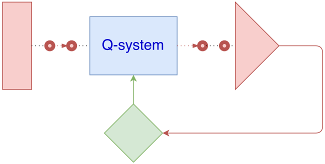
Linear feedback of continuous quantum measurements

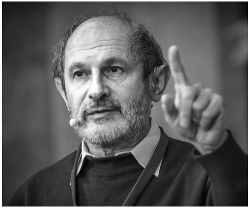
Antoine Tilloy
Max Planck Institute of Quantum Optics, Garching, Germany



22nd Claude Itzykson conference:
"Manipulation of simple quantum systems"
June 6-8, 2017, CEA-Saclay

Introduction

- Work in progress $\to$ `arXiv` hopefully soon
- Collaboration with **Lajos Diósi** (Wigner Research Centre, Budapest)



- Discussions with **Luca Ferialdi** (Ljubiana), **Pierre Rouchon** (Mines Paris) and others

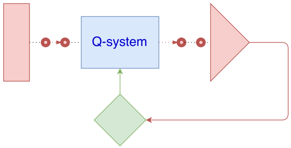
What?

Measurement-based feedback

A quantum system is *weakly* measured, the Hamiltonian depends on past measurements:

$$\partial_t|\psi_t\rangle = -iH(x_{t_1}, \cdots x_{t_n})|\psi_t\rangle,$$

with x_{t_i} the measurement outcomes and $t_n < t$.



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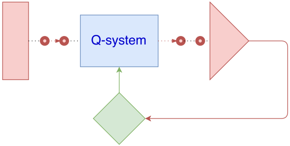
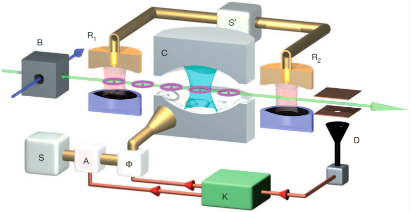


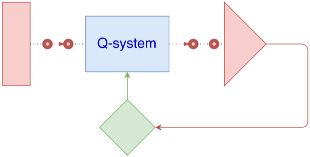
A famous example:



Gleyzes et al. Nature **446**, 297-300 (2007)

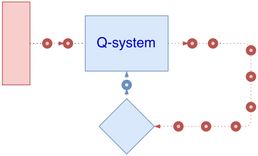
What not?

Measurement-based feedback



$\neq$

Coherent feedback



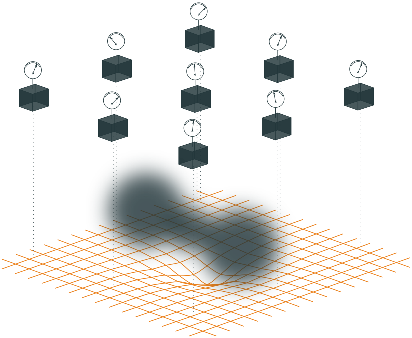
Lloyd ’00, fully **unitary**

Why feedback?

1. For **control**:
    - *fast* purification
    - *fast* measurement
    - state preparation/stabilization
    - continuous quantum error correction

Why feedback?

1. For **control**:
   - *fast* purification
   - *fast* measurement
   - state preparation/stabilization
   - continuous quantum error correction
2. For **fundamental** hybrid quantum-classical dynamics

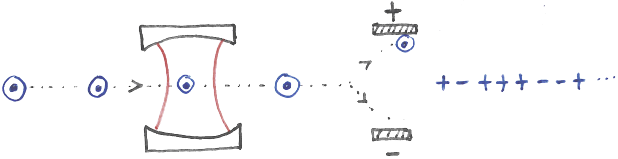


1509.08705 and 1706.01856

Outline

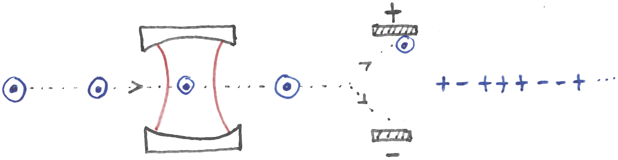
1. Continuous measurements from repeated interactions
2. Markovian feedback
3. Non-Markovian feedback
4. Derivation of the formula
5. Conclusion & perspectives

