-model for ^{18}O in *sd*-shell

Eigenvalues (g.s. and two excited 0^+ states):

$$\begin{array}{lll} E(0_1^+) = -12.602 \quad \text{MeV} & & E_{gs}(0_1^+) = 0 \quad \text{MeV} \\ E(0_2^+) = -8.097 \quad \text{MeV} & \Rightarrow & E_{ex}(0_2^+) = 4.505 \quad \text{MeV} \\ E(0_3^+) = 0.622 \quad \text{MeV} & & E_{ex}(0_3^+) = 13.224 \quad \text{MeV} \end{array}$$

Eigenstates:

$$|\Psi(0_1^+)\rangle = a_{11}|d_{5/2}^2\rangle_{01} + a_{21}|s_{1/2}^2\rangle_{01} + a_{31}|d_{3/2}^2\rangle_{01}$$

$$|\Psi(0_2^+)\rangle = a_{12}|d_{5/2}^2\rangle_{01} + a_{22}|s_{1/2}^2\rangle_{01} + a_{32}|d_{3/2}^2\rangle_{01}$$

$$|\Psi(0_2^+)\rangle = a_{13}|d_{5/2}^2\rangle_{01} + a_{23}|s_{1/2}^2\rangle_{01} + a_{33}|d_{3/2}^2\rangle_{01}$$

$$\sum_k a_{kp}^2 = 1$$

The full spectrum:

Repeat the same procedure for $J^\pi = 1^+, 2^+, 3^+, 4^+$.

Shell-model codes

m-scheme codes

- ANTOINE (E. Caurier)
  `http://www.iphc.cnrs.fr/nutheo/code_antoine/menu.html`
- NuShellX@MSU (W. Rae, B. A. Brown)
- MSHELL (T. Mizusaki)
- REDSTICK (W. E. Ormand, C. Johnson)
- ...

J-coupled codes

- NATHAN (E.Caurier, F.Nowacki)
- DUPSM (Novoselsky, Vallières)
- Ritsschil (Zwarts)
- ...

Features

- Matrix dimension: $\sim 10^{10}$ and beyond
- Lanczos diagonalization algorithm
- Calculation of the matrix elements on-the-fly

Lanczos algorithm

Creation of a tri-diagonal matrix:

$$\begin{aligned} \hat{H}|1\rangle &= E_{11}|1\rangle + E_{12}|2\rangle \\ \hat{H}|2\rangle &= E_{21}|1\rangle + E_{22}|2\rangle + E_{23}|3\rangle \\ &\dots \end{aligned}$$

Matrix elements:

$$\begin{aligned} E_{11} &= \langle 1|\hat{H}|1\rangle \\ E_{12}|2\rangle &= (\hat{H} - E_{11})|1\rangle \\ E_{21} &= E_{12},\ E_{22} = \langle 2|\hat{H}|2\rangle \\ E_{23}|3\rangle &= (\hat{H} - E_{22})|2\rangle - E_{21}|1\rangle \\ &\dots \end{aligned}$$

How to get the lowest states converged:

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & E_{23} \\ 0 & E_{32} & E_{33} \end{pmatrix} \Rightarrow \begin{pmatrix} E_{11} & E_{12} & 0 & 0 \\ E_{21} & E_{22} & E_{23} & 0 \\ 0 & E_{32} & E_{33} & E_{34} \\ 0 & 0 & E_{43} & E_{44} \end{pmatrix} \Rightarrow \dots$$

You may need only a few iterations to get the lowest state of a $(10^3 \times 10^3)$ matrix converged!

Calculation of observables

General scheme:

- Construct the basis: $|\Phi_k\rangle$
- Expand the wave function: $|\Psi_p\rangle = \sum_k a_{kp}|\Phi_k\rangle$ and compute the Hamiltonian matrix $\{H_{lk}\}$.
- Solution of the Shrödinger equation by Hamiltonian matrix diagonalization: $\{H_{lk}\}$ $\Rightarrow E_p$, $|\Psi_p\rangle$ (coefficients a_{kp})
- Calculation of matrix elements of the operators

$$T_{fi} \propto |\langle \Psi_f | \hat{O} | \Psi_i \rangle|^2$$

Electroweak operators:

$$\hat{O}(E, LM) = \sum_{k=1}^{A} e(k) r^L(k) Y_{LM}(\hat{r}(k))$$

$$\hat{O}(M, 1M) = \sum_{k=1}^{A} \mu_n (g_s(k)\vec{s}(k) + g_l(k)\vec{l}(k))$$

$$\hat{O}(F) = \sum_{k=1}^{A} \tau_{\pm}(k), \quad \hat{O}(GT) = \sum_{k=1}^{A} \vec{\sigma}(k)\tau_{\pm}(k) \quad \ldots$$

Exercise 3

Electromagnetic transitions in ^{17}O

Calculate the $B(E2; 1/2^+_1 \rightarrow 5/2^+_{g.s.})$ in ^{17}O modeled as a valence neutron in a $1s0d$ shell beyond the ^{16}O closed-shell core.
We take:

$$\int_0^\infty R_{1s_{1/2}}(r) r^2 R_{0d_{5/2}}(r) dr \approx 12\,\mathrm{fm}^2.$$

Compare your result to experimental value $B_{exp}(E2) = 6.3\ \mathrm{e}^2.\mathrm{fm}^4$. What can you conclude?

