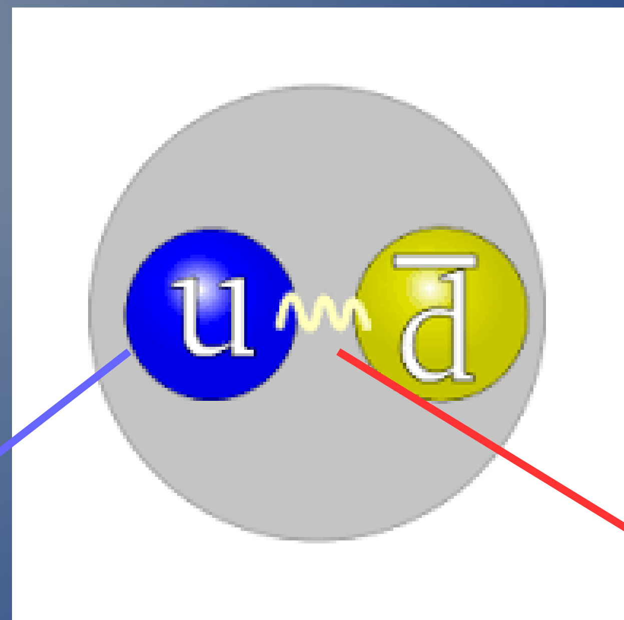
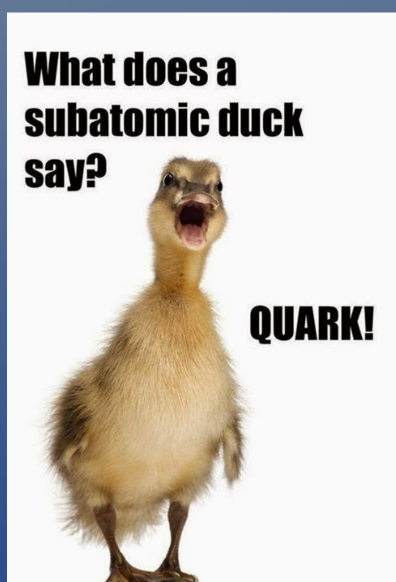
GPDs Modeling with LFWF: the pion's case





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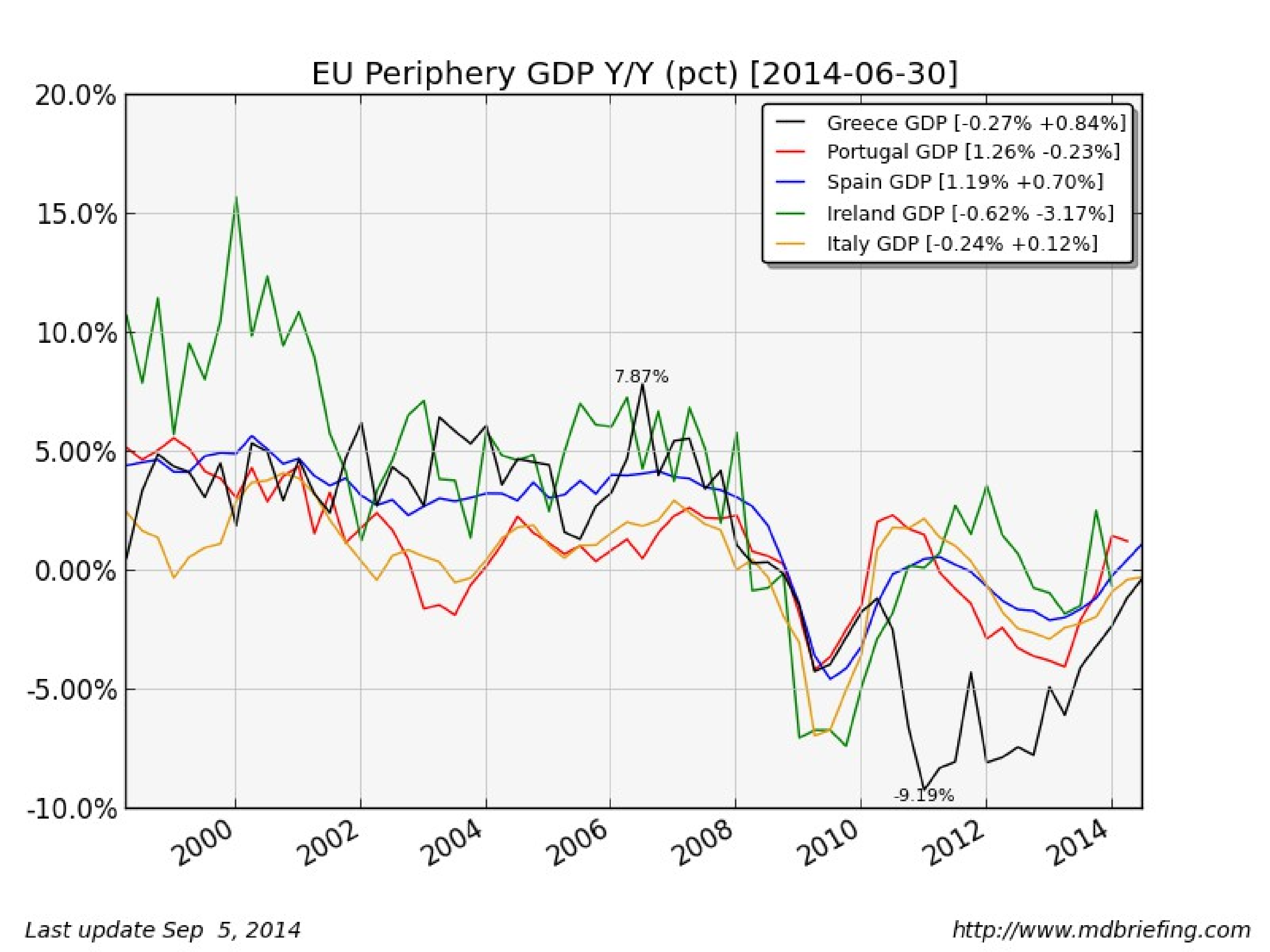
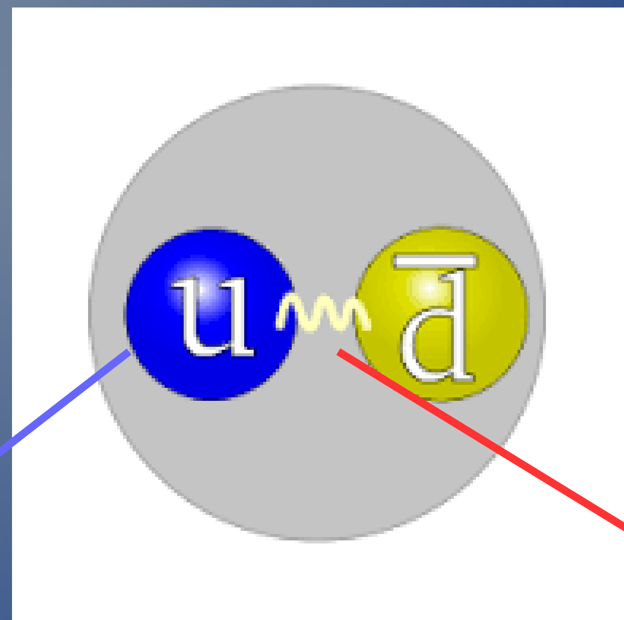
In collaboration with:
L. Chang, N. Chouika, C. Mezrag, H. Moutarde, C.D. Roberts, F. Sabatié

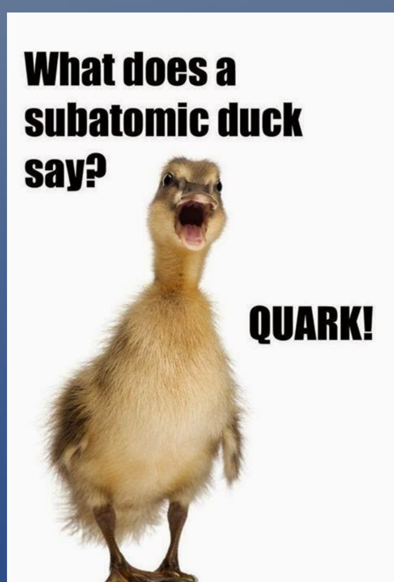
Nucleon and Resonance Structure with Hard Exclusive Processes 2017;
Orsay, 29-31 May

GDPs Modeling with LFWF: the Spain case



GPDs Modeling with LFWF: the pion's case





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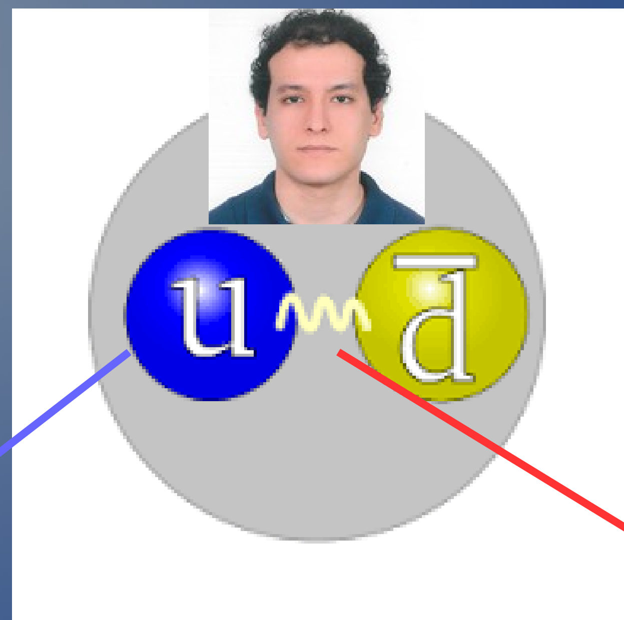
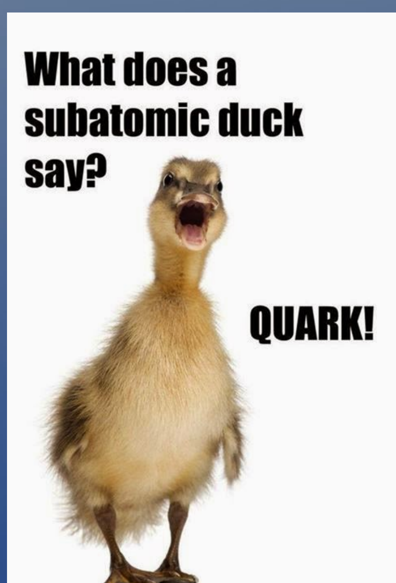
GPDs Modeling with LFWF: the pion's case





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Theoretical framework for

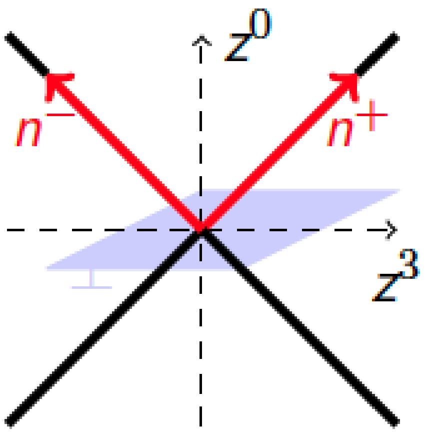
Pion GPD

Definition, constraints and symmetry properties:

$$H_\pi^q(x,\xi,t) \quad = $$

$$\frac{1}{2}\int\frac{dz^-}{2\pi}\,e^{ixP^+z^-}\left\langle \pi, P+\frac{\Delta}{2}\right|\bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right)\left|\pi, P-\frac{\Delta}{2}\right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t=\Delta^2$ and $\xi=-\Delta^+/(2P^+)$.



References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- From **isospin symmetry**, all the information about pion GPD is encoded in $H^u_{\pi^+}$ and $H^d_{\pi^+}$.
- Further constraint from **charge conjugation**: $H^u_{\pi^+}(x,\xi,t) = -H^d_{\pi^+}(-x,\xi,t)$.

Pion GPD

Definition, constraints and symmetry properties:

- PDF **forward limit**
- Form factor **sum rule**
- **Polynomiality** — Lorentz invariance
- **Positivity** — Positivity of Hilbert space norm
- H^q is an **even function** of ξ from time-reversal invariance.
- H^q is **real** from hermiticity and time-reversal invariance.
- H^q has support $x \in [-1, +1]$. — Relativistic Quantum mechanics
- **Soft pion theorem** (pion target) — Dinamical CSB

Numerous theoretical constraints on GPDs.

- There is no known GPD parameterization **relying only on first principles**.
- Modeling becomes a key issue.

Pion GPD

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Numerous theoretical constraints on GPDs.

- There is no known GPD parameterization **relying only on first principles**.
- Modeling becomes a key issue.

Focus here on polynomiality and positivity!

Polinomiality

Mixed constraint between Lorentz invariance and discrete symmetries

- Express Mellin moments of GPDs as **matrix elements**:

$$\int_{-1}^{+1} \mathrm{d}x\, x^m H^q(x,\xi,t) = \frac{1}{2(P^+)^{m+1}} \left\langle P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| P - \frac{\Delta}{2} \right\rangle$$

- Identify the **Lorentz structure** of the matrix element:

$$\text{linear combination of } (P^+)^{m+1-k}(\Delta^+)^k \text{ for } 0 \leq k \leq m+1$$

- Remember definition of **skewness** $\Delta^+ = -2\xi P^+$.
- Select **even powers** to implement time reversal.
- Obtain **polynomiality condition**:

$$\int_{-1}^{1} \mathrm{d}x\, x^m H^q(x,\xi,t) = \sum_{\substack{i=0 \\ \text{even}}}^{m} (2\xi)^i C^q_{mi}(t) + (2\xi)^{m+1} C^q_{mm+1}(t) .$$

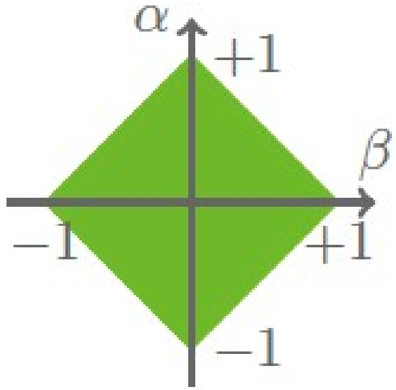
Double Distributions

A well fitted tool to encode GPD properties

- Define Double Distributions F^q and G^q as matrix elements of **twist-2 quark operators**:

$$\left\langle P+\frac{\Delta}{2}\right|\bar{q}(0)\gamma^{\{\mu}i\overleftrightarrow{D}^{\mu_1}\ldots i\overleftrightarrow{D}^{\mu_m\}}q(0)\left|P-\frac{\Delta}{2}\right\rangle = \sum_{k=0}^{m}\binom{m}{k}$$

$$\left[F^q_{mk}(t)2P^{\{\mu} - G^q_{mk}(t)\Delta^{\{\mu}\right]P^{\mu_1}\ldots P^{\mu_{m-k}}\left(-\frac{\Delta}{2}\right)^{\mu_{m-k+1}}\ldots\left(-\frac{\Delta}{2}\right)^{\mu_m\}}$$



with

$$F^q_{mk} = \int_\Omega \mathrm{d}\beta\mathrm{d}\alpha\,\alpha^k\beta^{m-k}F^q(\beta,\alpha)$$

$$G^q_{mk} = \int_\Omega \mathrm{d}\beta\mathrm{d}\alpha\,\alpha^k\beta^{m-k}G^q(\beta,\alpha)$$

[Muller et al., Fortschr.Phys. 42 (1994)101
[Radyshkin, Phys.Rev.D59(1999)014030;Phys.Lett.B499(1999)81

Double Distributions

Relation to Generalized Parton Distributions

- Representation of GPD:

$$H^q(x, \xi, t) = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta \mathrm{d}\alpha \, \delta(x - \beta - \alpha\xi)\big(F^q(\beta, \alpha, t) + \xi G^q(\beta, \alpha, t)\big)$$

- Support property: $x \in [-1, +1]$.
- Discrete symmetries: F^q is α-even and G^q is α-odd.
- **Gauge**: any representation (F^q, G^q) can be recast in one representation with a single DD f^q:

$$H^q(x, \xi, t) = x \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta \mathrm{d}\alpha \, f^q_{\mathrm{BMKS}}(\beta, \alpha, t) \delta(x - \beta - \alpha\xi)$$

[Belitsky et al., Phys.Rev.D64 (2001)110062]

Positivity and overlap representation

Relation to Generalized Parton Distributions

- Identify the matrix element defining a GPD as an **inner product** of two different states.
- Apply Cauchy-Schwartz inequality, and identify PDFs at specific kinematic points, *e.g.*:

$$|H^q(x,\xi,t)| \leq \sqrt{\frac{1}{1-\xi^2}\, q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- This procedures yields **infinitely many inequalities** stable under LO evolution.

Pobylitsa, Phys. Rev. **D66**, 094002 (2002)

- The **overlap representation** guarantees *a priori* the fulfillment of positivity constraints.

Positivity and overlap representation

A first-principle connection to Light Front Wave Functions

- Decompose an hadronic state $|H; P, \lambda\rangle$ in a Fock basis:

$$|H; P, \lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x\mathrm{d}\mathbf{k}_\perp]_N \psi_N^{(\beta,\lambda)}(x_1, \mathbf{k}_{\perp 1}, \ldots, x_N, \mathbf{k}_{\perp N}) \, |\beta, k_1, \ldots, k_N\rangle$$

- Derive an expression for the pion GPD in the DGLAP region $\xi \leq x \leq 1$:

$$H^q(x, \xi, t) \propto \sum_{\beta,j} \int [\mathrm{d}\bar{x}\mathrm{d}\bar{\mathbf{k}}_\perp]_N \delta_{j,q} \delta(x - \bar{x}_j) \psi_N^{(\beta,\lambda)*}(\hat{x}', \hat{\mathbf{k}}'_\perp) \psi_N^{(\beta,\lambda)}(\tilde{x}, \tilde{\mathbf{k}}_\perp)$$

with $\tilde{x}, \tilde{\mathbf{k}}_\perp$ (resp. $\hat{x}', \hat{\mathbf{k}}'_\perp$) generically denoting incoming (resp. outgoing) parton kinematics.