Repeated interactions



System $|\psi_n\rangle \in \mathscr{H}_s$ $\otimes$ Probe $|+\rangle_x \in \mathbb{C}^2$

Repeated interactions

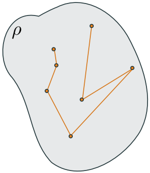


System $|\psi_n\rangle \in \mathscr{H}_s \otimes$ **Probe** $|+\rangle_x \in \mathbb{C}^2$

$$\rho_n = |\psi_n\rangle\langle\psi_n|$$

$$|\psi_n\rangle \otimes |+\rangle_x \xrightarrow{\text{interaction}} \hat{\Omega}_+|\psi_n\rangle \otimes |+\rangle_z + \hat{\Omega}_-|\psi_n\rangle \otimes |-\rangle_z$$

$$\xrightarrow{\text{measurement}} |\psi_{n+1}\rangle = \frac{\hat{\Omega}_\pm|\psi_n\rangle}{\sqrt{\langle\psi_n|\Omega_\pm^\dagger\hat{\Omega}_\pm|\psi_n\rangle}}$$



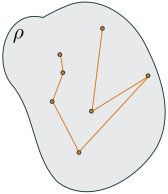
with

$$\hat{\Omega}_+^\dagger\hat{\Omega}_+ + \hat{\Omega}_-^\dagger\hat{\Omega}_- = \mathbb{1}$$

Quickly repeated *soft* interactions

Discrete quantum trajectories

A sequence $|\psi\rangle_n$ or ρ_n (random) and the corresponding measurement results $\delta_n = \pm 1$.



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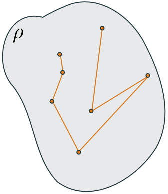

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$\Rightarrow$ Make the interactions soft and frequent:

$$\Omega_\pm = \frac{1}{\sqrt{2}} \left(\mathbb{1} \pm N\,\varepsilon + \#\,\varepsilon^2 + \cdots \right)$$

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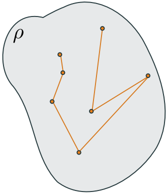

$\Rightarrow$ Make the interactions soft and frequent:

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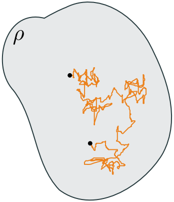
Continuous quantum trajectories

A continuous process $|\psi\rangle_t$ or ρ_t (random) and the corresponding measurement signal $s_t \approx \frac{\mathrm{d}x_t}{\mathrm{d}t}$:

$$x_t \propto \sqrt{\Delta t} \sum_{n=1}^{t/\Delta t} \delta_n$$



Result

Stochastic master equation

State ρ_t:

$$d\rho_t = -i[H,\rho_t]\,dt + \mathcal{D}[N](\rho_t)\,dt + \mathcal{H}[N](\rho_t)\,dW_t$$

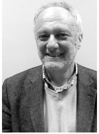
Signal $s_t \approx \frac{dx_t}{dt}$ with:

$$dx_t = \mathrm{tr}\left[(N+N^\dagger)\,\rho_t\right]\,dt + dW_t$$

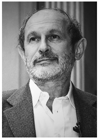
with:

- $\mathcal{D}[N](\rho) = N\rho N^\dagger - \frac{1}{2}\left(N^\dagger N\rho + \rho N^\dagger N\right)$ decoherence and dissipation
- $\mathcal{H}[N](\rho) = N\rho + \rho N^\dagger - \mathrm{tr}\left[(N+N^\dagger)\,\rho\right]\rho$ acquisition of information
- W_t Wiener process