Solution to exercise 3

Electromagnetic transitions in ^{17}O

The single-particle electric multipole operator reads

$$\hat{O}(E,2M) = er^2\, Y_{2M}(\theta,\phi)\,,$$

The reduced probability of the $\mathcal{E}L$-transition from the initial to the final state is

$$B(EL; J_i \to J_f) = \frac{1}{2J_i+1}|\langle J_f||O(EL)||J_i\rangle|^2\,.$$

In our case, there is one valence particle: $J_i = (1s_{1/2})$, $J_f = (0d_{5/2})$.

$$B(E2; 1/2^+ \to 5/2^+) = \frac{1}{2}|\langle 0d_{5/2}||\hat{O}(E2)||1s_{1/2}\rangle|^2 =$$

$$\frac{1}{2}|\langle n_f{=}0, l_f{=}2, s_f{=}\frac{1}{2}, j_f{=}\frac{5}{2}||er^2\,Y_2(\theta,\phi)||n_i{=}1, l_i{=}0, s_i{=}\frac{1}{2}, j_i{=}\frac{1}{2}\rangle|^2 =$$

$$\frac{1}{2}e^2\underbrace{\left(\int R^*_{0d_{5/2}}(r)r^2R_{1s_{1/2}}(r)dr\right)^2}_{\langle r^2\rangle^2}|\langle 2\tfrac{1}{2};\tfrac{5}{2}||Y_2(\theta,\phi)||0\tfrac{1}{2};\tfrac{1}{2}\rangle|^2\,.$$

$$B(EL; j_i \to j_f) = e^2\frac{1}{4\pi}\langle r^2\rangle^2(2j_f+1)(2l_i+1)(2L+1)(l_i0L0|l_f0)^2\left\{\begin{array}{ccc}\frac{1}{2} & l_f & j_f\\ L & j_i & l_i\end{array}\right\}^2$$

Solution to exercise 3

Electromagnetic transitions in ^{17}O (continued)

$$B(E2; 1/2^+ \to 5/2^+) = e^2 \tfrac{1}{4\pi} \langle r^2 \rangle^2 6 \times 5 (0020|20)^2 \left\{ \begin{array}{ccc} 1/2 & 2 & 5/2 \\ 2 & 1/2 & 0 \end{array} \right\}^2 = \\ = 34.4\, \mathrm{e}^2.\mathrm{fm}^4.$$

Experimental value $B_{exp}(E2; 1/2^+ \to 5/2^+) = 6.3\ \mathrm{e}^2.\mathrm{fm}^4$.

This means that the neutron should have an effective charge: $\tilde{e}_n \approx 0.43\, e$. because we work in a severely restricted model space (one valence nucleon !).

Standard effective $E2$ and $M1$ operators

$$\tilde{e}_\pi \approx 1.5\, e\,, \quad \tilde{e}_\nu \approx 0.5\, e$$

$$\tilde{g}_s(\pi) \approx 0.7 \underbrace{g_s(\pi)}_{5.586}\,, \quad \tilde{g}_l(\pi) = g_l(\pi) = 1$$

$$\tilde{g}_s(\nu) \approx 0.7 \underbrace{g_s(\nu)}_{-3.826}\,, \quad \tilde{g}_l(\nu) = g_l(\nu) = 0$$

Nuclear many-body problem

Non-relativistic Hamiltonian for A nucleons

$$\hat{H} = \sum_{i=1}^{A} \frac{\hat{\vec{p}}_i^2}{2m} + \sum_{i<j=1}^{A} \hat{W}(\vec{r}_i - \vec{r}_j)$$

$$\hat{H} = \underbrace{\sum_{i=1}^{A} \left[\frac{\vec{p}_i^2}{2m} + U(\vec{r}_i) \right]}_{\hat{H}^{(0)}} + \underbrace{\sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j) - \sum_{i=1}^{A} U(\vec{r}_i)}_{\hat{V}},$$

The residual interaction is assumed to have a two-body form

$$\hat{V} = \sum_{i<j=1}^{A} \hat{V}(\vec{r}_i - \vec{r}_j)$$

One and two-body Hamitonian

$$\hat{H} = \sum_{i=1}^{A} \hat{h}(\vec{r}_i) + \sum_{i<j=1}^{A} \hat{V}(\vec{r}_i - \vec{r}_j)$$

Occupation-number representation (second quantization)

Creation and annihilation operators

$$|\alpha\rangle = a_\alpha^\dagger |0\rangle \quad \langle\alpha| = \langle 0| a_\alpha$$

Wave function of a fermion in a quantum state α in coordinate space:

$$\langle \vec{r}|\alpha\rangle = \phi_\alpha(\vec{r})$$

(Anti-)commutation relations:

$$\left\{a_\alpha^\dagger, a_\beta\right\} = a_\alpha^\dagger a_\beta + a_\beta a_\alpha^\dagger = \delta_{\alpha\beta}$$
$$\left\{a_\alpha^\dagger, a_\beta^\dagger\right\} = \{a_\alpha, a_\beta\} = 0$$

Normalized and antisymmetric A-fermion state:

$$|\alpha_1\alpha_2\ldots\alpha_A\rangle = a_{\alpha_A}^\dagger a_{\alpha_{A-1}}^\dagger \ldots a_{\alpha_2}^\dagger a_{\alpha_1}^\dagger |0\rangle$$

Operators in the occupation-number formalism

One-body operators

$$\hat{O} = \sum_{k=1}^{A} \hat{O}(\vec{r}_k)$$

$$\langle \alpha | \hat{O} | \beta \rangle = \int \phi_\alpha^*(\vec{r}) \hat{O}(\vec{r}) \phi_\beta(\vec{r}) d\vec{r}$$

Second-quantized form of the one-body operator $\hat{O}$:

$$\hat{O} = \sum_{\alpha\beta} \langle \alpha | \hat{O} | \beta \rangle a_\alpha^\dagger a_\beta$$

For example, the number operator reads

$$\hat{N} = \sum_{\alpha\beta} \langle \alpha | \hat{1} | \beta \rangle a_\alpha^\dagger a_\beta = \sum_{\alpha} a_\alpha^\dagger a_\alpha$$

