Introduction to the Standard Model of particle physics

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Laboratoire de Physique Subatomique et de Cosmologie

Summer School in Particle and Astroparticle physics
Annecy-le-Vieux, 20-26 July 2017

LPSC

Laboratoire de Physique Subatomique et de Cosmologie

cnrs IN2P3 Les deux infinis

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Organisation

- 3h of lectures in English for students after the 3rd/4th year of studies
- Prerequisites
  - Lagrangian formalism, Electrodynamics, Quantum Mechanics
  - Group theory (basic knowledge)
- Use slides
- Please ask many questions during the lectures!
  - More interesting
  - Slows down the speed (necessary in particular when slides are used)
  - There are NO stupid questions!

Disclaimer

- I'm grateful to Benjamin Fuks and Guillaume Chalons. I took several slides from their lecture at the MADGRAPH workshop given for CERN Summer Students in 2015

- Many thanks also to Cedric Delauney who gave this lecture at the GraSPA school in 2015. I made use of several of his slides.

Literature

1) Michele Maggiore, *A Modern Introduction to Quantum Field Theory*, Oxford University Press

2) Matthew D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press

3) Francis Halzen, Alan D. Martin, *Quarks & Leptons*, Wiley

4) S. Weinberg, *The Quantum Theory of Fields I*, Cambridge Univ. Press

5) H. Georgi, *Lie algebras in particle physics*, Frontiers in Physics

6) Robert Cahn, *Semi-Simple Lie Algebras and Their Representations*, freely available on internet

7) R. Slansky, *Group Theory for Unified Model Building*, Phys. Rep. 79 (1981) 1-128

Plan

1. The Standard Model of particle physics (1st round)

2. Some Basics

3. The Standard Model of particle physics (2nd round)

   - Symmetries & Fields
   - Lagrangian terms
   - Higgs mechanism

4. From the SM to predictions at the LHC

   - Cross sections, Decay widths
   - Feynman rules
   - Parton Model

5. Beyond the Standard Model

1. The Standard Model of particle physics (1st round)

The ultimate goal (for some at least...)

A consistent view of the world

Daß ich erkenne, was die Welt
im Innersten zusammenhält...
(Goethe, Faust I)

AGE-OLD Questions

What are the fundamental constituents which comprise the universe?

AGE-OLD Questions

What are the fundamental constituents which comprise the universe?

How do they interact?

AGE-OLD Questions

What are the fundamental constituents which comprise the universe?

How do they interact?

What holds them together?

AGE-OLD Questions

What are the fundamental constituents which comprise the universe?

How do they interact?

What holds them together?

Who will win the next World Cup?

Periodic Table circa 425 BC

Earth

“The periodic table.”

Periodic Table circa 425 BC

Earth
Water

"The periodic table."

Periodic Table circa 425 BC

Earth
Water
Fire

"The periodic table."

Periodic Table circa 425 BC

Earth

Water

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Air

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Compact

Easy to remember

Fits on a T-shirt

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"Of course the elements are earth, water, fire and air. But what about chromium? Surely you can't ignore chromium."

Sidney Harris

Periodic Table circa 425 BC

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Physics Beyond the Standard Model?
The Higgs field?

Unification

Earth

Water

Fire

Air

"The periodic table."

Compact
Easy to remember
Fits on a T-shirt

Plato:
Since the four elements can transform into each other, it is reasonable to assume that there is only **one fundamental substance** and the four elements are just different manifestations of it!

Periodic Table circa 1900

TABLE DE MENDÉLÉEF

H=1	I	II	III	IV	III	II	I	II		
	Li 7,01	Gl 9,08	B 10,9	C 11,97	Az 14,01	O 15,88	Fl 19			
	Na 22,99	Mg 23,94	Al 27,04	Si 28	P 30,96	S 31,98	Cl 35,37			
	K 39,03	Ca 39,91	Sc 43,97	Ti 48	V 51,1	Cr 52,45	Mn 54,8	Fe 55.88	Ni 58,56	Co 58,74
	Cu 63,18	Zn 64,88	Ga 69,9	Ge 72,32	As 75	Se 78,87	Br 79,76			
	Rb 85,2	Sr 87,3	Y 89,6	Zr 90.4	Nb 93,7	Mo 95,9	—	Ru 101,5	Rh 103,2	Pd 106,3
	Ag 107,66	Cd 111,7	In 113,4	Sn 117,35	Sb 119,6	Te 126,3	I 126,54			
	Cs 132,7	Ba 136,86	La 138,5	Ce 141,2	Di 145	—	—			
	—	—	Yb 172,6	—	Ta 182	Tu 183,6	—	Os 190	Ir 192	Pt 194
	Au 196,2	Hg 199,8	Tl 203,7	Pb 206.39	Bi 207,5	—	—			
				Th 231,96		U 239,8	—			