V. Belavkin



A. Barchielli



L. Diósi

A bit of stochastic pedantry

Two possible definitions of stochastic integrals:

1. Itô integral:

$$\int_0^t f(u)\,\mathrm{d}W_u = \lim_{\Delta t \to 0} \sum_{k=1}^{n} f(t_k)\,[W(t_{k+1}) - W(t_k)]$$

2. Stratonovich integral:

$$\int_0^t f(u) \circ \mathrm{d}W_u = \lim_{\Delta t \to 0} \sum_{k=1}^{n} \frac{f(t_{k+1}) + f(t_k)}{2}\,[W(t_{k+1}) - W(t_k)]$$

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Principal characteristic:

- Itô: zero average

$$\mathbb{E}\left[\int_0^t f(u)\,\mathrm{d}W_u\right] = 0$$

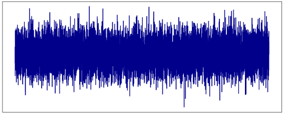
- Stratonovich: robust to smoothing (Wong-Zakaï theorem)

$$\int_0^t f(u)\dot{W}_u^{\varepsilon}\,\mathrm{d}u \xrightarrow[\varepsilon \to 0]{} \int_0^t f(u) \circ \mathrm{d}W_u$$

Comments

- The signal is a singular object:

$$s_t \approx \mathrm{tr}\left[(N + N^\dagger)\rho_t\right] + \frac{\mathrm{d}W}{\mathrm{d}t}$$



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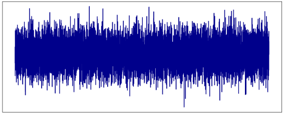

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- Taking the average $\bar{\rho}_t = \mathbb{E}[\rho_t]$ simply removes the stochastic term

$$\partial_t \bar{\rho}_t = -i[H, \bar{\rho}_t] + \mathcal{D}[N](\bar{\rho}_t)$$

the magic of Itô calculus

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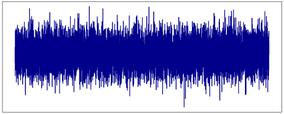

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the magic of Itô calculus

- Coincides with stochastic filtering (Kushner-Stratonovich equation) for diagonal density matrices

Markovian feedback

Stochastic master equation with Markovian potential:

$$d\rho_t = -i[H + \hat{V}_t, \rho_t]\, dt + \mathcal{D}[N](\rho_t)\, dt + \mathcal{H}[N](\rho_t)\, dW_t$$

Markovian feedback consists in adding an external potential $\hat{V}_t$ proportionnal to the **real-time** signal s_t:

$$\hat{V}_t \approx \hat{A}\, s_t \approx \hat{A}\, \frac{dx_t}{dt}$$

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- × Can we just add directly $\hat{V}_t dt = \hat{A}\, dx_t$? $\rightarrow$ would be incorrect (even in Strato!)
- ✓ It needs to act ε-after:

$$\rho_t + d\rho_t^{feedback} = e^{-iA\,dx_t}(\rho_t + d\rho_t)e^{iA\,dx_t},$$

The correct result is obtained by expanding to second order using physicist's Itô rule $dWdW = dt$.

Markovian feedback equations

Stochastic master equation

$$\mathrm{d}\rho_t^{feedback} = \mathrm{d}\rho_t - i[A, \rho_t]\,\mathrm{d}W_t + \mathcal{D}[A](\rho_t)\,\mathrm{d}t - i[A, N\rho_t + \rho_t N^\dagger]\,\mathrm{d}t$$

Milburn & Wiseman '93

Master equation

$$\partial_t \bar{\rho}_t = \underbrace{-i[H, \bar{\rho}_t]}_{\text{"free evol"}} + \underbrace{\mathcal{D}[N](\bar{\rho}_t)}_{\text{measurement}} + \underbrace{\mathcal{D}[A](\bar{\rho}_t)}_{\text{feedback decoherence}} \underbrace{-i[A, N\bar{\rho}_t + \bar{\rho}_t N^\dagger]}_{\text{feedback dissipation}}$$