Operators in the occupation-number formalism

Symmetric two-body operator acting on an A-fermion system

$$\hat{T} = \sum_{j<k=1}^{A} \hat{T}(\vec{r}_k, \vec{r}_j)$$

$$\langle \alpha\beta | \hat{T} | \gamma\delta \rangle = \int \phi^*_\alpha(\vec{r}_1)\phi^*_\beta(\vec{r}_2)\hat{T}(\vec{r}_1, \vec{r}_2)\left(1 - \hat{P}_{12}\right)\phi_\gamma(\vec{r}_1)\phi_\delta(\vec{r}_2)d\vec{r}_1 d\vec{r}_2 \,,$$

Second-quantized form of the two-body operator $\hat{T}$:

$$\hat{T} = \frac{1}{4}\sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | \hat{T} | \gamma\delta \rangle a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma$$

Nuclear many-body Hamiltonian in the occupation-number formalism

Non-relativistic Hamiltonian for A nucleons

$$\hat{H} = \sum_{i=1}^{A} \hat{h}(\vec{r}_i) + \sum_{i<j=1}^{A} \hat{V}(\vec{r}_i - \vec{r}_j) = \hat{H}^{(0)} + \hat{V}$$

$$\Downarrow$$

$$\hat{H} = \underbrace{\sum_{\alpha} \varepsilon_\alpha a_\alpha^+ a_\alpha}_{\text{one-body term}} + \underbrace{\frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | V | \gamma\delta \rangle a_\alpha^+ a_\beta^+ a_\delta a_\gamma}_{\text{two-body term}} ,$$

Two-body term in a JT-coupled form

$$\hat{V} = -\frac{1}{4} \sum_{j_\alpha j_\beta j_\gamma j_\delta} \langle j_\alpha j_\beta | V | j_\gamma j_\delta \rangle_{JT} \sqrt{(1+\delta_{\alpha\beta})(1+\delta_{\gamma\delta})}$$
$$\left[\left[a_{j_\alpha}^+ a_{j_\beta}^+ \right]^{(JT)} \left[\tilde{a}_{j_\gamma} \tilde{a}_{j_\delta} \right]^{(JT)} \right]^{(00)}$$

Necessary ingredients

Single-particle energies

$$\varepsilon_\alpha$$

from experimental spectra of A_{core} plus a neutron or a proton

Two-body matrix elements (TBMEs)

$$\langle j_\alpha j_\beta | V | j_\gamma j_\delta \rangle_{JT}$$

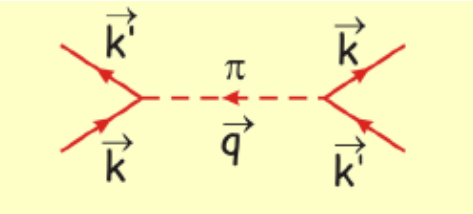
from theory ?

Bare nucleon-nucleon (NN) interaction

The NN interaction between two nucleons in the vacuum:
NN scattering data, deuteron bound states properties.

Elastic scattering in momentum space (Yukawa)

$$V_{\pi NN}(1,2) = \frac{g^2_{\pi NN}}{4M^2} \frac{\left(\vec{\sigma_1} \cdot \vec{q}\right)\left(\vec{\sigma_2} \cdot \vec{q}\right)}{\vec{q}^2 + m_\pi^2}$$



Potential (Fourier transform) in coordinate space

$$\begin{aligned} V^{OPEP}_{\pi NN}(1,2) &= \frac{g^2_{\pi NN}}{4M^2}\frac{m^3_\pi}{12}\left\{\vec{\sigma_1}\cdot\vec{\sigma_2}\right. \\ &+ \left.\left(1+\frac{3}{m_\pi r}+\frac{3}{(m_\pi r)^2}\right)\left(3\vec{\sigma_1}\cdot\vec{r}\vec{\sigma_2}\cdot\vec{r}-\vec{\sigma_1}\cdot\vec{\sigma_2}\right)\right\}\frac{e^{-m_\pi r}}{m_\pi r} \end{aligned}$$

Meson-exchange theories of NN potential: high-precision potentials (CD-Bonn, AV18, etc)

Concept of effective interaction (operators)

Effective in-medium nucleon-nucleon interaction

- In-medium effects (renormalization of the hard core)
- Truncated model space

$$\hat{H}\Psi = (\hat{H}^{(0)} + \hat{V})\Psi = E\Psi\,,$$

True wave function:

$$\Psi = \sum_{k=1}^{\infty} a_k \Phi_k\,.$$

Wave function in a model space:

$$\Psi' = \sum_{k=1}^{M} a_k \Phi_k\,.$$

$$\langle\Psi'|\hat{H}_{eff}|\Psi'\rangle = \langle\Psi|\hat{H}|\Psi\rangle = E$$

$$\langle\Psi'|\hat{O}_{eff}|\Psi'\rangle = \langle\Psi|\hat{O}|\Psi\rangle$$

Approaches to the problem: **phenomenological** or **microscopic**.

Effective Interaction

Practical approaches to effective interaction

- **Schematic** interaction (parametrized interaction between two nucleons in a nuclear medium)
- **Phenomenological** interaction (Fit of TBME's to energy levels of nuclei to be described within the chosen model space)
- **Microscopic** interaction (derived from a bare NN-force)

Schematic interaction

Some examples:

$$
\begin{aligned}
V(1,2) &= -V_0 \exp(\mu r)/(\mu r) \\
V(1,2) &= -V_0 \delta(\vec{r_1} - \vec{r_2}) \\
V(1,2) &= -V_0 \delta(\vec{r_1} - \vec{r_2})(1 + \alpha \vec{\sigma_1} \cdot \vec{\sigma_2}) \\
V(1,2) &= -V_0 \delta(\vec{r_1} - \vec{r_2}) \delta(r_1 - R) \\
&\dots \\
V(1,2) &= \chi Q \cdot Q \quad (Q = r^2 Y_{2\mu}(\Omega_r)) \\
&\dots
\end{aligned}
$$

A few parameters (interaction strengths) are fitted to reproduce energy levels in a certain region of (a few) neighboring nuclei $\Rightarrow$ local description only!