Diehl *et al.*, Nucl. Phys. **B596**, 33 (2001)

- Similar expression in the ERBL region $-\xi \leq x \leq \xi$, but with overlap of N- and $(N+2)$-body LFWF.

Positivity and overlap representation

Advantages and drawbacks

Then:

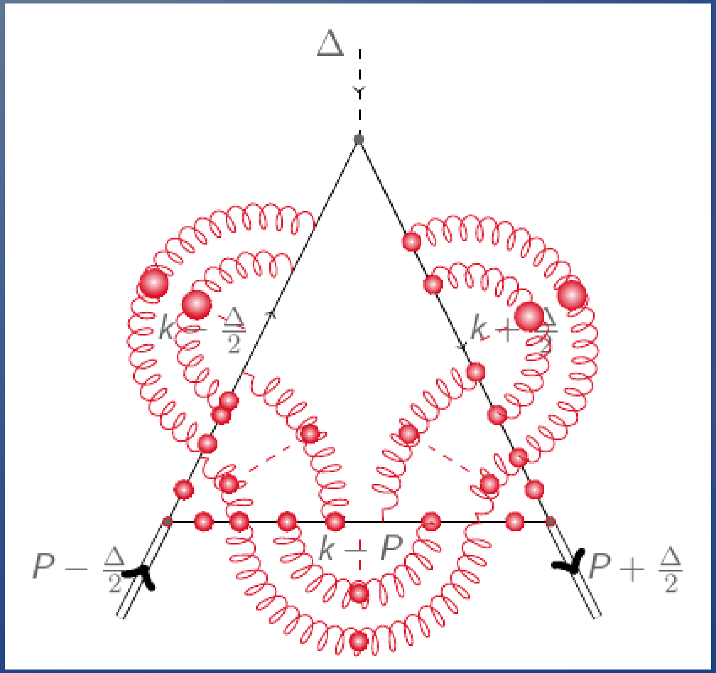
- Physical picture.
- Positivity relations are fulfilled **by construction**.
- Implementation of **symmetries of N-body problems**.

What is not obvious anymore

What is *not* obvious to see from the wave function representation is however the **continuity of GPDs at** $x = \pm\xi$ and the **polynomiality** condition. In these cases both the DGLAP and the ERBL regions must cooperate to lead to the required properties, and this implies **nontrivial relations between the wave functions** for the different Fock states relevant in the two regions. An *ad hoc* Ansatz for the wave functions would **almost certainly lead** to GPDs that **violate the above requirements**.

Diehl, Phys. Rept. **388**, 41 (2003)

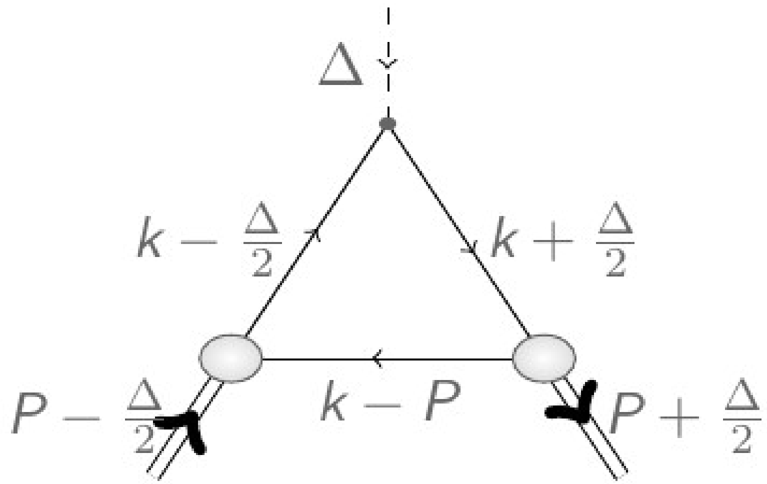
GPDs in the Bethe-Salpeter and Schwinger-Dyson approach



GPD in the DSE-BSE approach

Evaluation *via* the triangle diagram approximation:

$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD H.

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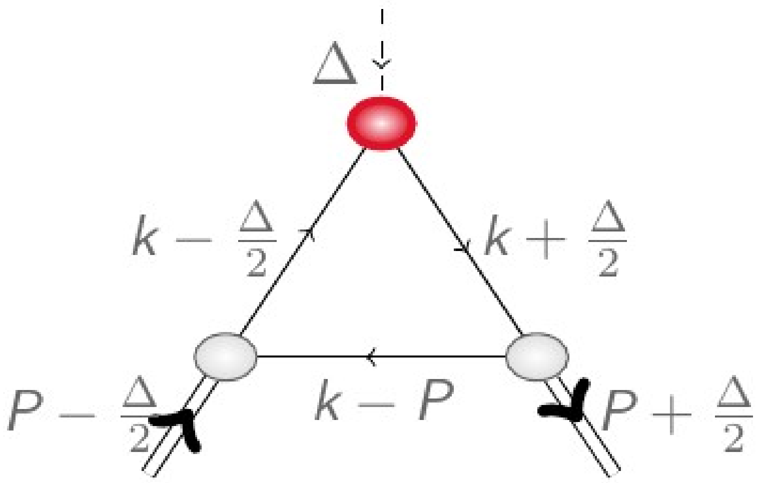

- Compute **Mellin moments** of the pion GPD H.
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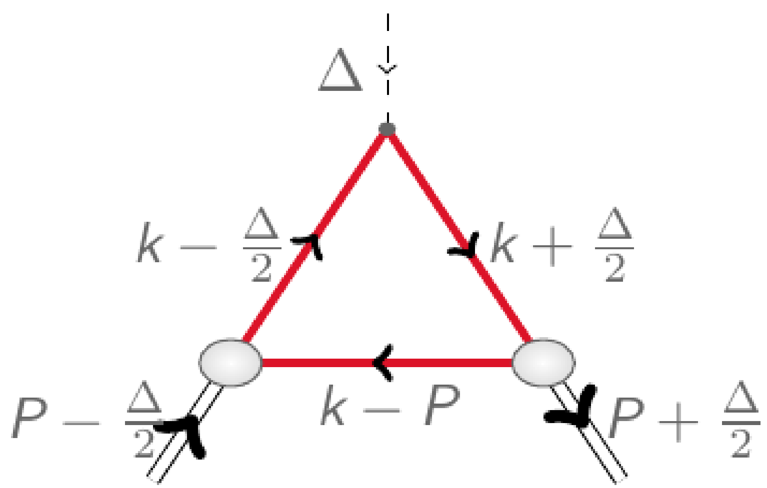
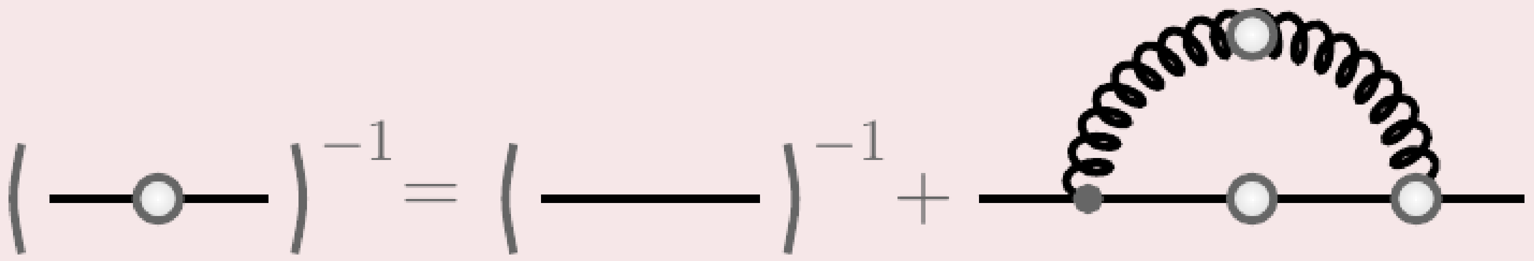


- Compute **Mellin moments** of the pion GPD *H*.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

Dyson - Schwinger equation



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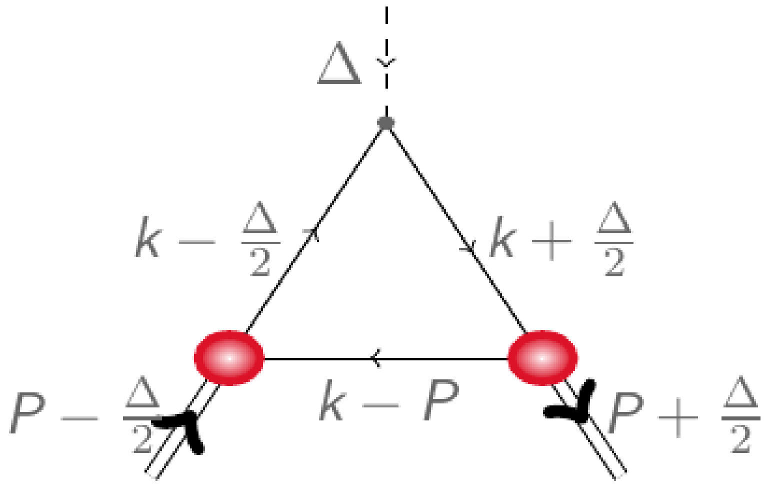

- Compute **Mellin moments** of the pion GPD H.
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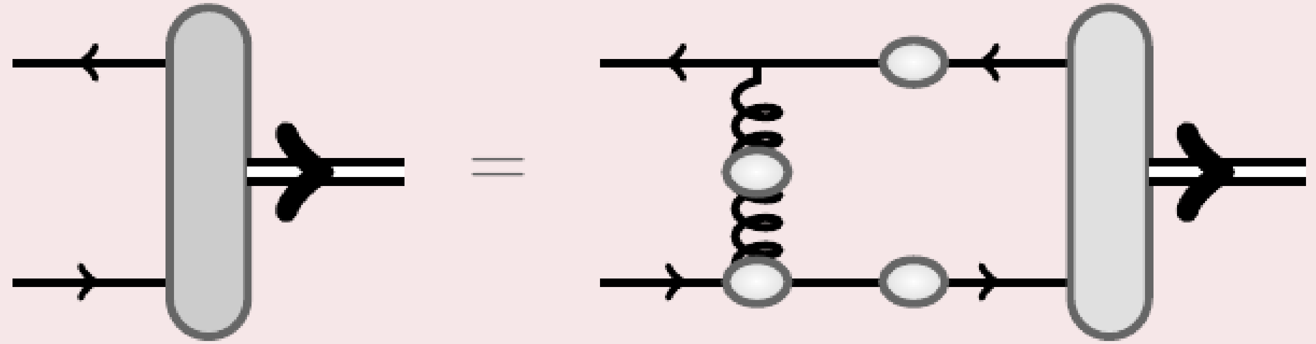
Bethe - Salpeter equation



GPD in the DSE-BSE approach

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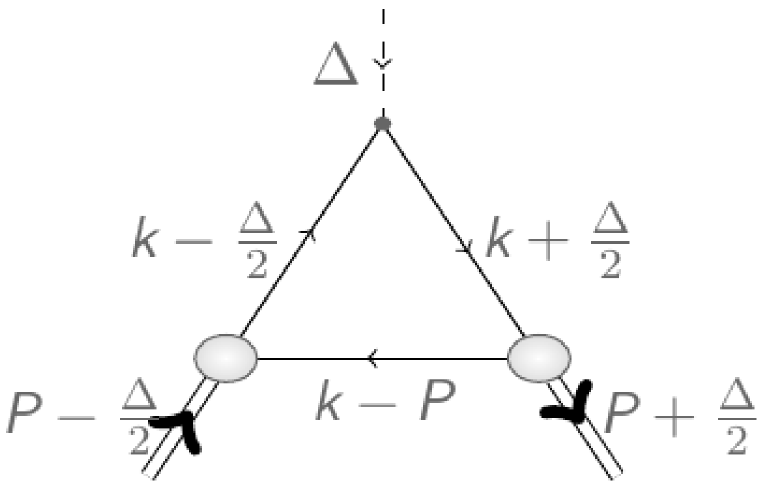

- Compute **Mellin moments** of the pion GPD *H*.
- Triangle diagram approx.
- Resum **infinitely many** contributions.
- **Nonperturbative** modeling.

- Most GPD properties **satisfied by construction**.
- Also compute crossed triangle diagram.

Mezrag *et al.*, arXiv:1406.7425 [hep-ph]
and Phys. Lett. **B741**, 190 (2015)

Algebraic DSE-BSE inspired GPD model

Have to deal with DSEs and BSEs solutions:

- Numerical resolution of gap and Bethe-Salpeter equations in Euclidean space.
- Analytic continuation to Minkowskian space required.
- **Ill-posed** problem in the sense of Hadamard.
- Parameterize solutions and fit to numerical solution:

Gap Complex-conjugate pole representation:

$$S(k) = \sum_{i=0}^{N} \left[\frac{z_i}{i\not{k} + m_i} + \frac{z_i^*}{i\not{k} + m_i^*} \right]$$

Bethe-Salpeter Nakanishi representation of amplitude $\mathcal{F}_\pi$:

$$\mathcal{F}_\pi(q^2, q \cdot P) = \int_{-1}^{+1} \mathrm{d}\alpha \int_0^\infty \mathrm{d}\lambda \frac{\rho(\alpha, \lambda)}{(q^2 + \alpha q \cdot P + \lambda^2)^n}$$

Algebraic DSE-BSE inspired GPD model

A first intermediate step before dealing with numerical solutions:

- Expressions for vertices and propagators:

$$\begin{aligned} S(p) &= \left[- i\gamma \cdot p + M \right] \Delta_M(p^2) \\ \Delta_M(s) &= \frac{1}{s + M^2} \\ \Gamma_\pi(k,p) &= i\gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} \mathrm{d}z \, \rho_\nu(z) \ \left[\Delta_M(k^2_{+z}) \right]^\nu \\ \rho_\nu(z) &= R_\nu (1 - z^2)^\nu \end{aligned}$$

  with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

  Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:

Algebraic DSE-BSE inspired GPD model

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- Only two parameters:
  - Dimensionful parameter M.

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  Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:
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  - Dimensionless parameter ν

Algebraic DSE-BSE inspired GPD model

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\rho_\nu(z) &= R_\nu (1-z^2)^\nu
\end{aligned}
$$

with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:
  - Dimensionful parameter M.
  - Dimensionless parameter ν. **Fixed to 1** to recover asymptotic pion DA.

Results for the pion GPD

Verification of the theoretical constraints:

- **Analytic expression** in the DGLAP region.

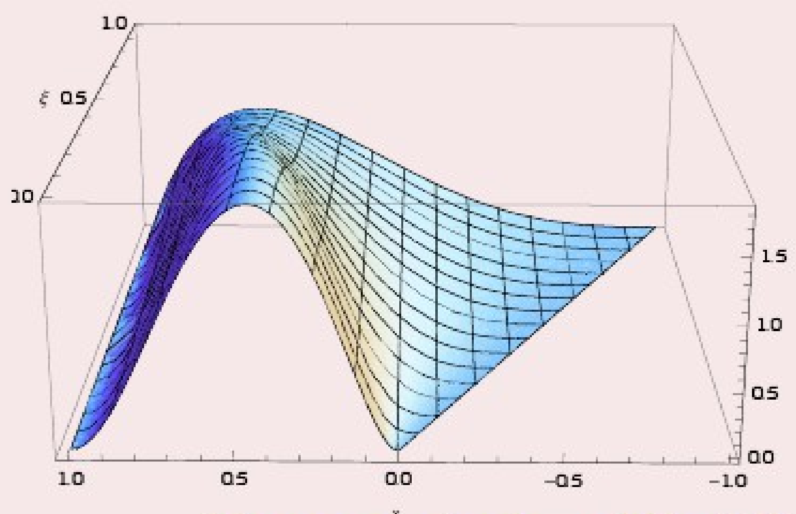
$$
\begin{aligned}
H^{u}_{x\geq\xi}(x,\xi,0) \quad = \quad & \frac{48}{5}\Bigg\{ \frac{3\left(-2(x-1)^4\left(2x^2-5\xi^2+3\right)\log(1-x)\right)}{20\left(\xi^2-1\right)^3} \\
& \frac{3\left(+4\xi\left(15x^2(x+3)+(19x+29)\xi^4+5(x(x(x+11)+21)+3)\xi^2\right)\tanh^{-1}\left(\frac{(x-1)}{x-\xi^2}\right.\right.}{20\left(\xi^2-1\right)^3} \\
& + \frac{3\left(x^3(x(2(x-4)x+15)-30)-15(2x(x+5)+5)\xi^4\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3} \\
& + \frac{3\left(-5x(x(x(x+2)+36)+18)\xi^2-15\xi^6\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3} \\
& + \frac{3\left(2(x-1)\left((23x+58)\xi^4+(x(x(x+67)+112)+6)\xi^2+x(x((5-2x)x+15)+\right.\right.}{20\left(\xi^2-1\right)^3} \\
& + \frac{3\left(\left(15(2x(x+5)+5)\xi^4+10x(3x(x+5)+11)\xi^2\right)\log\left(1-\xi^2\right)\right)}{20\left(\xi^2-1\right)^3} \\
& + \frac{3\left(2x(5x(x+2)-6)+15\xi^6-5\xi^2+3\right)\log\left(1-\xi^2\right)}{20\left(\xi^2-1\right)^3}\Bigg\}
\end{aligned}
$$

Results for the pion GPD

Verification of the theoretical constraints:

- **Analytic expression** in the DGLAP region.
- Similar expression in the ERBL region.
- **Explicit check** of **support property** and **polynomiality** with correct powers of ξ.
- Also direct verification using Mellin moments of H.