Caves & Milburn '87

- if $[N, A] \neq 0$, feedback adds **dissipation**
- if $[N, A] = 0$, unitary + decoherence

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→ **Why is it useful?**

Decoherence and dissipation

Not all master equations containing a "dissipator" are dissipative:

Master equation in **diagonal** Lindblad form:

$$\partial_t \rho_t = -i[H_0, \rho_t] + \sum_i \left[N_i \rho_t N_i^\dagger - \frac{1}{2}\{N_i^\dagger N_i, \rho_t\} \right]$$

Master equation in **non-diagonal** Kossakovski's form:

$$\partial_t \rho_t = -i[H_0, \rho_t] + \sum_{i,j} D_{ij} \left[A_i \rho_t A_j - \frac{1}{2}\{A_j A_i, \rho_t\} \right]$$

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1. A master equation is **non-dissipative** $\Leftrightarrow$ D real $\Leftrightarrow \forall\, k,\ N_k = N_k^\dagger$.
2. Non-dissipative $\Leftrightarrow$ pure classical noise:

$$\mathrm{d}|\psi_t\rangle = -iA_k \circ \mathrm{d}W_t^{(k)}\, |\psi_t\rangle$$

with $\mathrm{d}W_t^{(i)} \mathrm{d}W_t^{(j)} = D_{ij}\mathrm{d}t$. $\quad\rightarrow$ **not** a resource

Comments

Dissipation is a resource, decoherence is not

e.g.: *Quantum computation and quantum-state engineering driven by dissipation*
Verstraete, Wolf & Cirac, (2009)

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The alchemy of feedback:

Let $\partial_t \bar{\rho}_t = \mathscr{L}(\bar{\rho}_t)$ be a generic **dissipative** master equation. It can be obtained from **non-dissipative** measurement $+$ Markovian feedback

The Alchemist in Search of the Philosopher's Stone
Joseph Wright, 1771

Markovian feedback

Advantage:

- Very simple
- Generates tunable dissipation

Limits:

- Physically unrealistic (neglects delays, filters, etc.)
- Mathematically subtle to define directly in the continuum

non-Markovian feedback

Stochastic master equation with non-Markovian potential:

$$\mathrm{d}\rho_t = -i[H + \hat{V}_t, \rho_t]\,\mathrm{d}t + \mathcal{D}[N](\rho_t)\,\mathrm{d}t + \mathcal{H}[N](\rho_t)\,\mathrm{d}W_t \tag{1}$$

non-Markovian feedback consists in choosing $\hat{V}_t$:

$$\begin{aligned}\hat{V} &= \hat{A}_t \int_{-\infty}^{t} K(t,\tau)\, s_\tau \,\mathrm{d}\tau \\ &= \hat{A}_t \int_{-\infty}^{t} K(t,\tau)\,\mathrm{d}x_\tau\end{aligned}$$

where K can be anything

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- ✓ The resulting **stochastic master equation** is trivial, it is just (1)!
- × The **master equation** is not:

$$\bar{\rho}_t = \mathbb{E}[\rho_t] = ?$$

Question: can we write a master equation (even formal) for $\bar{\rho}_t$?