Dimitri Mendeleev (1834-1907)

Periodic Table circa 1900

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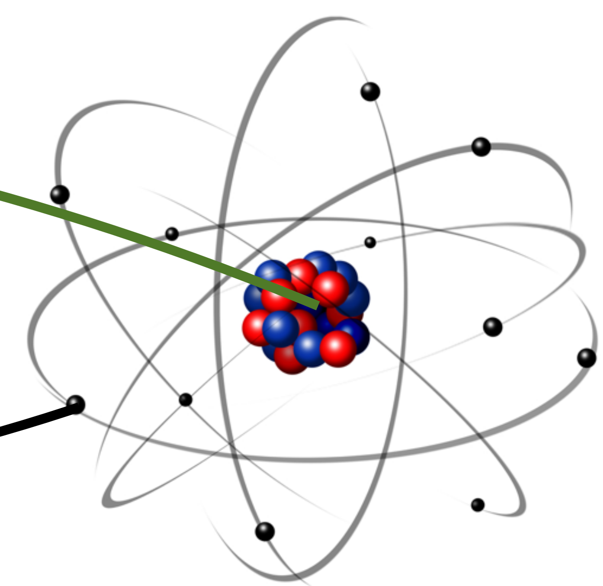
Dimitri Mendeleev (1834-1907)

66 elements!

Atoms

✦ At the atomic scale, matter is composed of atoms:

- A core: the nucleus, made of
  - ★ Protons ( )
  - ★ Neutrons ( )
- Peripheral electrons (•)



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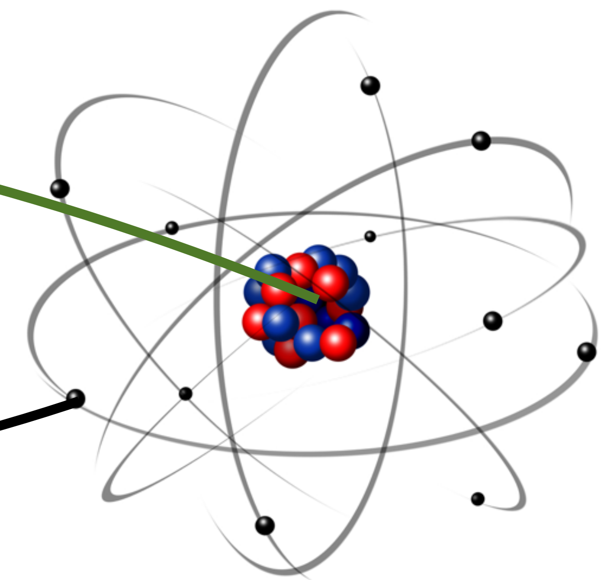
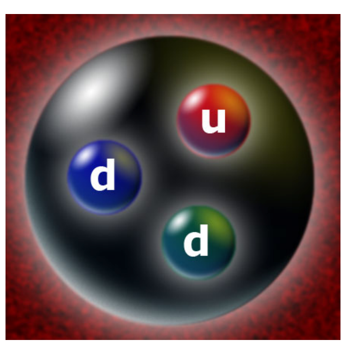
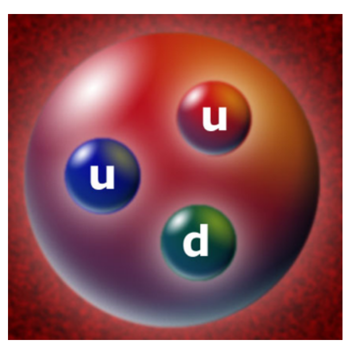
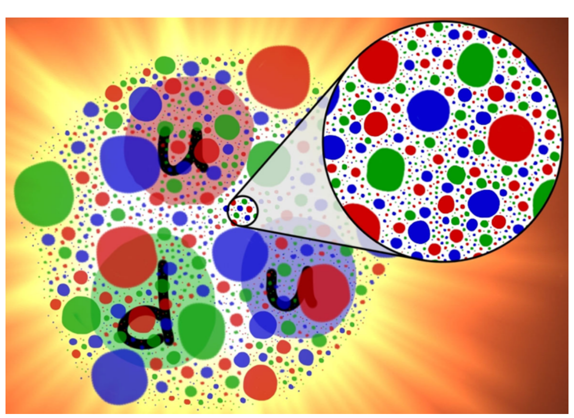






✦ Naively, protons and neutrons are composed objects:

- ✤ Proton: two up quarks and one down quark
- ✤ Neutron: one up quarks and two down quarks