Valence $H^u(x, \xi, t)$ as a function of x and ξ at vanishing t.



Mezrag *et al.*, arXiv:1406.7425 [hep-ph]

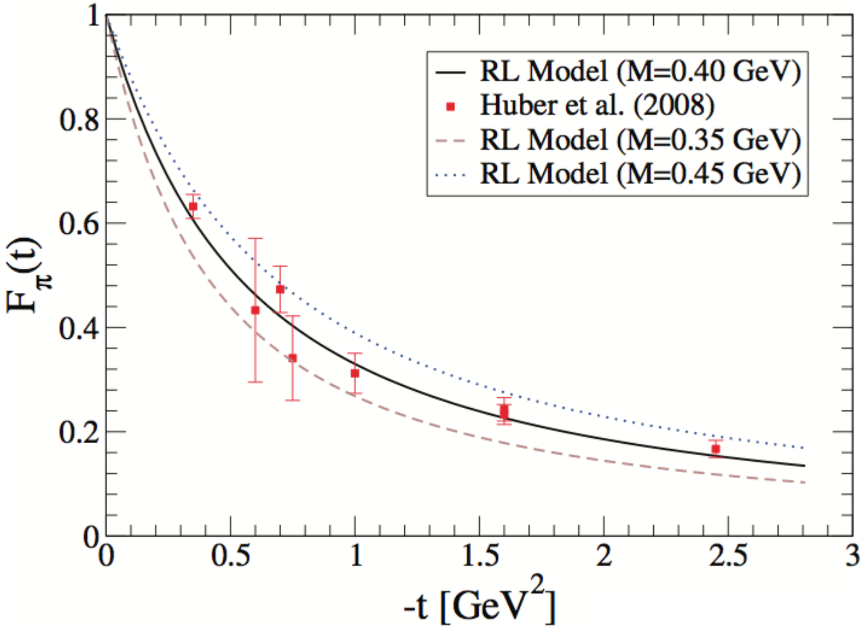
Results for the pion GPD

The form factor and the dimensionful parameter:

- Pion form factor obtained from isovector GPD:

$$\int_{-1}^{+1} \mathrm{d}x\, H^{I=1}(x,\xi,t) = 2F_\pi(t)$$

- Single dimensionful parameter $M \simeq 400$ MeV.



C. Mezrag et al., Phys. Lett. B190 (2015) 741

Results for the pion GPD

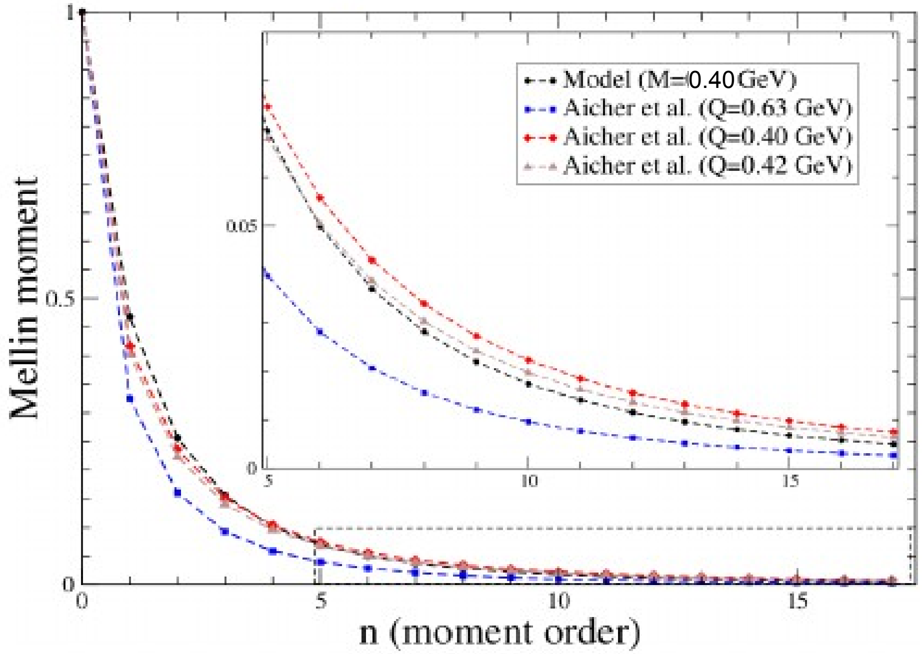
The parton distribution function:

- Pion PDF obtained from forward limit of GPD:

$$q(x) = H^q(x, 0, 0)$$

- Use LO DGLAP equation and compare to PDF extraction.

Aicher *et al.*, Phys. Rev. Lett. **105**, 252003 (2010)



Mezrag *et al.*, arXiv:1406.7425 [hep-ph]

- Find model initial scale $\mu \simeq 400$ MeV.

Results for the pion GPD

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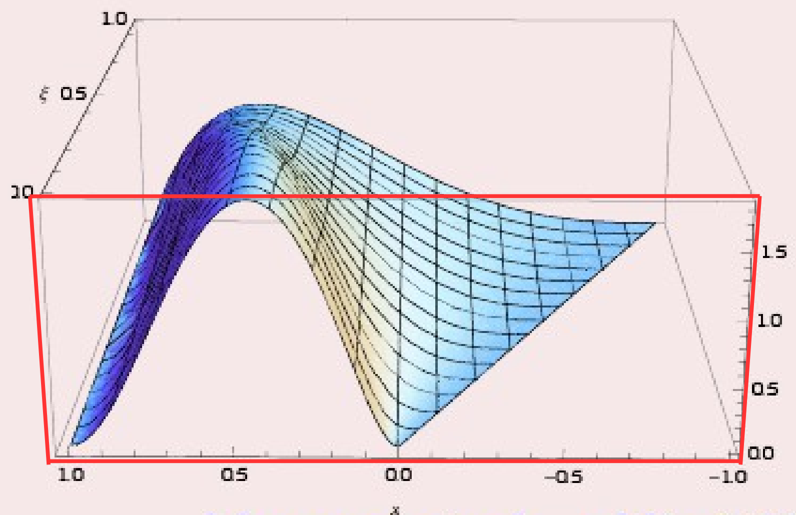
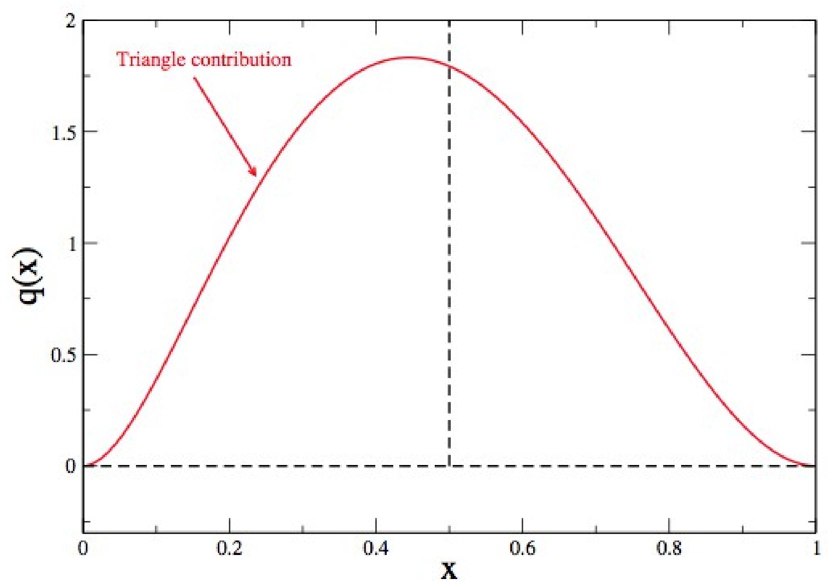


Mezrag *et al.*, arXiv:1406.7425 [hep-ph]

Results for the pion GPD

The two-body problem:

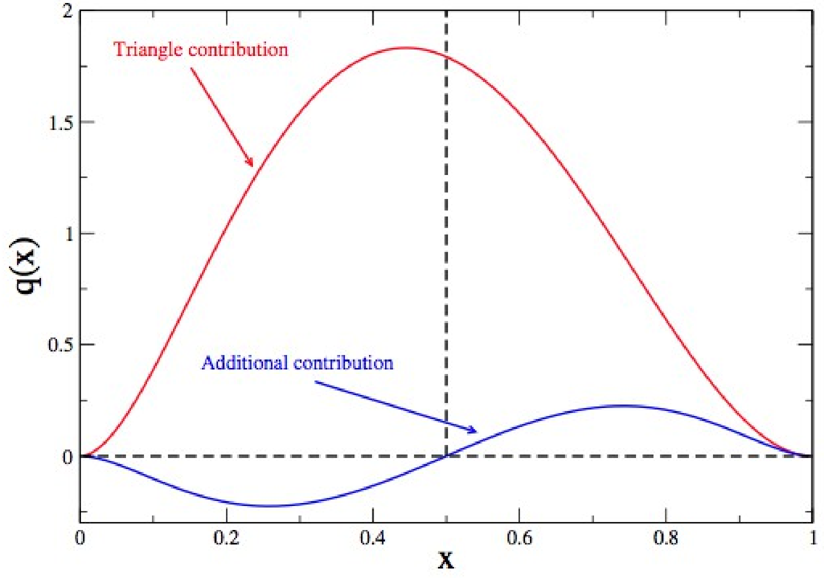


$$q_A^\pi(x) = n_q \Big[x^3(x[-2(x-4)x-15]+30)\ln(x) + (2x^2+3)$$
$$\times(x-1)^4\ln(1-x) + x[x(x[2x-5]-15)-3](x-1)\Big] \,,$$

- The PDF appears not to be symmetric around $x = \frac{1}{2}$.

Results for the pion GPD

The two-body problem:



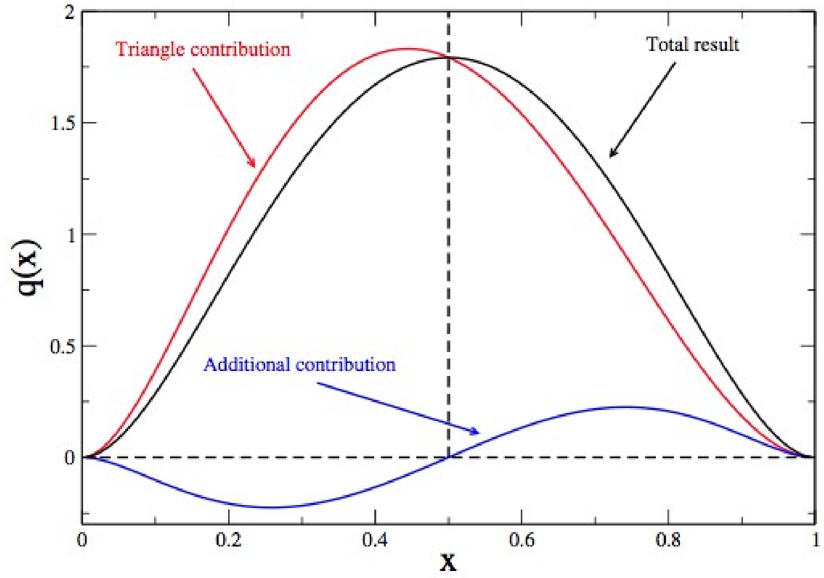
$$q_A^{\pi}(x) = n_q \Big[x^3(x[-2(x-4)x-15]+30)\ln(x) + (2x^2+3) \\ \times(x-1)^4 \ln(1-x) + x[x(x[2x-5]-15)-3](x-1)\Big] ,$$

$$q_{BC}^{\pi}(x) = n_q \Big[x^3(2x([x-3]x+5)-15)\ln(x) - (2x^3+4x+9) \\ \times(x-1)^3 \ln(1-x) - x(2x-1)([x-1]x-9)(x-1)\Big] . \quad (13)$$

- The PDF appears not to be symmetric around $x = \frac{1}{2}$.
- Part of the gluons contribution is neglected in the triangle diagram approach.

Results for the pion GPD

The two-body problem:



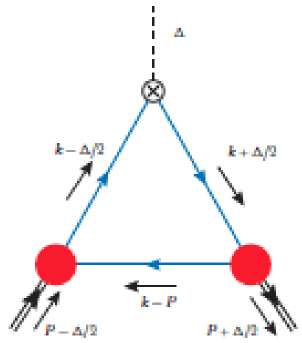
$$q_L^\pi(x) = \frac{72}{25}\Big[x^3(x[2x-5]+15)\ln(x) + (x[2x+1]+12) \times (1-x)^3\ln(1-x) + 2x(6-[1-x]x)(1-x)\Big].$$

- The PDF appears not to be symmetric around $x = \frac{1}{2}$.
- Part of the gluons contribution is neglected in the triangle diagram approach.
- Adding this contribution allows us to recover a symmetric PDF [L. Chang *et al.*, Phys.Lett.B737(2014)2329].

Results for the pion GPD

The off-forward (non-skewed) GPD:

The model:



$$2(P\cdot n)^{m+1}\,\langle x^m\rangle^u = \mathrm{tr}_{CFD}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)$$
$$S(k-\frac{\Delta}{2})\; i\gamma\cdot n\; S(k+\frac{\Delta}{2})$$
$$\tau_- i\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P),$$

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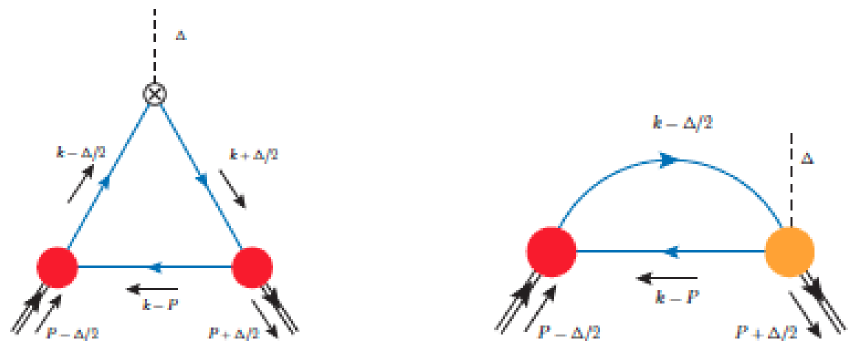
Results for the pion GPD

The off-forward (non-skewed) GPD:

The full model:



$$2(P\cdot n)^{m+1}\langle x^m\rangle^u = \mathrm{tr}_{CFD}\int\frac{d^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)$$
$$S(k-\frac{\Delta}{2})\ i\gamma\cdot n\ S(k+\frac{\Delta}{2})$$
$$\tau_- i\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P),$$

$$2(P\cdot n)^{m+1}\langle x^m\rangle^u = \mathrm{tr}_{CFD}\int\frac{d^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)$$
$$S(k-\frac{\Delta}{2})\tau_-\ \frac{\partial}{\partial k}\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P)$$

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Results for the pion GPD

The off-forward (non-skewed) GPD:

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$$S\left(k-\frac{\Delta}{2}\right)\tau_-\,\frac{\partial}{\partial k}\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P), P+\frac{\Delta}{2}\right) S(k-P)$$

$$F^{BC}(\beta,\alpha,t),\, G^{BC}(\beta,\alpha,t)$$

$$H^{BC}(x,\xi,t) = \int_{-1}^{1}\mathrm{d}\beta\int_{-1+|\beta|}^{1-|\beta|}\mathrm{d}\alpha\,\left(F^{BC}(\beta,\alpha,t) + \xi\, G^{BC}(\beta,\alpha,t)\right)\delta(x-\beta-\alpha\xi)$$

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Results for the pion GPD

The off-forward (non-skewed) GPD:

$$2(P\cdot n)^{m+1}\langle x^m\rangle^u = \mathrm{tr}_{CFD}\int \frac{\mathrm{d}^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right), P-\frac{\Delta}{2}\right)$$

$$S(k-\frac{\Delta}{2})\tau_-\,\frac{\partial}{\partial k}\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P), P+\frac{\Delta}{2}\right)S(k-P)$$

$$F^{BC}(\beta,\alpha,t),\ G^{BC}(\beta,\alpha,t)$$

$$H^{BC}(x,\xi,t)=\int_{-1}^{1}\mathrm{d}\beta\int_{-1+|\beta|}^{1-|\beta|}\mathrm{d}\alpha\,\left(F^{BC}(\beta,\alpha,t)+\xi\,G^{BC}(\beta,\alpha,t)\right)\delta(x-\beta-\alpha\xi)$$

$$\boxed{H^{BC}(x,0,0)=\int_{-1+|x|}^{1-|x|}\mathrm{d}\alpha\,F^{BC}(x,\alpha,0)}\equiv q^{\pi}_{BC}(x)$$

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Results for the pion GPD

The off-forward (non-skewed) GPD:

The pion GPD

$$H^q(x,0,t) = \int_{-1+|x|}^{1-|x|} \mathrm{d}\alpha \left(F^0(x,\alpha,t) + F^{BC}(x,\alpha,t)\right)$$

$$H(x,0,t) \doteq H(x,0,0)\mathcal{N}(t)C_\pi(x,t)F_\pi(t)\,,$$

$$1 = \mathcal{N}(t)\int_{-1}^{1} dx\; H(x,0,0)C_\pi(x,t)\,.$$

$$F(\beta,\alpha,t) = \frac{1}{\left(1+\frac{t}{4M^2}(1-\beta+\alpha)(1-\beta+\alpha)\right)^2} \times (F_S(\beta,\alpha) + t\,[\cdots])$$

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Results for the pion GPD

The off-forward (non-skewed) GPD:

The pion GPD

$$H^q(x,0,t) = \int_{-1+|x|}^{1-|x|} \mathrm{d}\alpha \, \left(F^0(x,\alpha,t) + F^{BC}(x,\alpha,t) \right)$$

$$H(x,0,t) = H(x,0,0)\mathcal{N}(t)C_\pi(x,t)F_\pi(t)\,,$$

$$1 = \mathcal{N}(t)\int_{-1}^{1} dx\, H(x,0,0)\, C_\pi(x,t)\,.$$

$$F(\beta,\alpha,t) = \frac{1}{\left(1+\frac{t}{4M^2}(1-\beta)(1-\beta)\right)^2} \times F_S(\beta,\alpha)$$

Simplified analytical model:

$$C(x,t) = \frac{1}{\left(1+\frac{t}{4M^2}(1-x)^2\right)^2}$$

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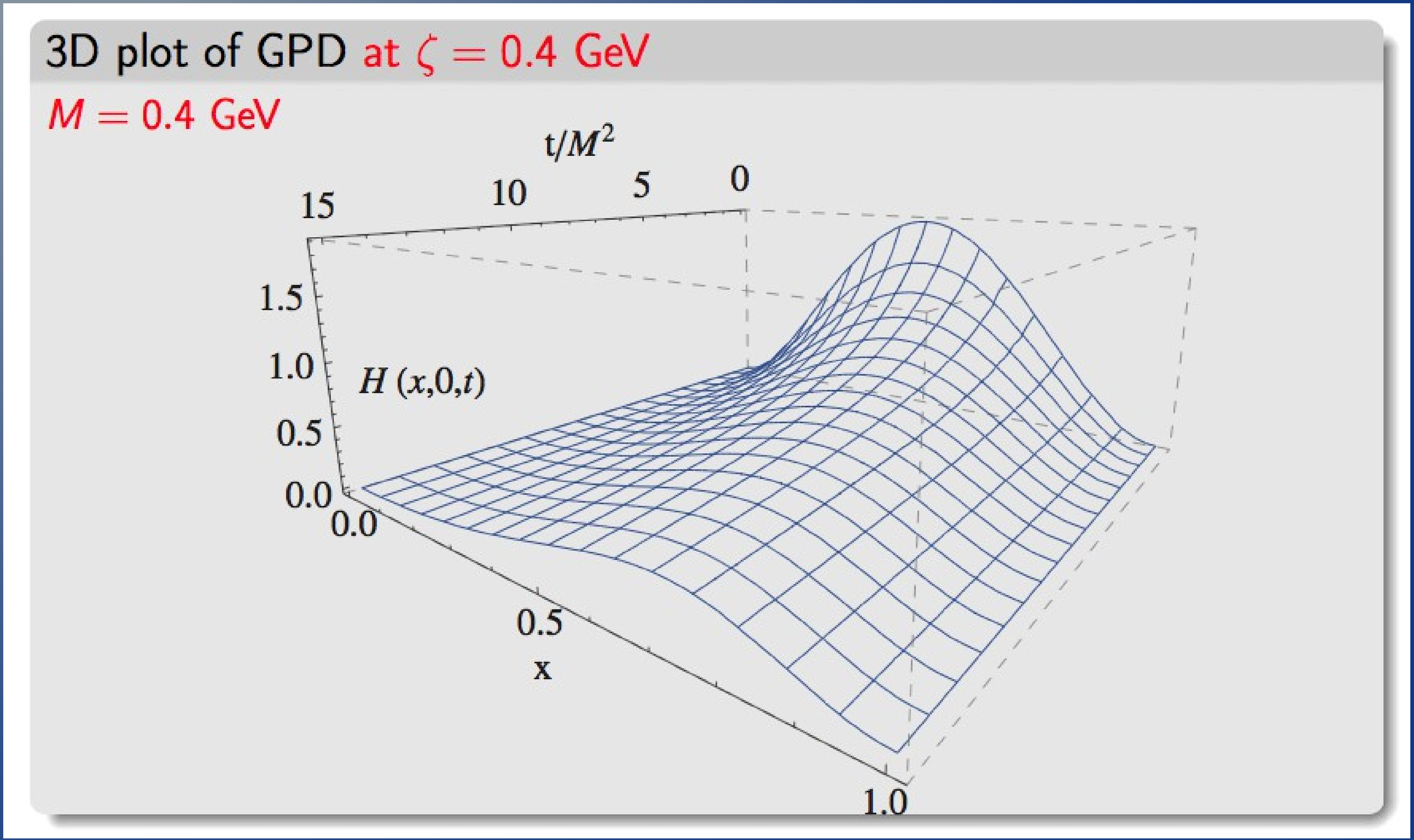
Results for the pion GPD

The off-forward (non-skewed) GPD:



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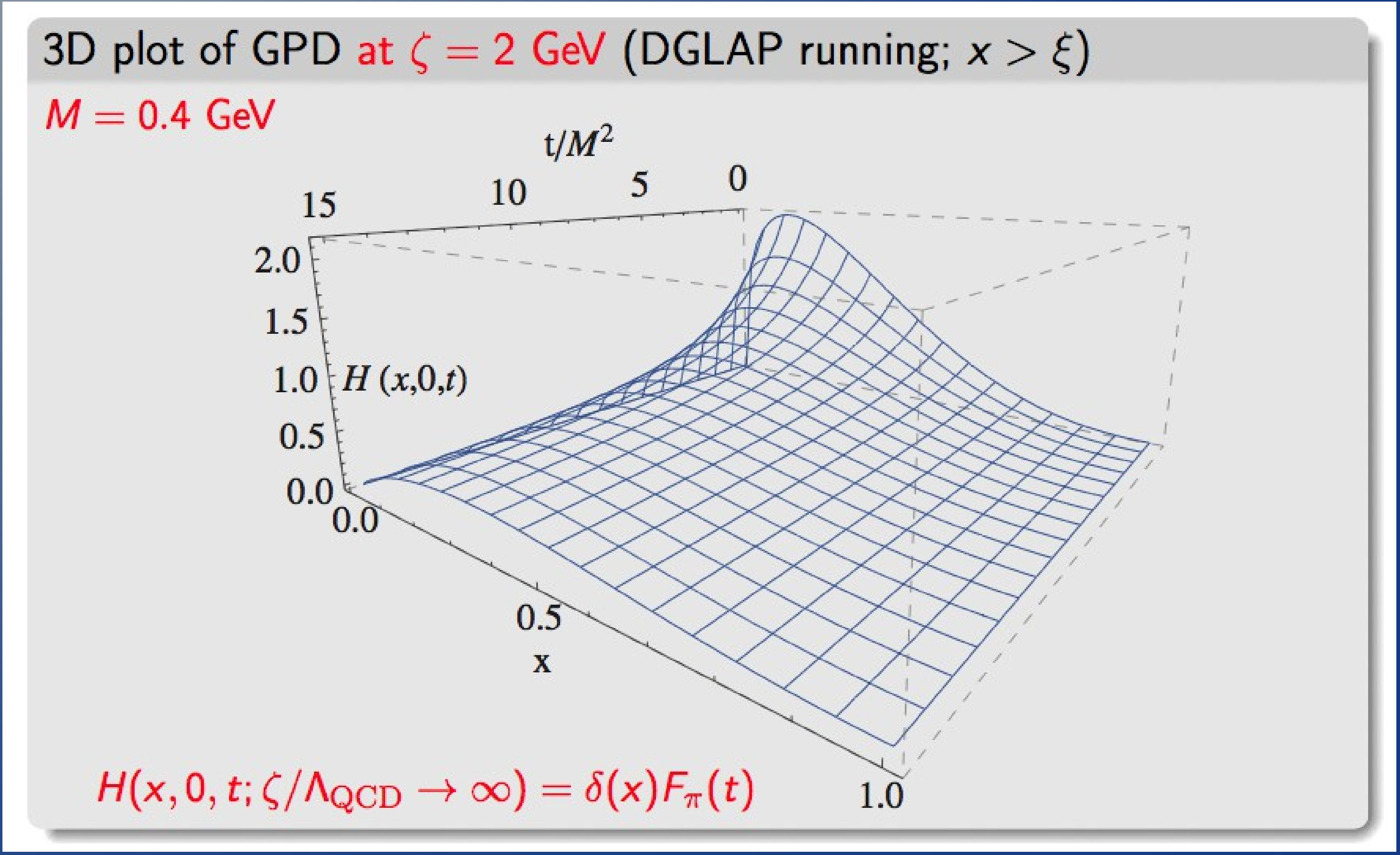

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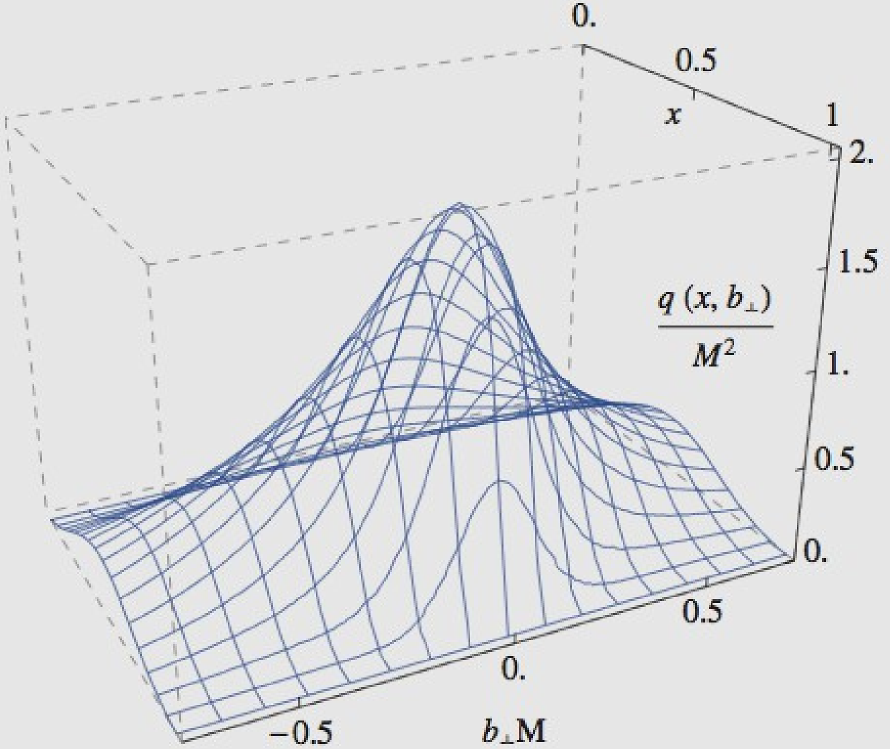
Results for the pion GPD

The off-forward (non-skewed) GPD:

$$q(x, |\vec{b}|) = \int \frac{d|\vec{\Delta}_\perp|}{2\pi} |\vec{\Delta}_\perp| J_0(|\vec{b}_\perp||\vec{\Delta}_\perp|) H(x, 0, -\Delta_\perp^2)$$

Impact parameter space GPD at $\zeta = 0.4$ GeV

$M = 0.4$ GeV



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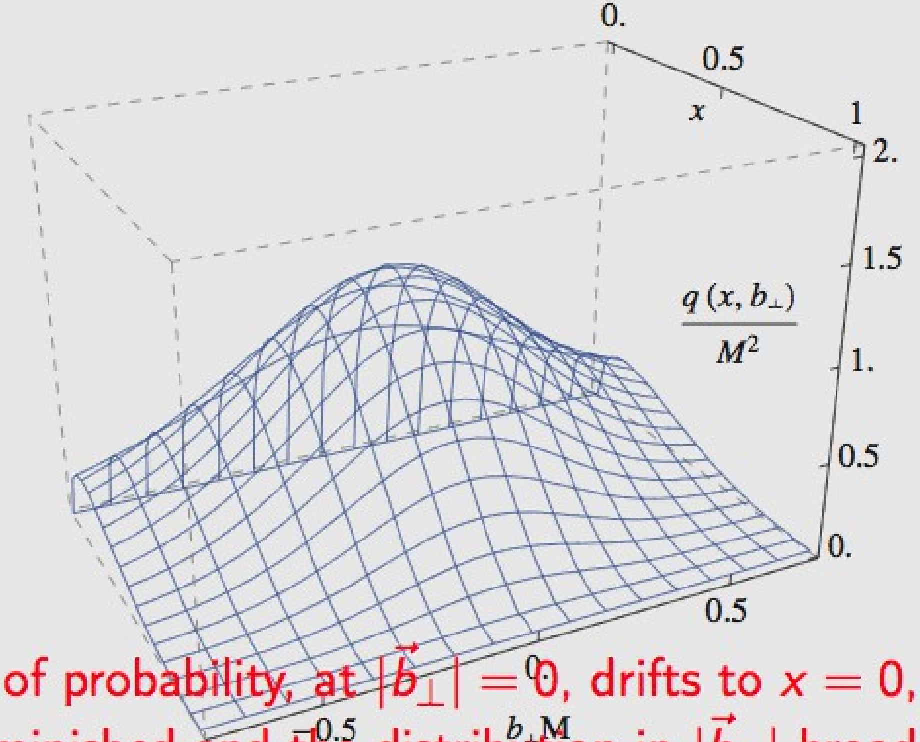
Results for the pion GPD

The off-forward (non-skewed) GPD:

$$q(x,|\vec{b}|) = \int \frac{d|\vec{\Delta}_\perp|}{2\pi}|\vec{\Delta}_\perp| J_0(|\vec{b}_\perp||\vec{\Delta}_\perp|) H(x,0,-\Delta_\perp^2)$$

Impact parameter space GPD at $\zeta = 2$ GeV

$M = 0.4$ GeV



The peak of probability, at $|\vec{b}_\perp| = 0$, drifts to $x = 0$, its height is diminished and the distribution in $|\vec{b}_\perp|$ broadens.

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Results for the pion GPD

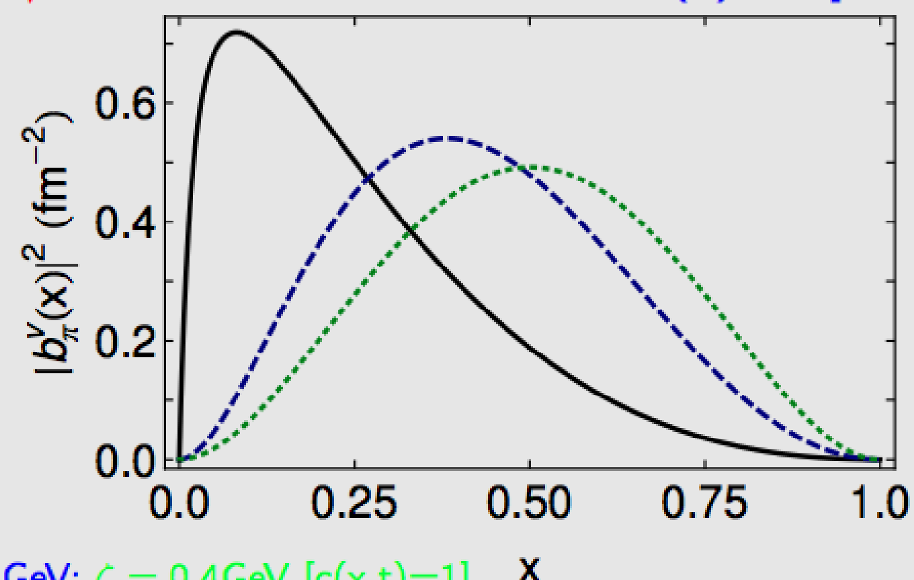
The off-forward (non-skewed) GPD:

$$q(x,|\vec{b}|) \;=\int \frac{d|\vec{\Delta}_\perp|}{2\pi}|\vec{\Delta}_\perp|J_0(|\vec{b}_\perp||\vec{\Delta}_\perp|)H(x,0,-\Delta_\perp^2)$$

$$\langle|\vec{b}_\perp|^2\rangle = \int_{-1}^{1} dx\; \langle|\vec{b}_\perp(x;\zeta)|^2\rangle = \int_{-1}^{1} dx \int_0^\infty d|\vec{b}_\perp||\vec{b}_\perp|^3 \int_0^\infty d\Delta\Delta J_0(\vec{b}_\perp|\Delta)F_\pi(\Delta^2)$$

Impact parameter space GPD

$r_\pi = \sqrt{3/2\langle|\vec{b}_\perp|^2\rangle} = 0.674$ fm $\Longleftrightarrow$ $r_\pi = 0.672(8)$ fm [PRD86(2012)010001]



$\zeta = 2$ GeV; $\zeta = 0.4$ GeV; $\zeta = 0.4$GeV [c(x,t)=1].

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GPDs in the overlap approach

c.f. Cedric's hadronic tourte!!!