Exercise 4: TBMEs of the δ-force

Multipole expansion of the delta-function

$$V(1,2) = -V_0 \delta(\vec{r_1} - \vec{r_2}),$$

$$\delta(\vec{r_1} - \vec{r_2}) = \sum_k \frac{\delta(r_1 - r_2)}{r_1 r_2} \frac{2k+1}{4\pi} P_k(\cos\theta_{12})$$

Diagonal TBMEs between normalized and antisymmetric states

$$\langle j_1 j_2 | V | j_1 j_2 \rangle_{JT} = I (2j_1+1)(2j_2+1) \begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 \frac{1+(-1)^{l_1+l_2+J}}{2}$$

$$\langle j^2 | V | j^2 \rangle_{JT} = I (2j+1)^2 \begin{pmatrix} j & j & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2, \qquad \text{if } j_1 = j_2 = j$$

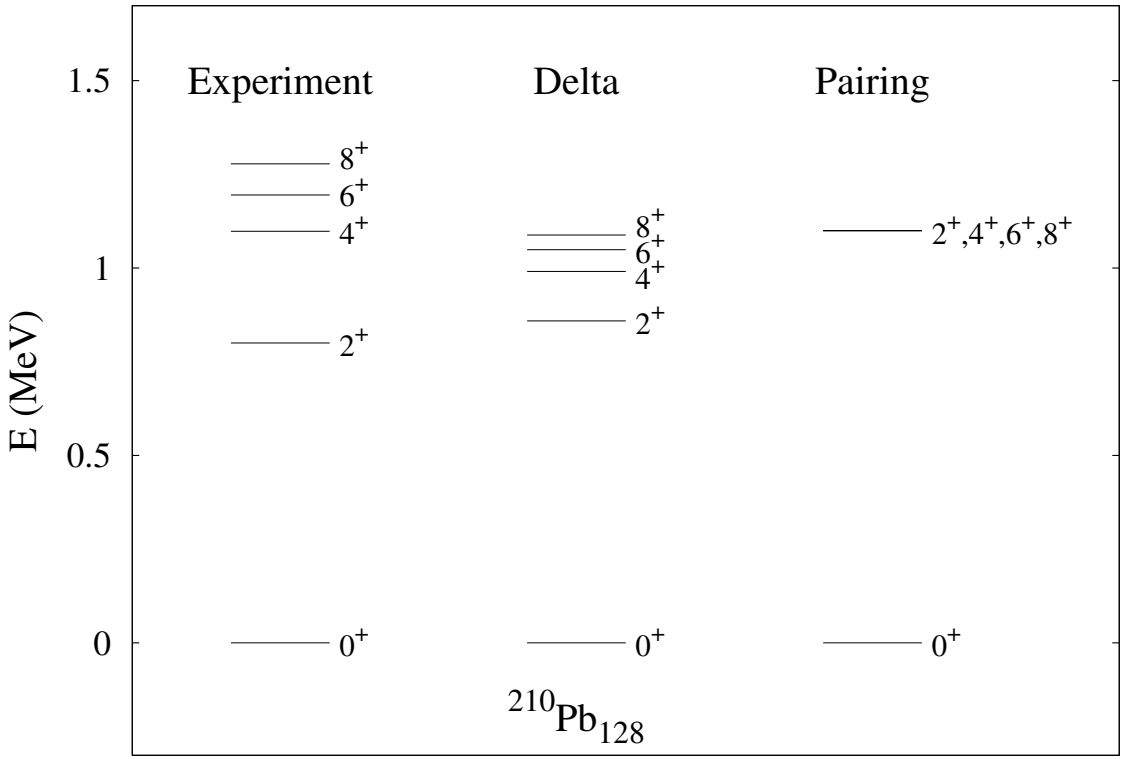
$$I = \frac{1}{4\pi} \int_0^\infty \frac{1}{r^2} \left[R_{n_1 l_1}(r) R_{n_2 l_2}(r) \right]^2 dr$$

Example 1: ^{210}Pb in $(\nu 0h_{9/2})^2$

$$V_{delta}(1,2) = -V_0\delta(\vec{r_1} - \vec{r_2})$$

$$V_{pairing}(1,2) = -G\hat{S}_+ \cdot \hat{S}_-$$

$$\langle j_a^2 \mid V_{pairing}(1,2) \mid j_b^2 \rangle_{01} = -(-1)^{l_a+l_b}\frac{1}{2}G\sqrt{(2j_a+1)(2j_b+1)}$$