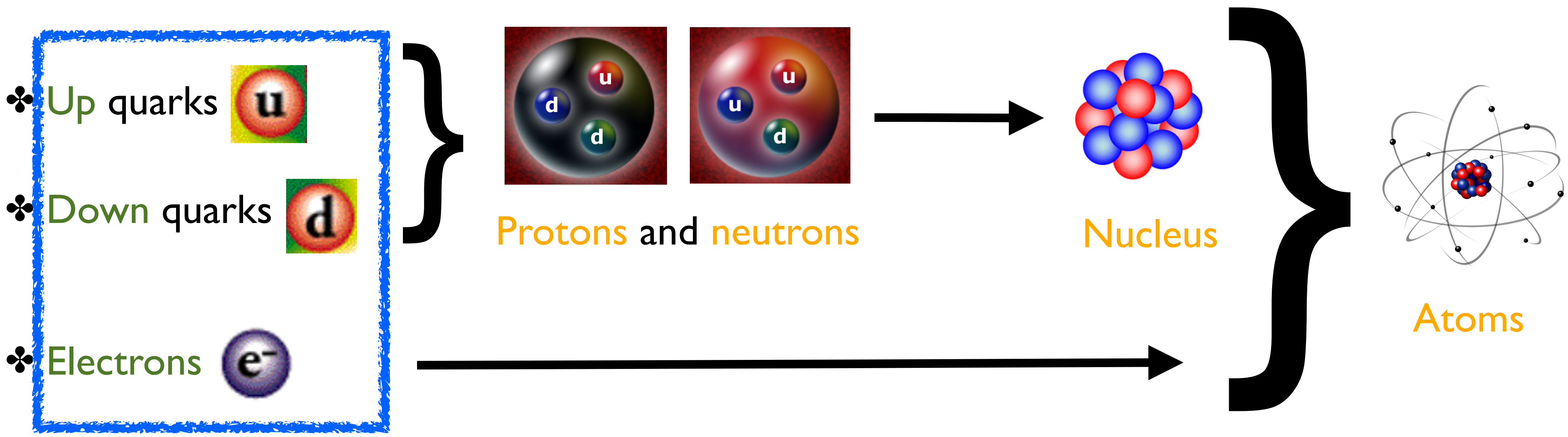


✦In reality, they are dynamical objects:

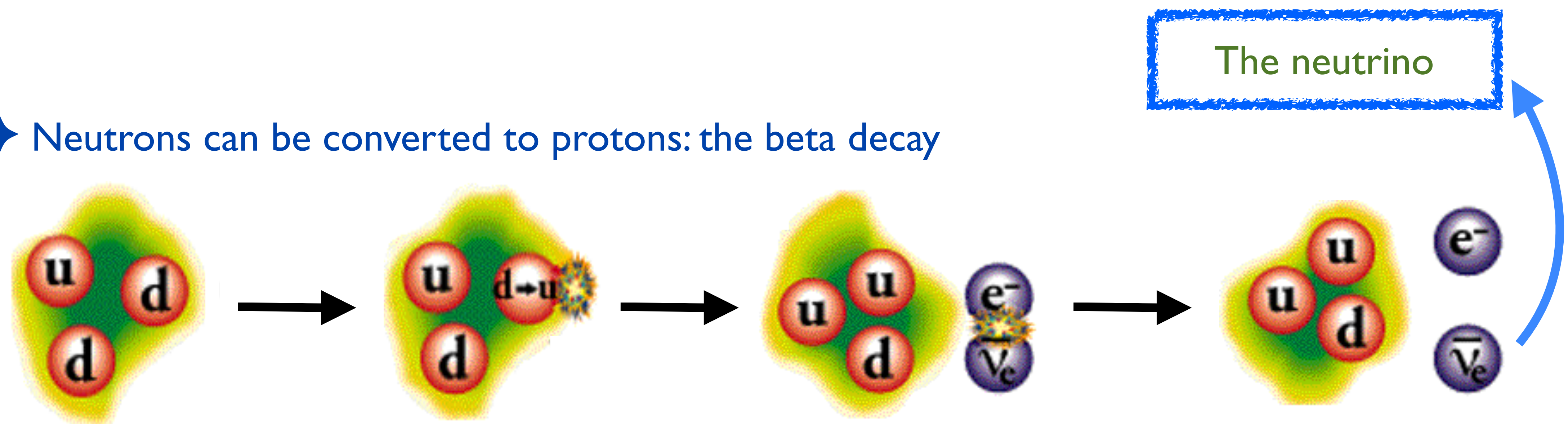
- ✤ Made of many interacting quarks and gluons (see later)