The overlap approach

First step: Pion Light Cone Wave Functions

$$|H;P,\lambda\rangle = \sum_{N,\beta}\int [\mathrm{d}x]_N [\mathrm{d}^2\mathbf{k}_\perp]_N \Psi^{\lambda}_{N,\beta}(\Omega)\,|N,\beta,k_1\cdots k_N\rangle$$

$$\Omega = (x_1,\mathbf{k}_{\perp 1},\cdots,x_N,\mathbf{k}_{\perp N})$$

$$[\mathrm{d}x]_N = \prod_{i=1}^{N}\mathrm{d}x_i\,\delta\left(1-\sum_{i=1}^{N}x_i\right),$$

$$[\mathrm{d}^2\mathbf{k}_\perp]_N = \frac{1}{(16\pi^3)^{N-1}}\prod_{i=1}^{N}\mathrm{d}^2\mathbf{k}_{\perp i}\,\delta^2\left(\sum_{i=1}^{N}\mathbf{k}_{\perp i}-\mathbf{P}_\perp\right)$$

N-partons LCWF for the hadron H

$$\sum_{N,\beta}\int [\mathrm{d}x]_N [\mathrm{d}^2\mathbf{k}_\perp]_N |\Psi^{\lambda}_{N,\beta}(\Omega)|^2 = 1.$$

Let's consider the two-body pion LCWF:

$$|\pi^+,P\rangle|^{2\text{-body}}_{\uparrow\downarrow} = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^3}\frac{\mathrm{d}x}{\sqrt{x(1-x)}}\Psi_{\uparrow\downarrow}(k^+,\mathbf{k}_\perp)\Big[b^\dagger_{u\uparrow}(x,\mathbf{k}_\perp)d^\dagger_{d\downarrow}(1-x,-\mathbf{k}_\perp) + b^\dagger_{u\downarrow}(x,\mathbf{k}_\perp)d^\dagger_{d\uparrow}(1-x,-\mathbf{k}_\perp)\Big]|0\rangle,$$

$$\Gamma_\pi(k,P) = S^{-1}(-k_2)\,\chi(k,P)\,S^{-1}(k_1).$$

BS wave function

$$\boxed{2P^+\Psi_{\uparrow\downarrow}(k^+,\mathbf{k}_\perp) = \int\frac{\mathrm{d}k^-}{2\pi}\mathrm{Tr}\left[\gamma^+\gamma_5\chi(k,P)\right]}$$

S. Brodsky and G. Lepage, PRD 22,(1980)

The overlap approach

First step: Pion Light Cone Wave Functions

$$\boxed{2P^+\Psi_{\uparrow\downarrow}(k^+,\mathbf{k}_\perp) = \int \frac{\mathrm{d}k^-}{2\pi}\mathrm{Tr}\left[\gamma^+\gamma_5\chi(k,P)\right]}$$

BS wave function

$$\Gamma_\pi(k,P) = S^{-1}(-k_2)\,\chi(k,P)\,S^{-1}(k_1).$$

- Expressions for vertices and propagators:

$$\begin{aligned}
S(p) &= \left[-i\gamma\cdot p + M\right]\Delta_M(p^2)\\
\Delta_M(s) &= \frac{1}{s+M^2}\\
\Gamma_\pi(k,p) &= i\gamma_5\frac{M}{f_\pi}M^{2\nu}\int_{-1}^{+1}\mathrm{d}z\,\rho_\nu(z)\,\left[\Delta_M(k_{+z}^2)\right]^\nu\\
\rho_\nu(z) &= R_\nu(1-z^2)^\nu
\end{aligned}$$

with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

Keeping so contact with the previous “covariant” approach” based on DSE and BSE.

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.}$$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{[\mathbf{k}_\perp^2+M^2]^{\nu+1}}x^\nu(1-x)^\nu.}$$

GPD in the overlap approach:

$$H(x,\xi,t) = \sqrt{2}\sum_{N,N'}\sum_{\beta,\beta'}\int [\mathrm{d}\hat{x}']_{N'}[\mathrm{d}^2\hat{\mathbf{k}}'_\perp]_{N'}[\mathrm{d}\tilde{x}]_N[\mathrm{d}^2\tilde{\mathbf{k}}_\perp]_N\Psi^*_{N',\beta'}(\hat{\Omega}')\Psi_{N,\beta}(\tilde{\Omega})$$

$$\times\int\frac{\mathrm{d}z^-}{2\pi}e^{iP^+z^-}\langle N',\beta,k'_1\cdots k'_N|\,\phi^{q\dagger}\left(-\frac{z}{2}\right)\phi^q\left(\frac{z}{2}\right)|N,\beta,k_1\cdots k_N\rangle$$

$$=\sum_N\sqrt{1-\xi}^{\,2-N}\sqrt{1+\xi}^{\,2-N}\sum_{\beta=\beta'}\sum_j\delta_{s_jq}$$

In DGLAP kinematics: $\zeta\leqslant x\leqslant 1$

$$\times\int[\mathrm{d}\bar{x}]_N[\mathrm{d}^2\bar{\mathbf{k}}_\perp]_N\delta(x-\bar{x}_j)\Psi^*_{N,\beta'}(\hat{\Omega}')\Psi_{N,\beta}(\tilde{\Omega})$$

$$=\int[\mathrm{d}\bar{x}]_2[\mathrm{d}^2\bar{\mathbf{k}}_\perp]_2\delta(x-\bar{x}_j)\Psi^*_{\uparrow\downarrow}(\hat{\Omega}')\Psi_{\uparrow\downarrow}(\tilde{\Omega})$$

In the pion 2-body case

$+$ Helicity-1 component

$$\frac{x-\zeta}{1-\zeta} \qquad \frac{x+\zeta}{1+\zeta}$$

$$=\frac{\Gamma(2\nu+2)}{\Gamma(\nu+2)^2}\int\mathrm{d}u\mathrm{d}v\,u^\nu v^\nu\delta(1-u-v)\,\frac{\left(2M^{2\nu}4^\nu R_\nu\right)^2\hat{x}^\nu(1-\hat{x})^\nu\tilde{x}^\nu(1-\tilde{x})^\nu}{\left(t\,uv\frac{(1-x)^2}{1-\xi^2}+M^2\right)^{2\nu+1}}.$$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\xi,t) = \frac{\Gamma(2\nu+2)}{\Gamma(\nu+2)^2}\int \mathrm{d}u\mathrm{d}v\, u^\nu v^\nu \delta(1-u-v)\frac{\left(2M^{2\nu}4^\nu R_\nu\right)^2 \hat{x}^\nu(1-\hat{x})^\nu \tilde{x}^\nu(1-\tilde{x})^\nu}{\left(t\,uv\frac{(1-x)^2}{1-\xi^2}+M^2\right)^{2\nu+1}}, \qquad \xi \leqslant x \leqslant 1$$

$$\hat{x} = \frac{x-\xi}{1-\xi} \qquad \tilde{x} = \frac{x+\xi}{1+\xi}$$

$$= \boxed{30\frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)}$$

$$\boxed{z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}}$$

Encoding the correlations of kinematical variables

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.}$$

GPD in the overlap approach:

$$H(x,\xi,t) = 30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+\cancel{z})^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-\cancel{2z}}{1+\cancel{z}}-\frac{\text{arct}\ldots}{\sqrt{1+z}}\right)$$

0

1

$0 \leqslant x \leqslant 1$

Forward limit

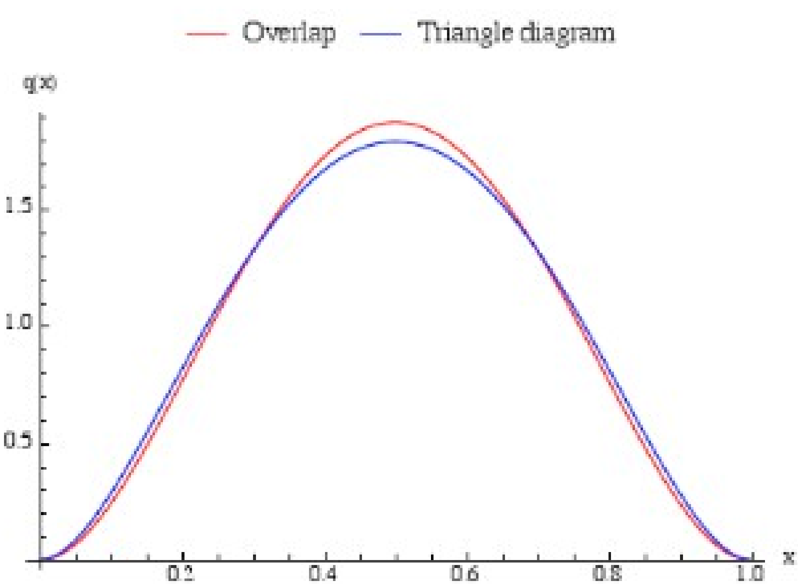
$$\boxed{z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}}$$

Encoding the correlations of kinematical variables

PDF:

$$H(x,0,0) = q(x) = 30x^2(1-x)^2$$

Compares numerically very well with the results obtained from the Triangle diagram!!!



The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x, \mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \frac{M^{2\nu+1} 4^\nu R_\nu}{\left[\mathbf{k}_\perp^2 + M^2\right]^{\nu+1}} x^\nu (1-x)^\nu .$$

GPD in the overlap approach:

$$H(x, \underset{\downarrow\,0}{\xi}, t) = \boxed{30 \frac{(1-x)^2 (x^2 - \cancel{\xi^2})}{(1-\cancel{\xi^2})^2} \frac{1}{(1+z)^2} \left(\frac{3}{4} + \frac{1}{4} \frac{1-2z}{1+z} \frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}} \right)}$$

$$0 \leqslant x \leqslant 1$$

$$\boxed{z = \frac{t}{4M^2} \frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables

Within this simple approach, we got an analytical expression for the pion form factor:

$$F_\pi(t) = \int_0^1 dx\, H(x, 0, t) = \boxed{45 \left(\frac{4M^2}{t}\right)^2 \left(1 - \sqrt{1 + \frac{t}{4M^2}} \frac{\operatorname{arctanh}\sqrt{\frac{\frac{t}{4M^2}}{1+\frac{t}{4M^2}}}}{\sqrt{\frac{t}{4M^2}}} + \frac{1}{3} \operatorname{arctanh}^2 \sqrt{\frac{\frac{t}{4M^2}}{1+\frac{t}{4M^2}}} \right)}$$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.}$$

GPD in the overlap approach:

$$H(x,\underset{\downarrow 0}{\xi},t) = \boxed{30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)}$$

$$0 \leqslant x \leqslant 1$$

$$\boxed{z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables

Within this simple approach, we got an analytical expression for the pion form factor:

$$F_\pi(t) = \int_0^1 dx\, H(x,0,t) = \boxed{1-\frac{16}{21}\frac{t}{4M^2}+\frac{16}{28}\left(\frac{t}{4M^2}\right)^2+\ldots}$$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu .$$

GPD in the overlap approach:

$$H(x,\xi,t) = 30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)$$

$\xi \to 0$

$$0 \leqslant x \leqslant 1$$

$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}$$

Encoding the correlations of kinematical variables

$$M = \sqrt{\frac{24}{21\langle r_\pi^2\rangle}} = 0.32\,GeV$$

$$r_\pi = 0.672(8)\,fm$$

[PRD86(2001)10001]

$$\langle r_\pi^2\rangle = -6\frac{d}{dt}F_\pi(t) = \frac{24}{21M^2}$$

Within this simple approach, we got an analytical expression for the pion form factor:

$$F_\pi(t) = \int_0^1 dx\, H(x,0,t) = 1 - \frac{16}{21}\frac{t}{4M^2} + \frac{16}{28}\left(\frac{t}{4M^2}\right)^2 + \ldots$$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.$$

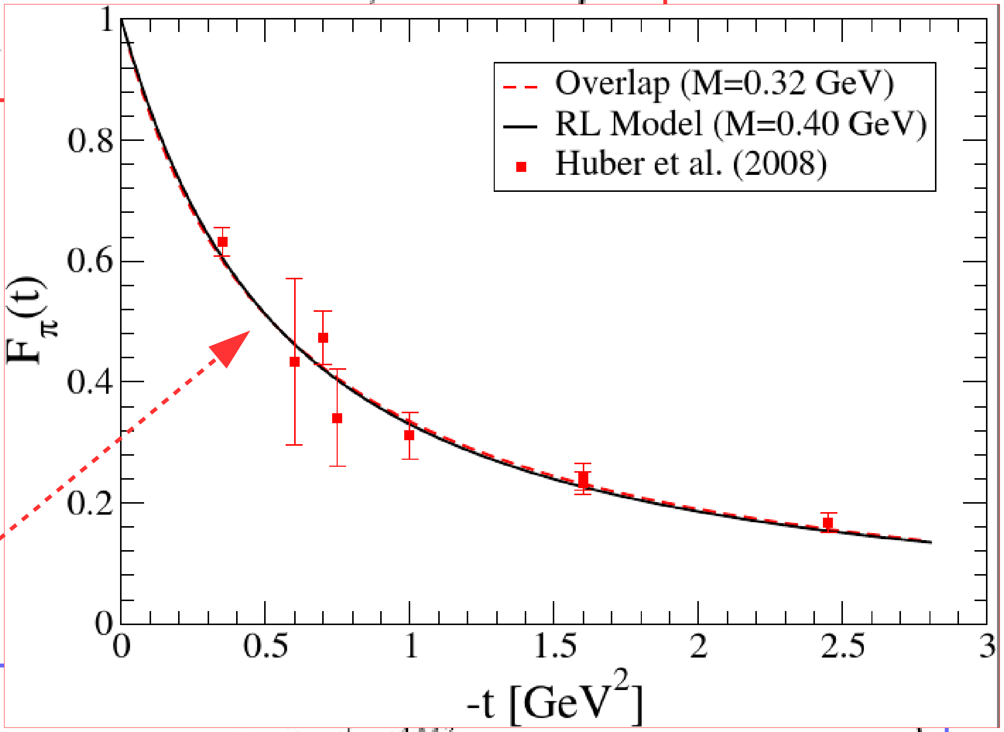
GPD in the overlap approach:

$$H(x,\underset{0}{\xi},t) = 30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{z}}\right.$$

$$0 \leqslant x \leqslant 1$$

$$M=\sqrt{\frac{24}{21\langle r_\pi^2\rangle}}=0.32\,GeV$$

Very good agreement with the pion form factor experimental data when no fit is needed!!!



$$F_\pi(t)=\int_0^1 dx\,H(x,0,t) = 45\left(\frac{4M^2}{t}\right)^2\left(1-\sqrt{1+\frac{t}{4M^2}}\,\frac{\operatorname{arctanh}\sqrt{\frac{\frac{t}{4M^2}}{1+\frac{t}{4M^2}}}}{\sqrt{\frac{t}{4M^2}}}+\frac{1}{3}\operatorname{arctanh}^2\sqrt{\frac{\frac{t}{4M^2}}{1+\frac{t}{4M^2}}}\right)$$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.}$$

GPD in the overlap approach:

$$H(x,\underset{\downarrow\ 0}{\xi},t) = \boxed{30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)} \qquad 0\leqslant x\leqslant 1$$

$$\boxed{z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables

RL-inspired model:

$$H(x,0,t) = H(x,0,0)\mathcal{N}(t)C_\pi(x,t)F_\pi(t),$$

$$1 = \mathcal{N}(t)\int_{-1}^{1}dx\, H(x,0,0)\,C_\pi(x,t)$$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{[\mathbf{k}_\perp^2+M^2]^{\nu+1}}x^\nu(1-x)^\nu.}$$

GPD in the overlap approach:

$$H(x,\underset{\downarrow\,0}{\xi},t) = \boxed{30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\text{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)} \qquad 0 \leqslant x \leqslant 1$$

$$\boxed{z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables

RL-inspired model:

$$H(x,0,t) = H(x,0,0)\cancel{\mathcal{N}(t)}C_\pi(x,t)\cancel{F_\pi(t)},$$

$$\boxed{1 = \mathcal{N}(t)\int_{-1}^{1}dx\, H(x,0,0)\, C_\pi(x,t)}$$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.}$$

GPD in the overlap approach:

$$H(x,\underset{\downarrow\,0}{\xi},t) = \boxed{30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)}$$

$$0 \leqslant x \leqslant 1$$

$$\boxed{z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables

RL-inspired model:

$$H(x,0,t) = H(x,0,0)\cancel{\mathcal{N}(t)}C_\pi(x,t)\cancel{F_\pi(t)},$$

$$\boxed{1 = \mathcal{N}(t)\int_{-1}^{1} dx\, H(x,0,0)\, C_\pi(x,t)}$$

Simplified analytical model:

$$C(x,t) = \frac{1}{\left(1+\frac{t}{4M^2}(1-x)^2\right)^2}$$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{[\mathbf{k}_\perp^2+M^2]^{\nu+1}}x^\nu(1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\underset{\downarrow\, 0}{\xi},t) = 30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(1-\frac{2}{3}z+\frac{7}{15}z^2+\dots\right) \qquad 0\leqslant x\leqslant 1$$

$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}$$

Encoding the correlations of kinematical variables

RL-inspired model:

$$H(x,0,t) = H(x,0,0)\cancel{\mathcal{N}(t)}C_\pi(x,t)\cancel{F_\pi(t)},$$

$$\boxed{1 = \mathcal{N}(t)\int_{-1}^{1}dx\, H(x,0,0)\, C_\pi(x,t)}$$

The common ingredient to both approaches is the BSE amplitude, which fixes the way the kinematical variables correlate to each other.

Simplified analytical model:

$$C(x,t) = \frac{1}{(1+z)^2}$$

Both approaches are near consistent when the previous spurious correlations are properly removed!!!