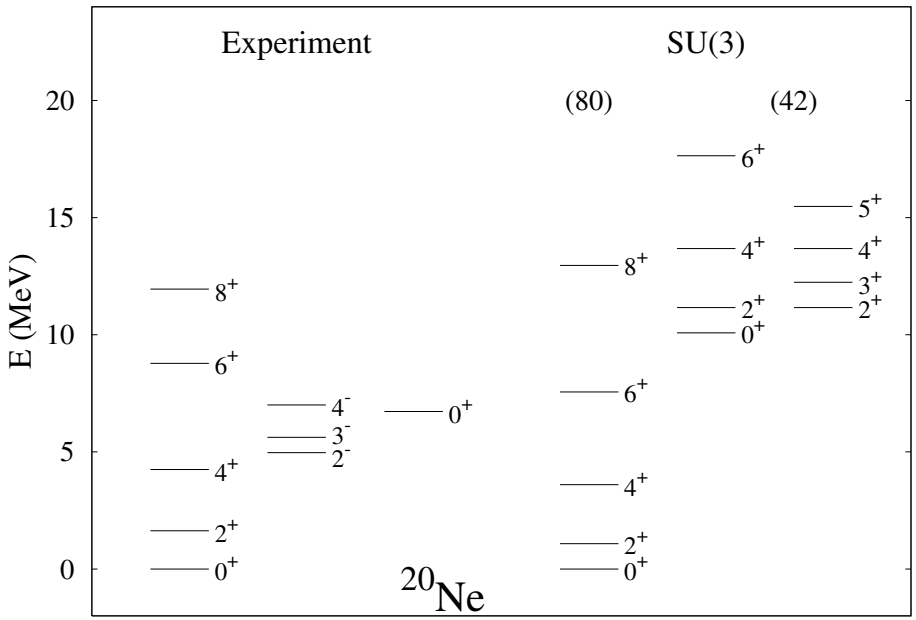


Example 2: ^{20}Ne and SU(3) model of Elliott

$$\hat{H} = \sum_{i=1}^{A} \left[-\frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 r_i^2 \right] - \chi Q \cdot Q$$

Q is an algebraic quadrupole operator (Q_μ, L_μ are SU(3) generators) *J.P.Elliott (1958)*

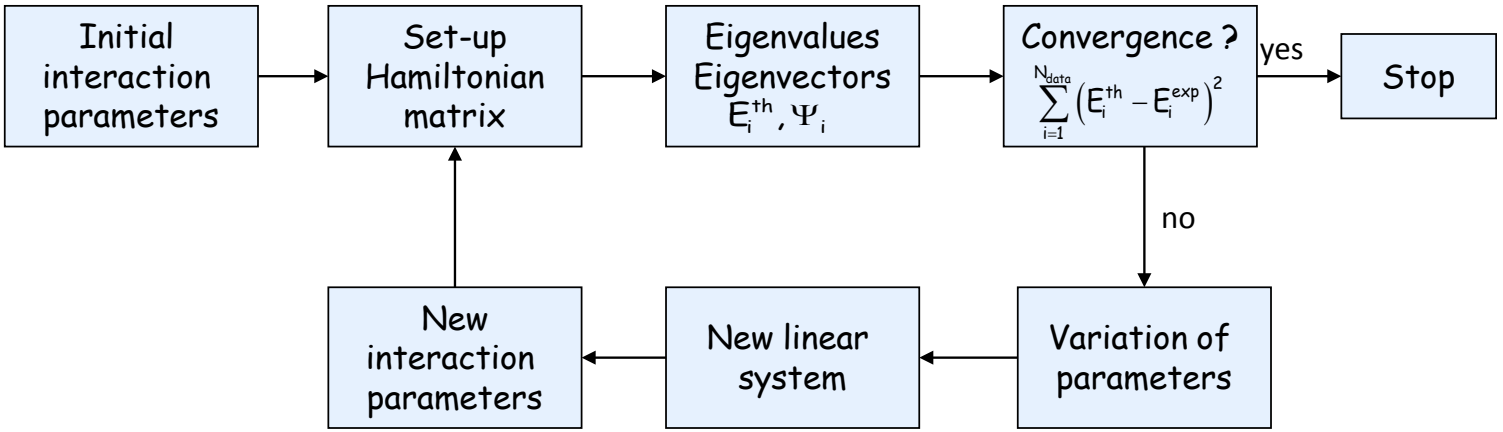
Group-theoretical classification of nuclear states (analytical solution) — see *lecture of H. Molique*.



Rotational classification of nuclear states as mixing of many spherical configurations

Empirical interaction (least-squares-fit method)

All TBME's $\langle j_a j_b | V | j_c j_d \rangle_{JT}$ are considered as free parameters !



Examples:

- 0*p*-shell: ^{4}He–^{16}O (15 TBME's) *Cohen, Kurath (1965)*
- 1*s*0*d*-shell: ^{16}O–^{40}Ca (63 TBME's) *Brown, Wildenthal (1988)*
- 1*p*0*f*-shell: ^{40}Ca–^{80}Zr (195 TBME's) *Honma et al (2002, 2004)*

Microscopic effective interaction

A bare NN potential (CD-Bonn, AV18, chiral N3LO, etc) requires regularization and modification to be applied for many-body calculations in a restricted model space.
Renormalization schemes (*see lecture of L. Bonneau*)

- *G* matrix followed by the many-body perturbation theory
- V_{low-k}
- SRG (IM-SRG)
- Okubo-Lee-Suzuki transformation

Successful, but still lack precision of the empirical interactions, mainly due to behavior of centroids. One of possible reasons: absence of 3N forces (*A. Poves, A.P. Zuker, 1981; A.P. Zuker, 2003*)