Master equation for non-Markovian feedback

Main result

In interaction representation:

$$\bar{\rho}(t) = T\,\exp\left\{ \int_0^t \int_0^t \mathrm{d}\tau_1\,\mathrm{d}\tau_2\;\mathscr{L}(\tau_1,\tau_2) \right\} \cdot \rho_0$$

with

$$\begin{aligned}\mathscr{L}(\tau_1,\tau_2) =& \mathcal{D}[N(\tau_1)]\,\delta(\tau_1-\tau_2) \\ & - \frac{1}{2}\,\underbrace{A_\Delta(\tau_1)(K * K^T)(\tau_1,\tau_2)A_\Delta(\tau_2)}_{\text{non-Markovian feedback decoherence}} - \underbrace{iN_+(\tau_1)K(\tau_2,\tau_1)A_\Delta(\tau_2)}_{\text{non-Markovian feedback dissipation}}\end{aligned}$$

using the superoperator notations

$$B_\Delta \cdot \rho = B\rho - \rho B^\dagger$$

$$B_+ \cdot \rho = B\rho + \rho B^\dagger$$

First comments

- **Markovian limit:** when $K(\tau_1, \tau_2) \to \delta(\tau_1, \tau_2)$:

$$\mathscr{L}(\tau_1, \tau_2) \longrightarrow \left[\mathcal{D}[N(\tau_1)] - \frac{1}{2} A_\Delta(\tau_1) A_\Delta(\tau_2) - i N_+(\tau_1) A_\Delta(\tau_2)\right] \delta(\tau_1 - \tau_2)$$

we get back Markovian feedback

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- **Inexplicit**: time ordered exponentials with two times τ_1, τ_2 verify no simple differential equation:

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- **Dyson expansion:** The most immediate way to make sense of the formula is by expanding the exponential.
- **Trivial iif all the operators commute**

Aparté on Gaussian master equations

Consider the most general linear coupling with a continuum of harmonic oscillators

$$H_{\text{int}}(t) = \sum_k \hat{A}_k(t) \otimes \hat{\phi}_k(t)$$

where $\phi_k(t)$ is whatever Hermitian linear combination of creation and annihilation operators:

$$\phi_k(t) = \int \mathrm{d}\omega\, f_{kl}(\omega) e^{-i\omega t} a_l^\dagger(\omega) + h.c.$$

Aparté on Gaussian master equations

Consider the most general linear coupling with a continuum of harmonic oscillators

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Gaussian master equation

The density matrix obeys the following formal master equation (Diósi & Ferialdi '15)

$$\rho(t) = T \exp\left\{ \int_0^t \int_0^t \mathrm{d}\tau_1 \mathrm{d}\tau_2 \, \mathscr{L}(\tau_1, \tau_2) \right\} \cdot \rho(0)$$

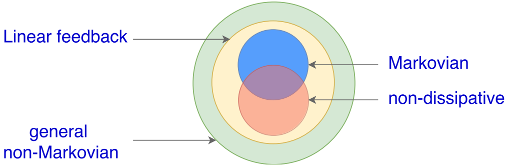
with

$$\mathscr{L}(\tau_1,\tau_2) = D_{ij}(\tau_1,\tau_2)\left(A_i^L(\tau_1) A_j^R(\tau_2) + \frac{\theta_{\tau_1,\tau_2} A_i^L(\tau_1) A_j^L(\tau_2) + \theta_{\tau_2,\tau_1} A_i^R(\tau_1) A_j^R(\tau_2)}{2} \right)$$

operator notation of Feynman Vernon influence functional

Master equation for non Markovian feedback

Our feedback equation is a special case of a **Gaussian master equation**:



We can import the standard techniques of open system theory

- Projection operator techniques
- Time convolutionless master equations
- Exact solutions in a few cases (Ferialdi '16)
- For an exponential kernel $\rightarrow$ Immamoglu's decaying oscillator technique is numerically efficient

In any case, the difficulty is more **algebraic** than **analytic**
$\Rightarrow$ "feedback + Monte-Carlo" to solve hard open-system problems?