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.}$$

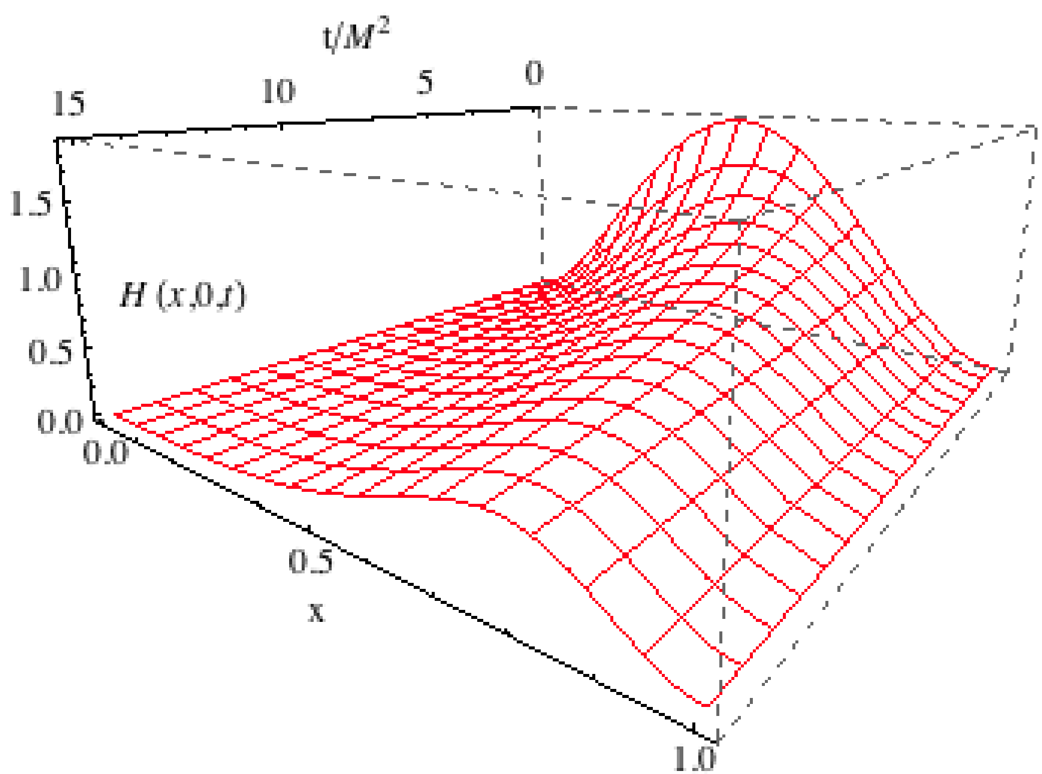
GPD in the overlap approach:

$$H(x,\underset{\substack{\downarrow\\ 0}}{\xi},t) = \boxed{30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)}$$

$0 \leqslant x \leqslant 1$

$$\boxed{z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables



3-d GPD plot

$\zeta = 0.51\ GeV$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.}$$

GPD in the overlap approach:

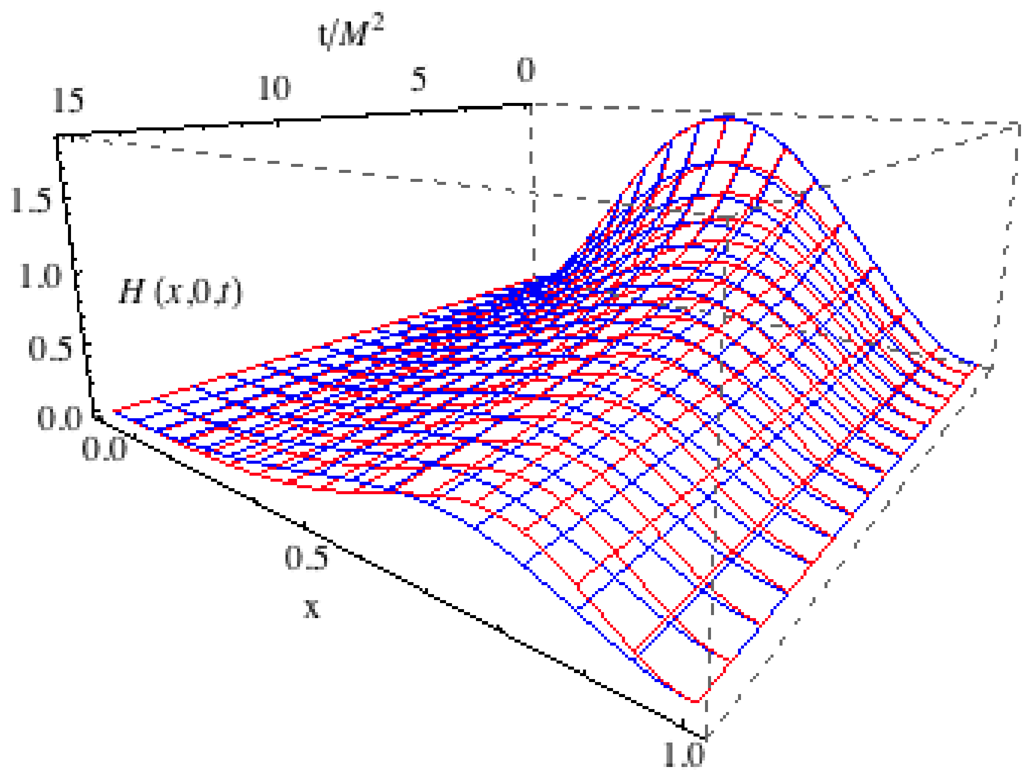
$$H(x,\xi,t) = \boxed{30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\,\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\,\frac{1-2z}{1+z}\,\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)}$$

0

$$0 \leqslant x \leqslant 1$$

$$\boxed{z = \frac{t}{4M^2}\,\frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables



3-d GPD plot

$\zeta = 0.51\ GeV$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

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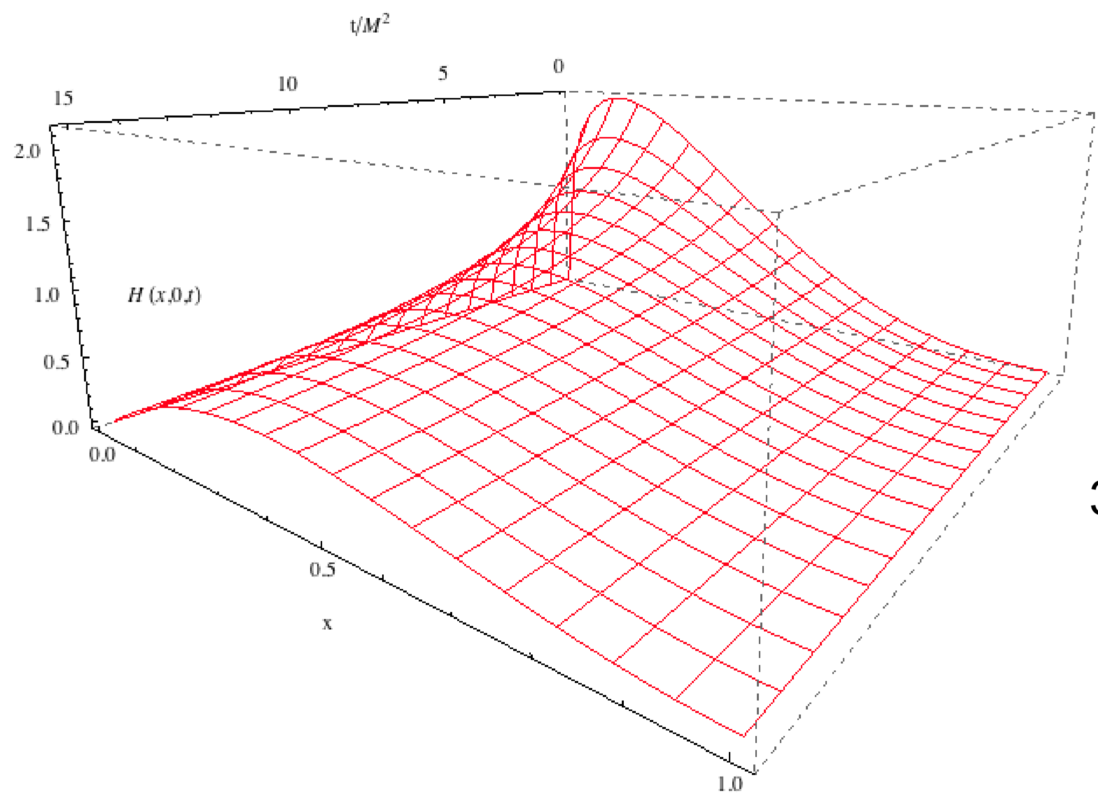
GPD in the overlap approach:

$$H(x,\underset{\downarrow\, 0}{\xi},t) = \boxed{30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)}$$

$0 \leqslant x \leqslant 1$

$$\boxed{z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables



3-d GPD plot

$\zeta = 2\ GeV$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

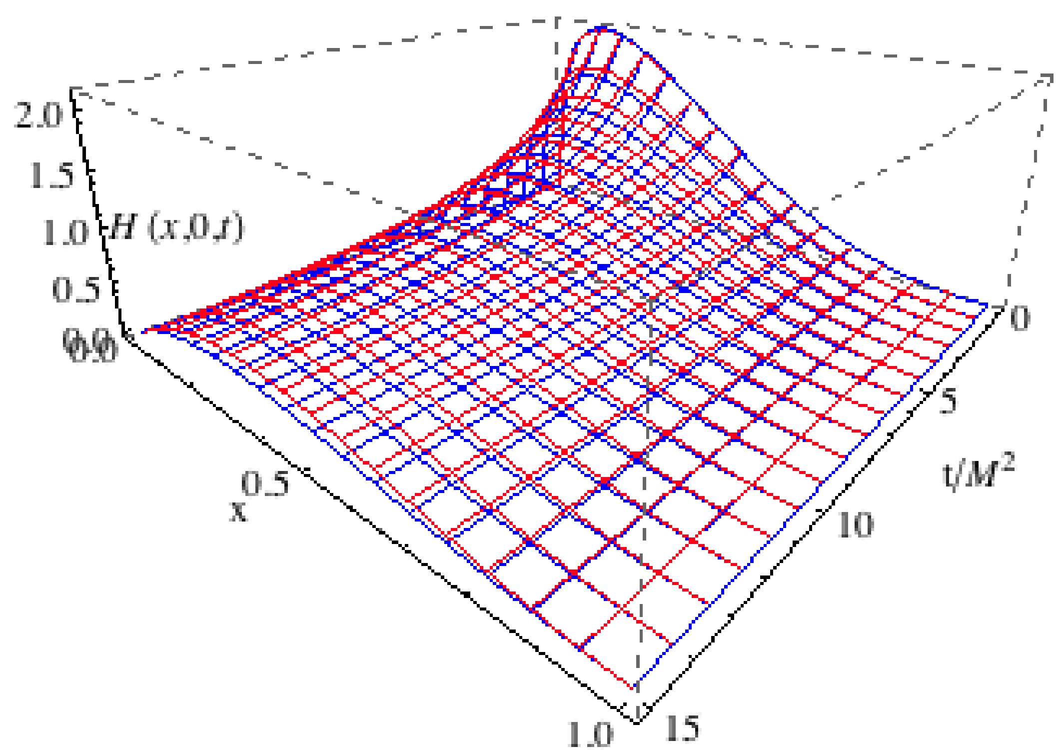
Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{[\mathbf{k}_\perp^2+M^2]^{\nu+1}}x^\nu(1-x)^\nu .}$$

GPD in the overlap approach:

$$H(x,\underset{\downarrow\,0}{\xi},t) = \boxed{30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)}$$

$0 \leqslant x \leqslant 1$



$$\boxed{z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables

3-d GPD plot

$\zeta = 2\ GeV$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.}$$

GPD in the overlap approach:

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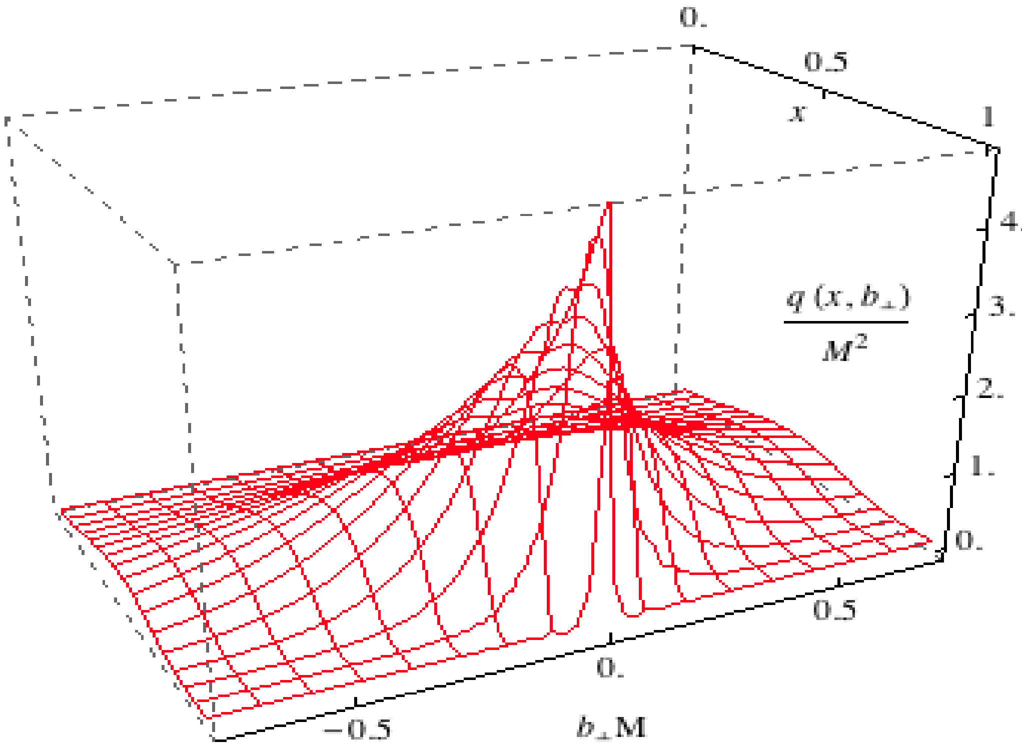
0

$$\boxed{z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables



3-d IPD GPD plot

$\zeta = 0.51\ GeV$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.$$

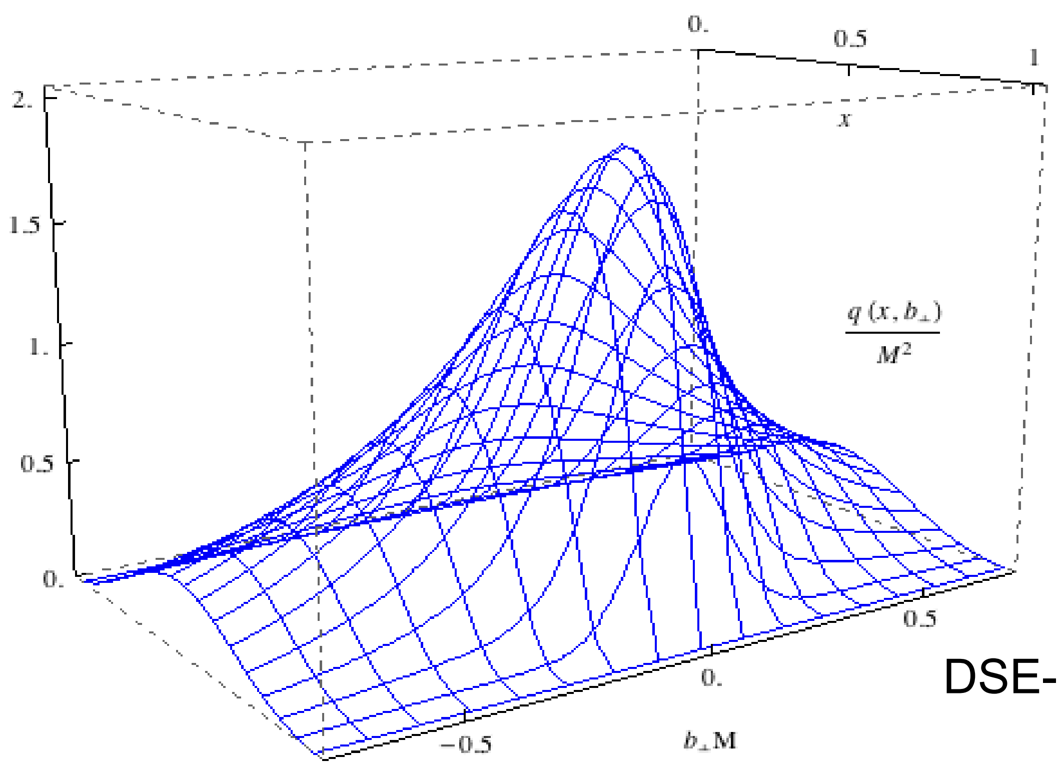
GPD in the overlap approach:

$$H(x,\underset{\downarrow\, 0}{\xi},t) = 30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)$$

$0 \leqslant x \leqslant 1$

$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\cancel{\xi^2}}$$

Encoding the correlations of kinematical variables



3-d IPD GPD plot

$\zeta = 0.51\ GeV$

DSE-BSE approach

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

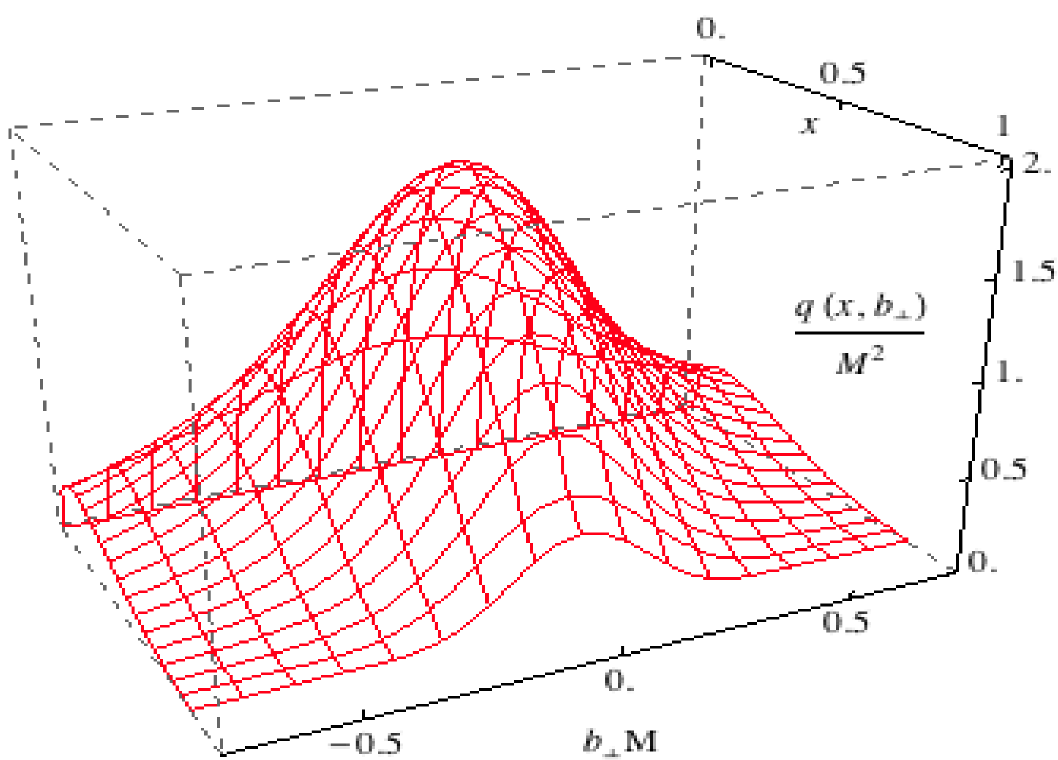
$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\underset{\downarrow\,0}{\xi},t) = \boxed{30\frac{(1-x)^2(x^2-\cancel{\xi^2})}{(1-\cancel{\xi^2})^2}\,\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\,\frac{1-2z}{1+z}\,\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)} \qquad 0\leqslant x\leqslant 1$$

$$\boxed{z = \frac{t}{4M^2}\,\frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables



3-d IPD GPD plot

$\zeta = 2\ GeV$

The overlap approach

Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\boxed{\Psi_{\uparrow\downarrow}(x, \mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \frac{M^{2\nu+1} 4^\nu R_\nu}{\left[\mathbf{k}_\perp^2 + M^2\right]^{\nu+1}} x^\nu (1-x)^\nu.}$$

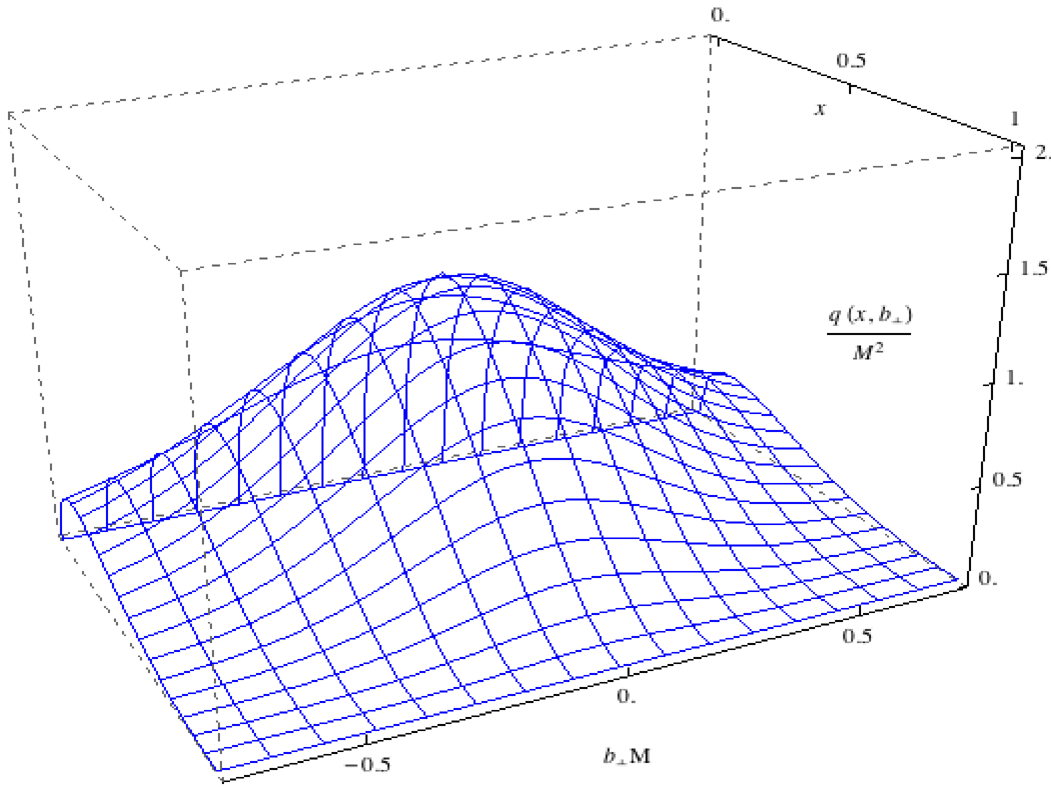
GPD in the overlap approach:

$$H(x, \underset{0}{\xi}, t) = \boxed{30 \frac{(1-x)^2 (x^2 - \cancel{\xi^2})}{(1-\cancel{\xi^2})^2} \frac{1}{(1+z)^2} \left(\frac{3}{4} + \frac{1}{4} \frac{1-2z}{1+z} \frac{\operatorname{arctanh} \sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}} \right)}$$

$0 \leqslant x \leqslant 1$

$$\boxed{z = \frac{t}{4M^2} \frac{(1-x)^2}{1-\cancel{\xi^2}}}$$

Encoding the correlations of kinematical variables



3-d IPD GPD plot

$\zeta = 2\ GeV$

The overlap approach

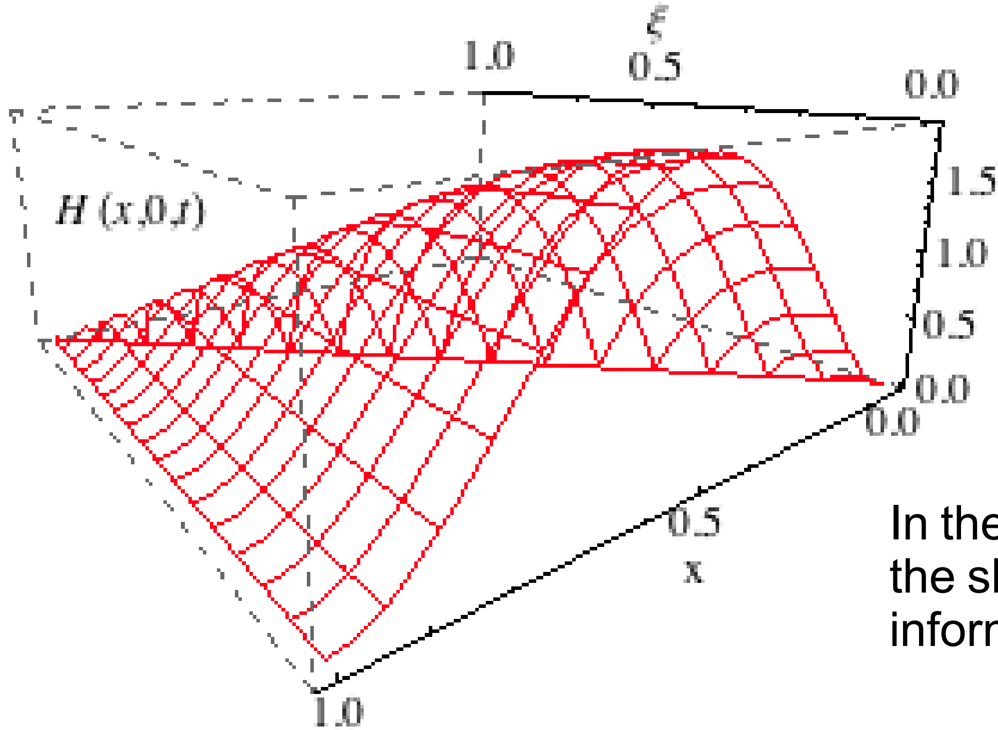
Second step: DGLAP GPD from Light Front Wave Functions

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x, \mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \frac{M^{2\nu+1} 4^\nu R_\nu}{\left[\mathbf{k}_\perp^2 + M^2\right]^{\nu+1}} x^\nu (1-x)^\nu.$$

GPD in the overlap approach:

$$H(x, \xi, t) = \boxed{30 \frac{(1-x)^2 (x^2-\xi^2)}{(1-\xi^2)^2} \frac{1}{(1+z)^2} \left(\frac{3}{4} + \frac{1}{4} \frac{1-2z}{1+z} \frac{\operatorname{arctanh} \sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}} \right)} \qquad 0 \leqslant x \leqslant 1$$



$$\boxed{z = \frac{t}{4M^2} \frac{(1-x)^2}{1-\xi^2}}$$

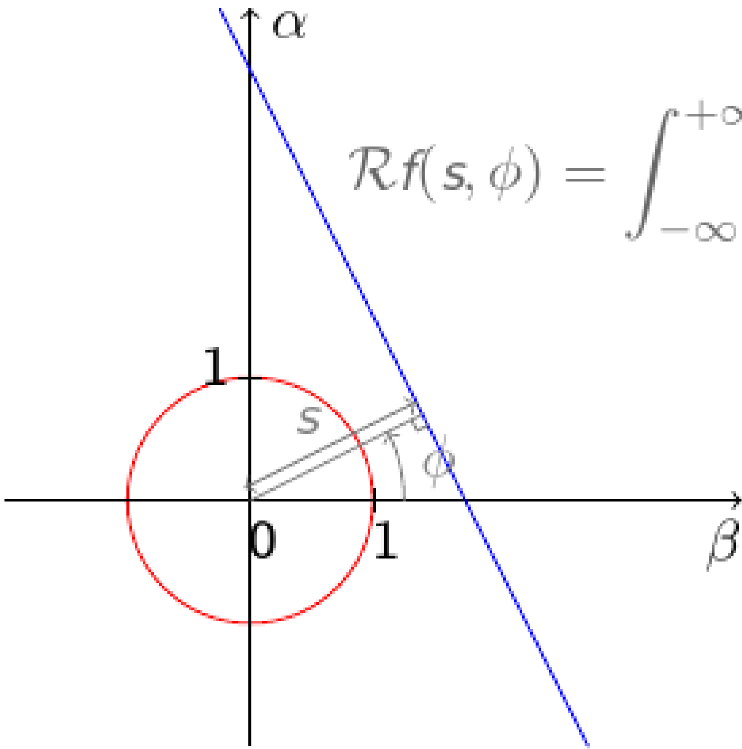
Encoding the correlations of kinematical variables

In the full DGLAP region (parity in the skewness provides us with the information in $-1 < x < -\xi < 0$)

The overlap approach

Third step: beyond DGLAP via Radon transform

Definitions and properies of the Radon transform:



For $s > 0$ and $\phi \in [0, 2\pi]$:

$$\mathcal{R}f(s,\phi) = \int_{-\infty}^{+\infty} \mathrm{d}\beta \mathrm{d}\alpha\, f(\beta,\alpha)\delta(s - \beta\cos\phi - \alpha\sin\phi)$$

and:

$$\mathcal{R}f(-s,\phi) = \mathcal{R}f(s,\phi \pm \pi)$$

Relation to GPDs:

$$x = \frac{s}{\cos\phi} \text{ and } \xi = \tan\phi$$

Relation between GPD and DD in Belistky *et al.* gauge

$$\frac{\sqrt{1+\xi^2}}{x} H(x,\xi) = \mathcal{R}f_{\mathrm{BMKS}}(s,\phi)$$

The overlap approach

Radon transform: polinomiality and Ludwig-Helgason condition

- The Mellin moments of a Radon transform are **homogeneous polynomials** in $\omega = (\sin\phi, \cos\phi)$.
- The converse is also true:

Theorem (Hertle, 1983)

Let $g(s,\omega)$ an even compactly-supported distribution. Then g is itself the Radon transform of a compactly-supported distribution if and only if the ***Ludwig-Helgason consistency condition*** *hold:*

(i) *g is C^∞ in ω,*

(ii) *$\int \mathrm{d}s\, s^m g(s,\omega)$ is a homogeneous polynomial of degree m for all integer $m \geq 0$.*

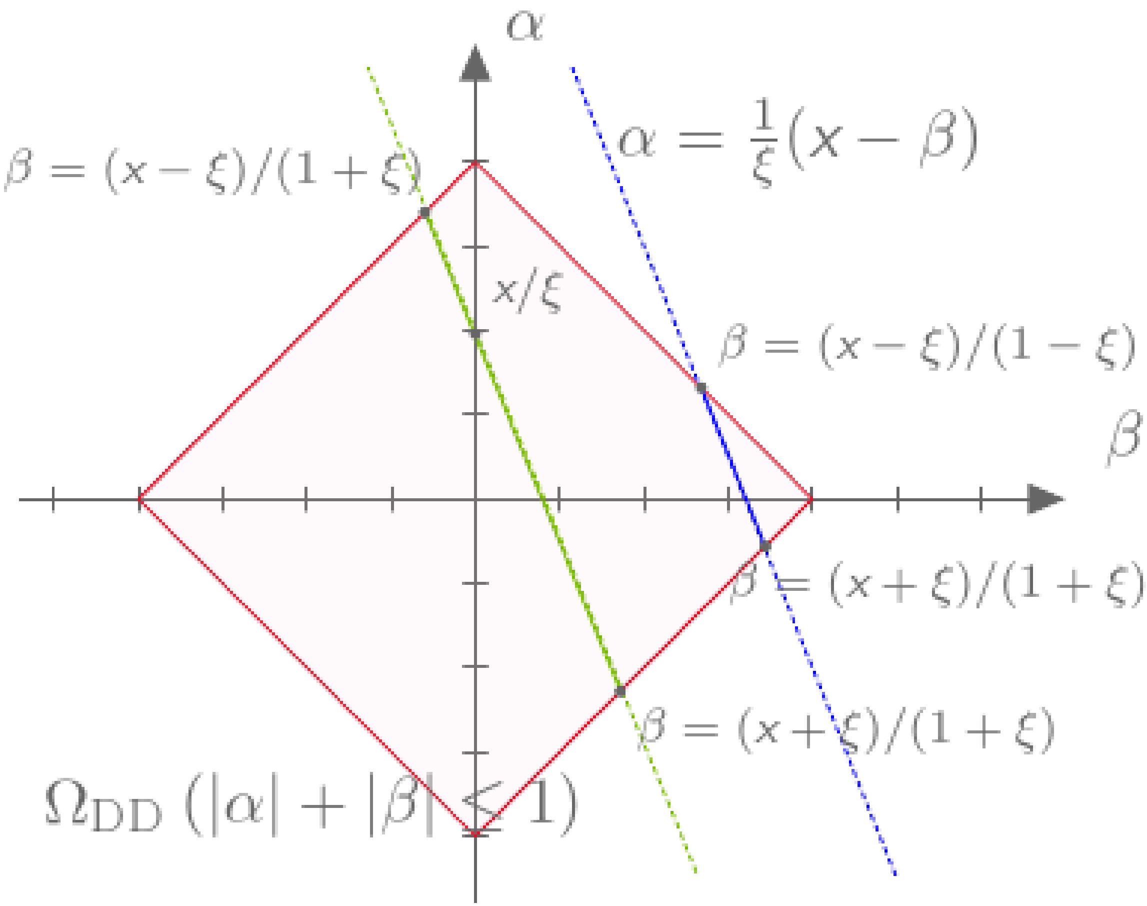
- Double Distributions and the Radon transform are the **natural solution** of the polynomiality condition.

The overlap approach

Radon transform: from GPD DGLAP to the the whole GPD domain

DGLAP and ERBL regions

$$(x,\xi) \in \text{ DGLAP} \quad \Leftrightarrow \quad |s| \geq |\sin\phi| \,,$$
$$(x,\xi) \in \text{ ERBL} \quad \Leftrightarrow \quad |s| \leq |\sin\phi| \,.$$



- Each point (β, α) with $\beta \neq 0$ contributes to **both** DGLAP and ERBL regions.
- Expressed in **support theorem**.

The overlap approach

Radon transform: from GPD DGLAP to the the whole GPD domain

The GPD is the Radon transform of the following DD in the Pobylitsa gauge:

$$
\begin{aligned}
H(x,\xi) &= (1-x)\int_{\Omega^>} f(\beta,\alpha)\,\delta(x-\beta-\alpha\xi)\,\mathrm{d}\beta\mathrm{d}\alpha \\
&+ (1+x)\int_{\Omega^<} f(\beta,\alpha)\,\delta(x-\beta-\alpha\xi)\,\mathrm{d}\beta\mathrm{d}\alpha\,.
\end{aligned}
$$

$$
\begin{aligned}
\Omega^> &= \{|\beta|+|\alpha|<1,\ \beta>0\} \\
\Omega^< &= \{|\beta|+|\alpha|<1,\ \beta<0\}
\end{aligned}
$$

The overlap approach

Radon transform: from GPD DGLAP to the whole GPD domain

The GPD is the Radon transform of the following DD in the Pobylitsa gauge:

$$H(x,\xi) = (1-x)\int_{\Omega^>} f(\beta,\alpha)\,\delta(x-\beta-\alpha\xi)\,\mathrm{d}\beta\mathrm{d}\alpha + (1+x)\int_{\Omega^<} f(\beta,\alpha)\,\delta(x-\beta-\alpha\xi)\,\mathrm{d}\beta\mathrm{d}\alpha\,.$$

$$\Omega^> = \{|\beta|+|\alpha|<1\,,\ \beta>0\}$$
$$\Omega^< = \{|\beta|+|\alpha|<1\,,\ \beta<0\}$$

DGLAP GPD

$$30\frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}\,\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\,\frac{1-2z}{1+z}\,\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)$$

Inverse Radon transform

$$f(\beta,\alpha,t) = \frac{120\left(4\left(3\alpha^2-3\beta^2+2\beta-1\right)+\frac{t\left(\alpha^4-2\alpha^2\beta^2+\beta^4-4\beta^2+4\beta-1\right)}{M^2}\right)}{\left(\frac{t\left(\alpha^2-(\beta-1)^2\right)}{M^2}-4\right)^3}$$

For this particular simple algebraic model, the Pobylitsa DD can be analytically obtained!!