Modern nuclear shell model and beyond

Interacting shell model

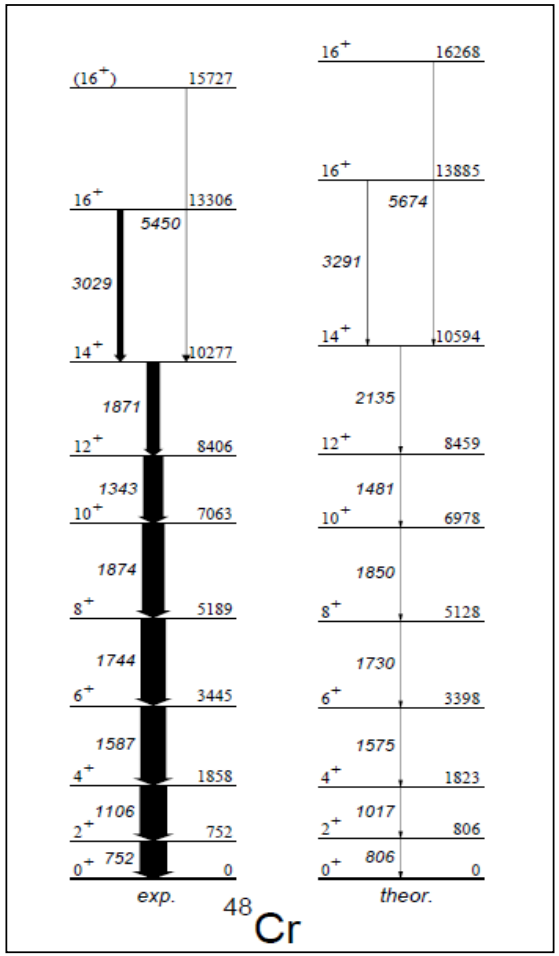
Oscillator-based shell model with accurate realistic interactions formulated in one or two harmonic-oscillator shells model spaces (large-scale diagonalization).

- Detailed information on individual states and transitions at low energies
- Conservation of principal symmetries

Numerous applications to nuclear structure, weak interaction and astrophysics

E. Caurier et al, Rev. Mod. Phys. 77, 427 (2005)

State-of-the-art calculations: backbending in ^{48}Cr



J<10: collective rotation
J=10-12 : backbending phenomenon (competition between rotation and alignment of $0f_{7/2}$ particles)
J>12 : spherical states

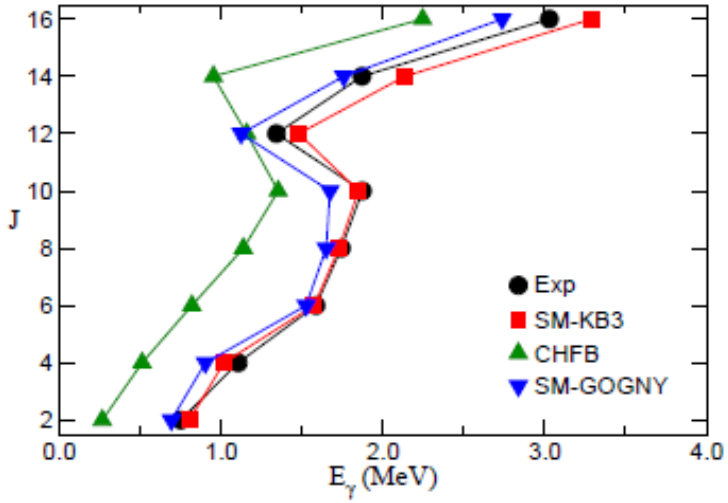
KB3 (semi-empirical interaction in *pf*-shell model space) Strasbourg-Madrid

For J<10 :

$$E_J \sim J(J+1)$$

$$Q_0 = \frac{(J+1)(2J+3)}{3K^2 - J(J+1)} Q_{spec}(J), \quad K \neq 1$$

$$B(E2; J \rightarrow J-2) = \frac{5}{16\pi} e^2 \left| (JK20 \mid J-2, K) \right|^2 Q_0^2$$



E. Caurier et al, Rev. Mod. Phys. 77 (2005) 427

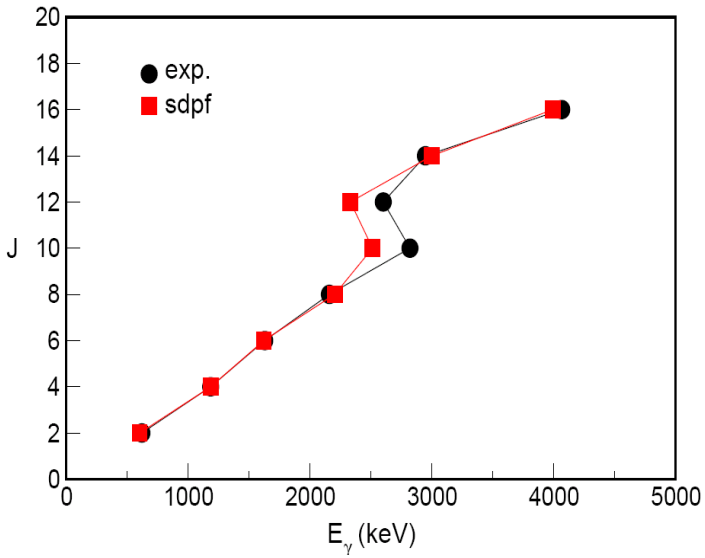
State-of-the-art calculations: superdeformation in ^{36}Ar

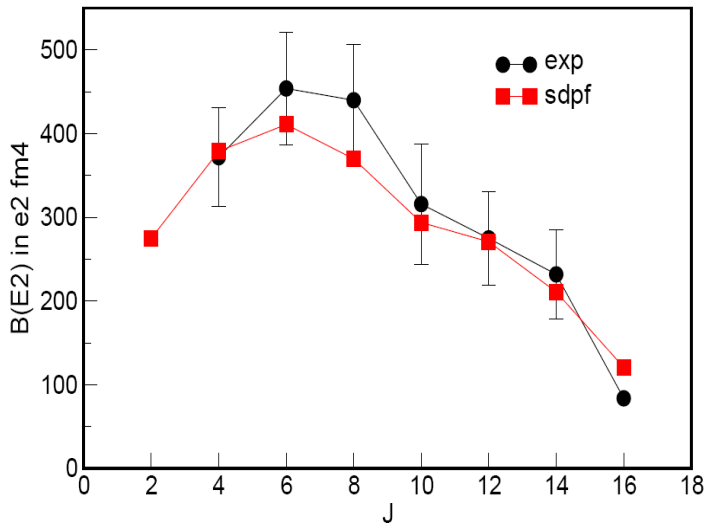
Intruder np-nh configurations can lead even to superdeformation !