Derivation I

Idea: use linear trajectories

1. Write a linear stochastic differential equation for ρ^ℓ such that $\rho = \rho^\ell/\mathrm{tr}[\rho^\ell]$
2. Pass the equation in Stratonovich form
3. Formally integrate as a time-ordered exponential
4. Use Girsanov's theorem to change the probability measure
5. Finally average using Wick's theorem

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1. Equivalent linear stochastic master equation:

$$\mathrm{d}\rho_t^\ell = -[H + \hat{V}_t, \rho_t^\ell]\,\mathrm{d}t + \mathcal{D}[N](\rho_t^\ell)\,\mathrm{d}t + \left(N\rho_t^\ell + \rho_t^\ell N^\dagger\right)\mathrm{d}x_t$$

2. Stratonovich form

$$\mathrm{d}\rho_t^\ell = -[H + \hat{V}_t, \rho_t^\ell]\,\mathrm{d}t + \left(N\rho_t^\ell + \rho_t^\ell N^\dagger\right) \circ \mathrm{d}x_t - \frac{1}{2}\left[(N^2 + N^\dagger N)\rho_t^\ell + \rho_t^\ell (N^{\dagger 2} + N^\dagger N)\right]\,\mathrm{d}t$$

Derivation II

2. Stratonovich form

$$\begin{aligned} d\rho_t^\ell &= -[H + \hat{V}_t, \rho_t^\ell]dt + \left(N\rho_t^\ell + \rho_t^\ell N^\dagger\right) \circ dx_t - \frac{1}{2}\left[(N^2 + N^\dagger N)\rho_t^\ell + \rho_t^\ell(N^{\dagger 2} + N^\dagger N)\right] dt \\ &= \mathcal{A} \cdot \rho_t^\ell \, dt + \mathcal{B} \cdot \rho_t^\ell \circ dx_t \end{aligned}$$

3. Formal integration

In interaction representation:

$$\begin{aligned} \rho^\ell(t) &= T \exp\left\{ \int_0^t \mathcal{A}(\tau)d\tau + \mathcal{B}(\tau)dx_\tau \right\} \cdot \rho_0 \\ &= T \exp\left\{\text{something linear in } x_t\right\} \cdot \rho_0 \end{aligned}$$

→ would be possible to average if x_t were Gaussian!

Derivation II

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4. Change of measure
We want to compute:

$$\bar{\rho}_t = \mathbb{E}[\rho_t] = \mathbb{E}[\rho_t^\ell \cdot \text{tr}(\rho_t^\ell)^{-1}]$$

Define $\mathbb{E}_t[\,\cdot\,] = \mathbb{E}[\,\cdot\,\text{tr}(\rho_t^\ell)^{-1}]$. Girsanov's theorem $\Rightarrow$ x_t is **Gaussian** for $\mathbb{E}_t$.

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5. Average using either Wick's (or Isserlis') theorem or Gaussian functional integration

Master equation for non-Markovian feedback

Main result (reminder)

In interaction representation:

$$\bar{\rho}(t) = T \exp\left\{ \int_0^t \int_0^t \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \, \mathscr{L}(\tau_1, \tau_2) \right\} \cdot \rho_0$$

with

$$\begin{aligned} \mathscr{L}(\tau_1, \tau_2) =& \mathcal{D}[N(\tau_1)] \, \delta(\tau_1 - \tau_2) \\ & - \underbrace{\frac{1}{2} A_\Delta(\tau_1)(K * K^T)(\tau_1, \tau_2) A_\Delta(\tau_2)}_{\text{non-Markovian feedback decoherence}} - \underbrace{i N_+(\tau_1) K(\tau_2, \tau_1) A_\Delta(\tau_2)}_{\text{non-Markovian feedback dissipation}} \end{aligned}$$

using the superoperator notations

$$B_\Delta \cdot \rho = B\rho - \rho B^\dagger$$

$$B_+ \cdot \rho = B\rho + \rho B^\dagger$$

Conclusion

Summary

- Markovian feedback is simple and can yield dissipation
- Non-Markovian feedback can be “solved” formally
- Same structure as general open-system evolutions

Options to go forward

- Compute non-Markovianity measures
- Estimate stationnary states
- non-Markovian measurement (repeated interactions with MPS????)