The overlap approach

Radon transform: from GPD DGLAP to the whole GPD domain

The GPD is the Radon transform of the following DD in the Pobylitsa gauge:

$$
\begin{aligned}
H(x,\xi) &= (1-x)\int_{\Omega^>} f(\beta,\alpha)\,\delta(x-\beta-\alpha\xi)\,\mathrm{d}\beta\mathrm{d}\alpha \\
&+ (1+x)\int_{\Omega^<} f(\beta,\alpha)\,\delta(x-\beta-\alpha\xi)\,\mathrm{d}\beta\mathrm{d}\alpha\,.
\end{aligned}
$$

$$
\begin{aligned}
\Omega^> &= \{|\beta|+|\alpha|<1\,,\ \beta>0\} \\
\Omega^< &= \{|\beta|+|\alpha|<1\,,\ \beta<0\}
\end{aligned}
$$

Radon transform

$$
f(\beta,\alpha,t) = \frac{120\left(4\left(3\alpha^2-3\beta^2+2\beta-1\right)+\frac{t\left(\alpha^4-2\alpha^2\beta^2+\beta^4-4\beta^2+4\beta-1\right)}{M^2}\right)}{\left(\frac{t\left(\alpha^2-(\beta-1)^2\right)}{M^2}-4\right)^3}
$$

$$
\begin{aligned}
H(x,\xi,t)\big|_{|x|<\xi} &= \frac{60(1-x)\left(\xi^2-x^2\right)}{\sqrt{t}\left(-4\xi^2+t(x-1)^2+4\right)^{5/2}\left(4\xi^2+t\left(\xi^2-x^2\right)\right)^2}\,, \qquad (19 \\
&\times \Bigg(\Big(t^2(x-1)\left(\xi^3(3\xi-2)+3x^4-4\xi x^3-6(\xi-1)\xi x^2+2\xi\left(\xi^2-1\right)x\right) \\
&+ 4\xi t\left(\xi^2\left(\xi^2-6\xi+4\right)+(5-6\xi)x^3+\left(3\xi^2+6\xi-8\right)x^2+\left(6\xi^3-9\xi^2+4\right)x\right) \\
&+ 16\left(\left(\xi^2-3\xi+2\right)\xi^3+\left(3\xi^3-5\xi^2+2\right)\xi x\right)\Big)\sqrt{t\left(-4\xi^2+t(x-1)^2+4\right)} \\
&+ 2\left(2\left(\xi^2-1\right)+t(x-1)^2\right)\left(4\xi^2+t\left(\xi^2-x^2\right)\right)^2\tanh^{-1}\left(\frac{\left(x-\xi^2\right)\sqrt{\frac{t}{-4\xi^2+t(x-1)^2+4}}}{\xi}\right) \\
&- 2\left(2\left(\xi^2-1\right)+t(x-1)^2\right)\left(4\xi^2+t\left(\xi^2-x^2\right)\right)^2\tanh^{-1}\left((x-1)\sqrt{\frac{t}{-4\xi^2+t(x-1)^2+4}}\right)\Bigg)
\end{aligned}
$$

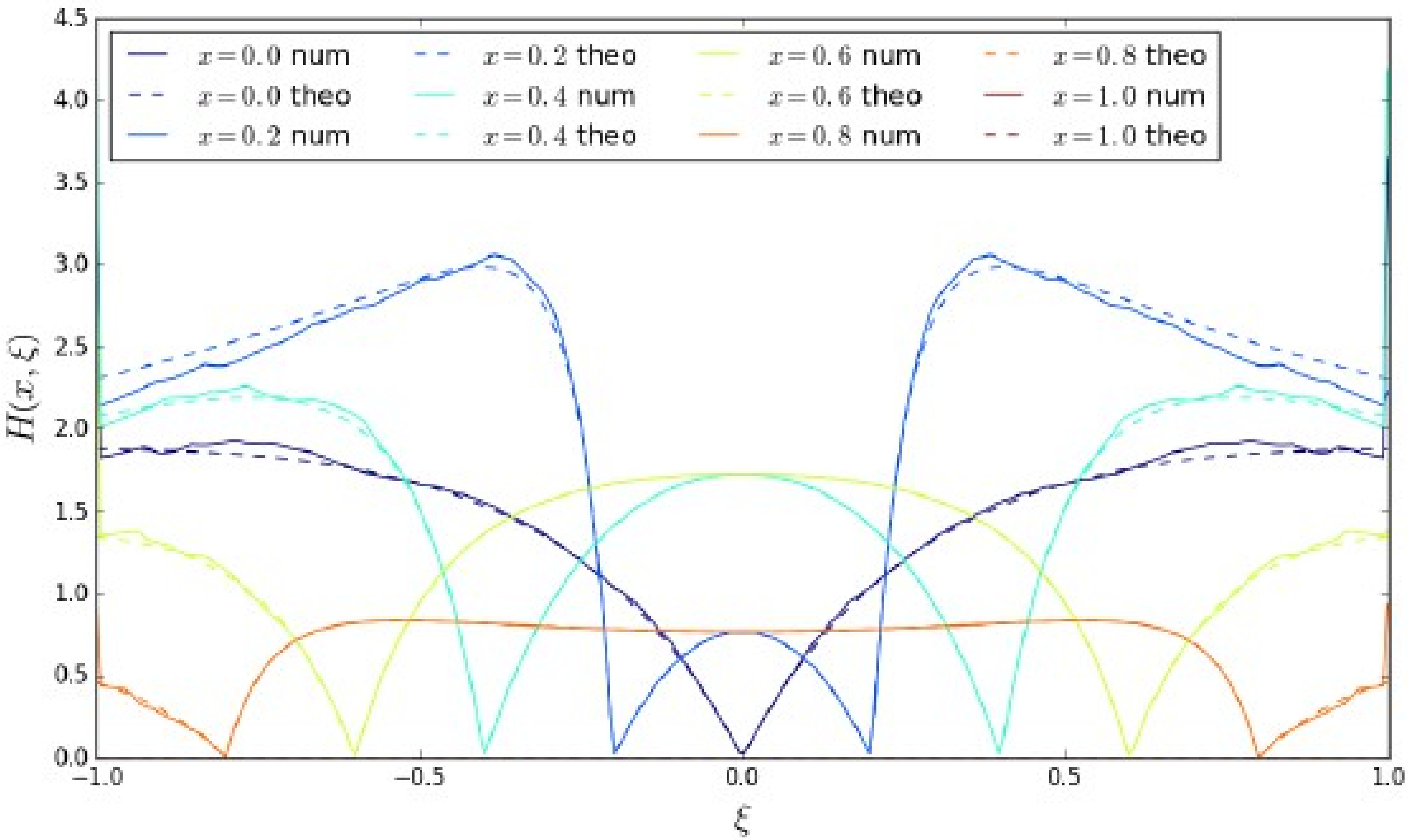
GPD in the ERBL region

The Radon transform makes possible a covariant extension from DGLAP (overlap) GPD to the ERBL region!!!

The overlap approach

Radon transform: from GPD DGLAP to the whole GPD domain

Some results for t=0 of the the GPD both reconstructed numerically and obtained analytically:



The overlap approach

Radon transform: from GPD DGLAP to the whole GPD domain

Other (more phenomenologically inspired) example of LCWFs:

$$\boxed{\varphi\left(x, \mathbf{k}_\perp^2\right) = \frac{2\sqrt{6}\pi}{M\left(1 + \frac{\mathbf{k}_\perp^2}{4M^2(1-x)x}\right)}}$$

DGLAP overlap GPD

$$\boxed{H\left(x, \xi, t\right) = \frac{48\, M^2\, (1-x)(x^2-\xi^2)\tanh^{-1}\left(\frac{\sqrt{64M^4(\xi^3+\xi-2\xi x)^2+16M^2(\xi^2-1)t(x-1)(\xi^2(x-2)+x)+(\xi^2-1)^2t^2(x-1)^2}}{8M^2(\xi^2(x-2)+x)+(\xi^2-1)t(x-1)}\right)}{\sqrt{64M^4\left(\xi^3+\xi-2\xi x\right)^2+16M^2\left(\xi^2-1\right)t(x-1)\left(-2\xi^2+\xi^2 x+x\right)+\left(\xi^2-1\right)^2 t^2(x-1)^2}}}$$

$$\boxed{\psi\left(x, \mathbf{k}_\perp^2\right) = \frac{2\sqrt{30}\pi\sqrt{(1-x)x}}{M\left(1 + \frac{\mathbf{k}_\perp^2}{4M^2(1-x)x}\right)}}$$

DGLAP overlap GPD

$$\boxed{\begin{aligned} H\left(x, \xi, t\right) &= \frac{240\, M^2 (x-1)^2\, (x^2-\xi^2)^{3/2}}{(1-\xi^2)} \\ &\times \frac{\tanh^{-1}\left(\frac{\sqrt{64M^4(\xi^3+\xi-2\xi x)^2+16M^2(\xi^2-1)t(x-1)(\xi^2(x-2)+x)+(\xi^2-1)^2t^2(x-1)^2}}{8M^2(\xi^2(x-2)+x)+(\xi^2-1)t(x-1)}\right)}{\sqrt{64M^4\left(\xi^3+\xi-2\xi x\right)^2+16M^2\left(\xi^2-1\right)t(x-1)\left(-2\xi^2+\xi^2 x+x\right)+\left(\xi^2-1\right)^2 t^2(x-1)^2}} \end{aligned}}$$

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DGLAP overlap GPD

$$H^u_{\pi^+}(x,{\color{red}0},t)\big|_{\,\mathrm{DGLAP}} \;=\; q(x)\,\frac{\mathrm{arctanh}\,\frac{\sqrt{z^2-1}}{z}}{\sqrt{z^2-1}}$$

$$\psi\left(x,\mathbf{k}_\perp^2\right) = \frac{2\sqrt{30}\pi\sqrt{(1-x)x}}{M\left(1+\frac{\mathbf{k}_\perp^2}{4M^2(1-x)x}\right)}$$

DGLAP overlap GPD

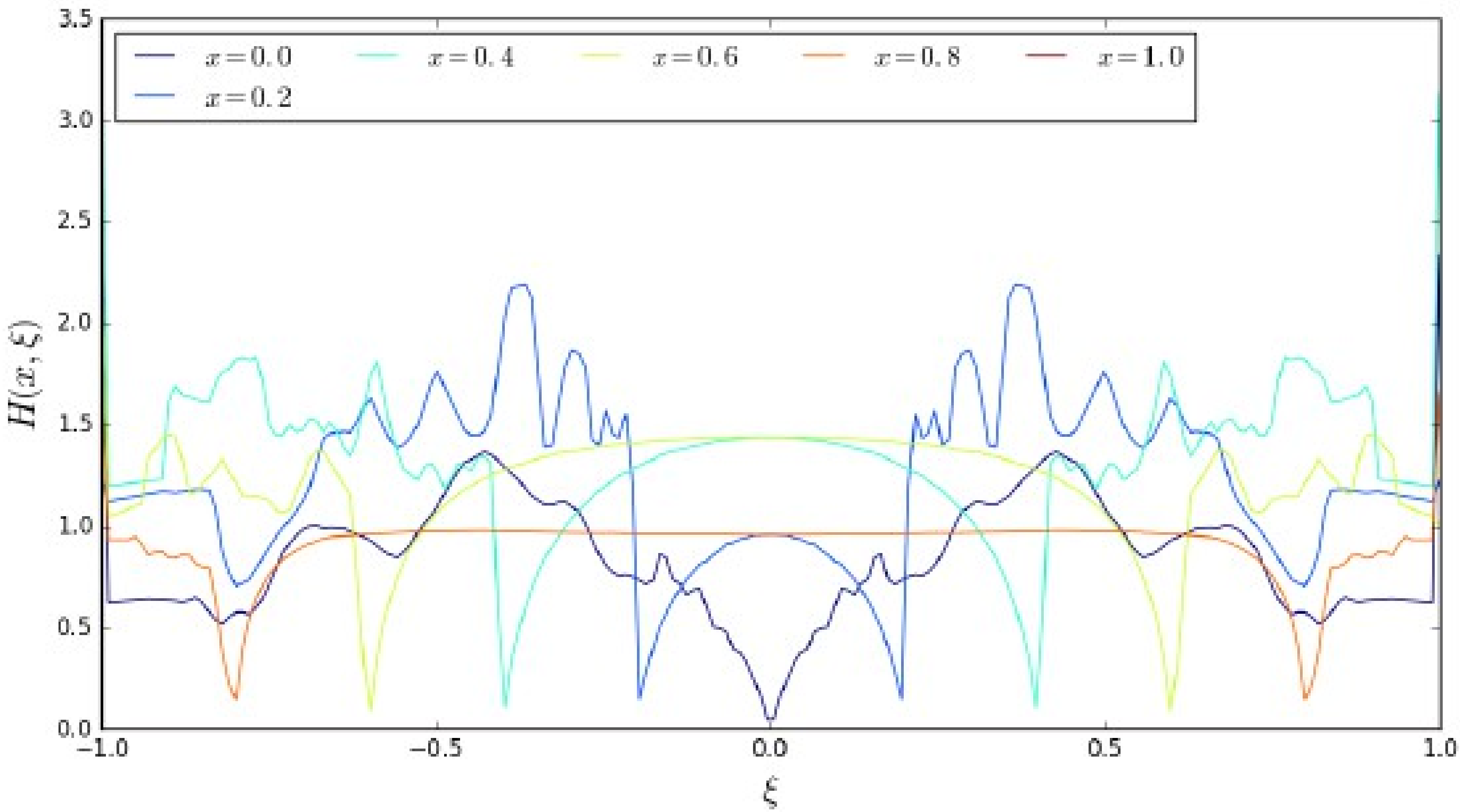
$$z \;=\; 1+\frac{t}{4M^2}\,\frac{1-x}{2x}$$

The overlap approach

Radon transform: from GPD DGLAP to the whole GPD domain

No analytical results in those cases, but a numerical reconstruction is possible (although, from a mathematically rigorous point of view, the problem is ill-posed)

GPD numerically reconstructed:



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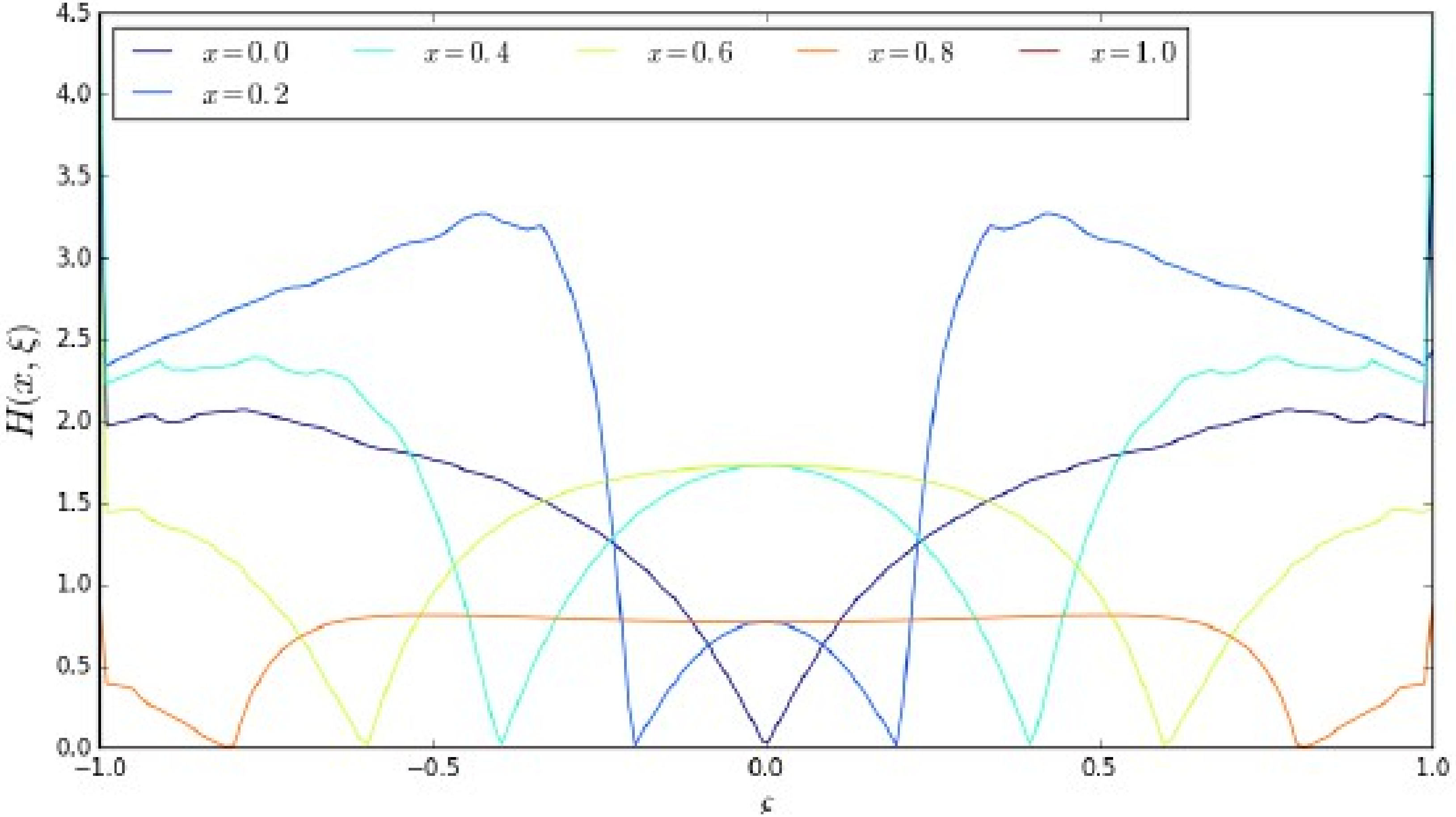

Conclusions:

Just made a few modest steps in a very long way!!!

- **Nonperturbative** computation of GPDs, DDs, LFWFs,...from Dyson-Schwinger equations.
- **Explicit check** of several theoretical constraints, including polynomiality, support property and soft pion theorem.
- **Systematic** procedure to construct GPD models from any "reasonable" Ansatz of LFWFs.
- **Characterization** of the **existence** and **uniqueness** of the extension from the DGLAP to the ERBL region.

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...much work in progress and to do!!!

Thank you.