$[sd]^{16}[pf]^{0}$ - 0p0h - spherical configuration

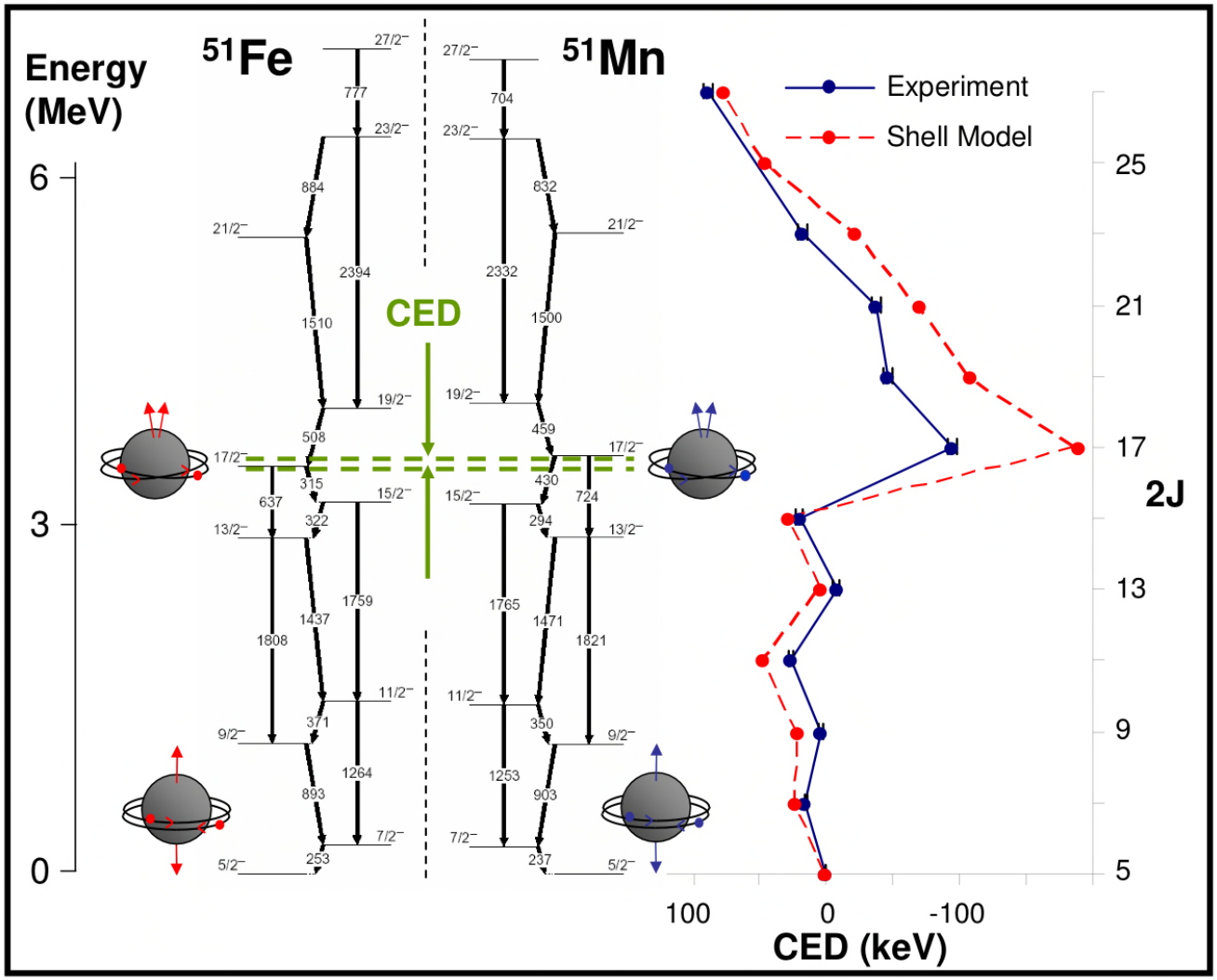
$[sd]^{12}[pf]^{4}$ - 4p4h - deformed configuration

E. Caurier et al, Phys.Rev.Lett. 95, 042502 (2005)





State-of-the-art calculations: mirror bands in $A = 51$



S. Lenzi, A. Zuker, E. Caurier et al

No-Core Shell Model

Non-relativistic Hamiltonian for A nucleons in many $N\hbar\Omega$ harmonic oscillator space

$$\hat{H} = \sum_{i=1}^{A} \frac{\vec{p}_i^2}{2m} + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j)$$

Problem: excitation of the center-of-mass of the system.

Center-of-mass coordinates

$$\vec{R} = \frac{1}{A}\sum_{i=1}^{A} \vec{r}_i\,; \qquad \vec{P} = \sum_{i=1}^{A} \vec{p}_i$$

Translational-invariant Hamiltonian

$$\hat{H} = \sum_{i=1}^{A} \frac{\vec{p}_i^2}{2m} - \frac{\vec{P}^2}{2mA} + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j)$$

Separation of the harmonic oscillator Hamiltonian into center-of-mass and intrinsic Hamiltonians

$$H_{ho} = \sum_{i=1}^{A} \left[\frac{\vec{p}_i^2}{2m} + \frac{m\Omega^2 \vec{r}_i^2}{2} \right]$$

$$= \underbrace{\frac{1}{2mA} \sum_{i<j=1}^{A} \left(\vec{p}_i - \vec{p}_j\right)^2 + \frac{m\Omega^2}{2A} \sum_{i<j=1}^{A} \left(\vec{r}_i - \vec{r}_j\right)^2}_{\hat{H}_{int}} + \underbrace{\frac{\vec{P}^2}{2mA} + \frac{mA\Omega^2 \vec{R}^2}{2}}_{\hat{H}_{CoM}}$$

No-Core Shell Model

Translational-invariant Hamiltonian

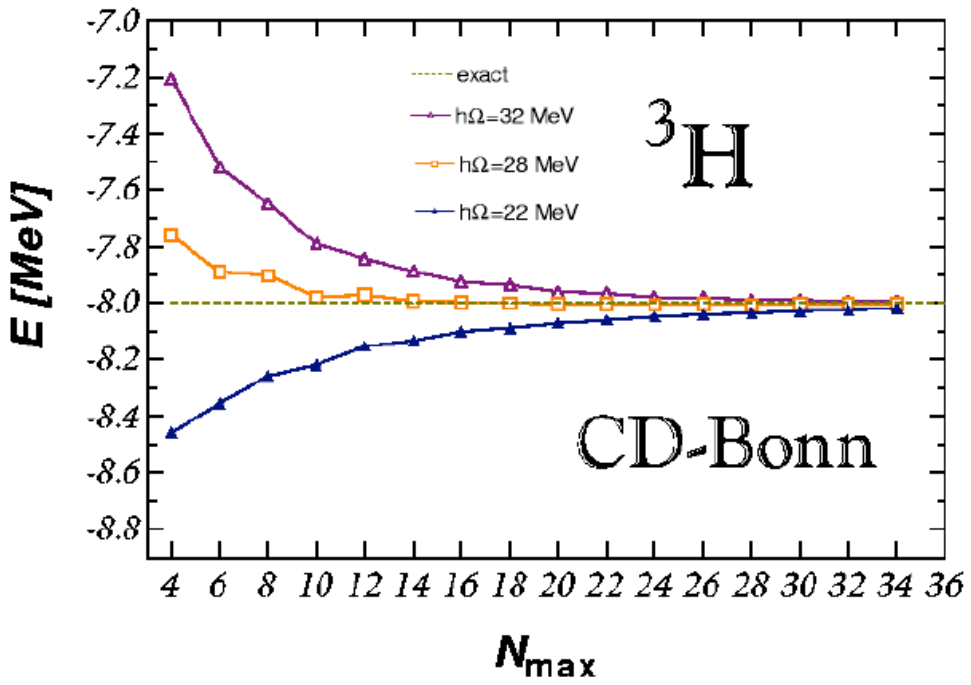
$$\hat{H} = \sum_{i=1}^{A} \left[\frac{\vec{p}_i^2}{2m} + \frac{m\Omega^2 \vec{r}_i^2}{2} \right] - \frac{\vec{P}^2}{2mA} + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j) - \sum_{i=1}^{A} \frac{m\Omega^2 \vec{r}_i^2}{2}$$

$$\hat{H} = \sum_{i=1}^{A} \left[\frac{\vec{p}_i^2}{2m} + \frac{m\Omega^2 \vec{r}_i^2}{2} \right] + \sum_{i<j=1}^{A} W(\vec{r}_i - \vec{r}_j) - \frac{m\Omega^2}{2A} \sum_{i<j=1}^{A} (\vec{r}_i - \vec{r}_j)^2 - \hat{H}_{CoM}$$

Separation of the Center-of-Mass motion

$$\hat{H}^{\Omega} = \hat{H} + \beta \left(\hat{H}_{CoM} - \frac{3}{2}\hbar\Omega \right)$$

^{3}H with CD-Bonn and N^{3}LO



Potential	E_{gs} (MeV)
AV18	-7.62
N^{3}LO	-7.86
CD-Bonn	-8.00
INOY	-8.48
Exp	-8.48

B.R. Barrett, P. Navratil, J. P. Vary, "*Ab-initio* no core shell model", Prog. Part. Nucl. Phys. 69 (2013) 131 — and refs. therein.

See lecture of R. Lazauskas for alternative methods.

Summary

The Nuclear Shell Model

- The Nuclear Shell Model is a powerful microscopic approach to nuclear structure
- Very good description of energies and transitions at low energies
- High predictive power.
- Important applications :
  - structure of nuclei far from stability (proton rich or neutron-rich nuclei)
  - calculation of weak interaction processes on nuclei for the tests of the Standard Model
  - calculation of the nuclear structure input (masses, half-lives, reaction rates, etc) relevant for astrophysical simulations (r-processes, *rp*-process, etc..).

Some references on the Shell model theory

Books

- P. J. Brussaard, P. W. M. Glaudemans, *Shell-Model Applications in Nuclear Spectroscopy* (North-Holland, Amsterdam, 1977).
- K. Heyde, *The Nuclear Shell Model* (Springer-Verlag, Heidelberg, 2004)
- A. DeShalit, I. Talmi, *Nuclear Shell Theory* (Academic Press, New York and London, 1963)
- I. Talmi, *Simple Models of Complex Nuclei, The Shell Model and the Interacting Boson Model* (Harwood Academic Publishers, New York, 1993)
- R. D. Lawson, *Theory of the Nuclear Shell Model* (Clarendon Press, 1980).

Reviews

- E. Caurier et al, *The shell model as a unified view of nuclear structure*, Rev. Mod. Phys. **77** (2005) 427.
- B. R. Barrett, P. Navratil, J. P. Vary, *Ab-initio no core shell model*, Prog. Part. Nucl. Phys. **69** (2013) 131