Elementary Matter Constituents I

✦ Elementary matter constituents

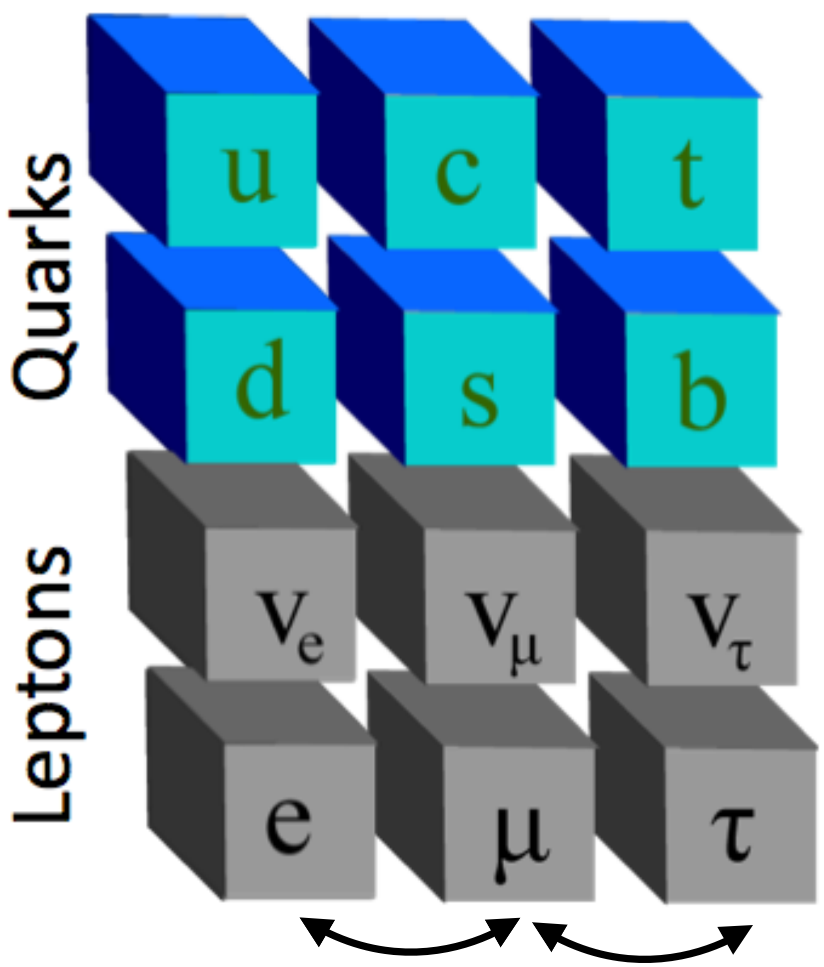


✦ Neutrons can be converted to protons: the beta decay



Elementary Matter Constituents II

✦ Elementary matter constituents: we have <u>three</u> families



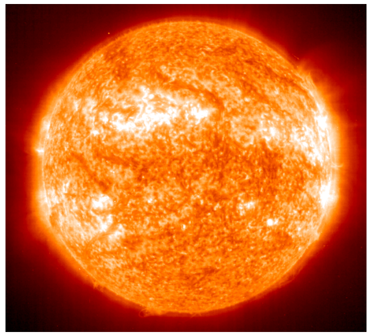
The only differences are the masses
All other properties are identical

- ✤ Three up-type quarks
  - ★ Up (u)
  - ★ Charm (c)
  - ★ Top (t)
- ✤ Three down-type quarks
  - ★ Down (d)
  - ★ Strange (s)
  - ★ Bottom (b)
- ✤ Three neutrinos
  - ★ Electron (ν_e)
  - ★ Muon (ν_μ)
  - ★ Tau (ν_τ)
- ✤ There charged leptons
  - ★ Electron (e)
  - ★ Muon (μ)
  - ★ Tau (τ)

Four fundamental Interactions

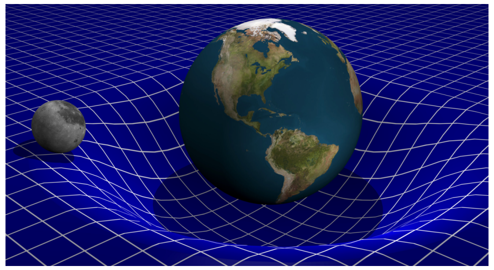
- ✦ Electromagnetism
  - ✤ Interactions between charged particles (quarks, charged leptons)
  - ✤ Mediated by massless photons γ

- ✦ Weak interactions
  - ✤ Interactions between all matter fields
  - ✤ Mediated by massive weak W-bosons and Z-bosons



- ✦ Strong interactions
  - ✤ Interactions between colored particles (quarks)
  - ✤ Mediated by massless gluons g
  - ✤ Responsible for binding protons and neutrons within the nucleus

- ✦ Gravity
  - ✤ Not included in the Standard Model



The Higgs boson

- The masses of the particles
  - Elegant mechanism to introduce them
  - Price to pay: a new particle, the so-called Higgs boson

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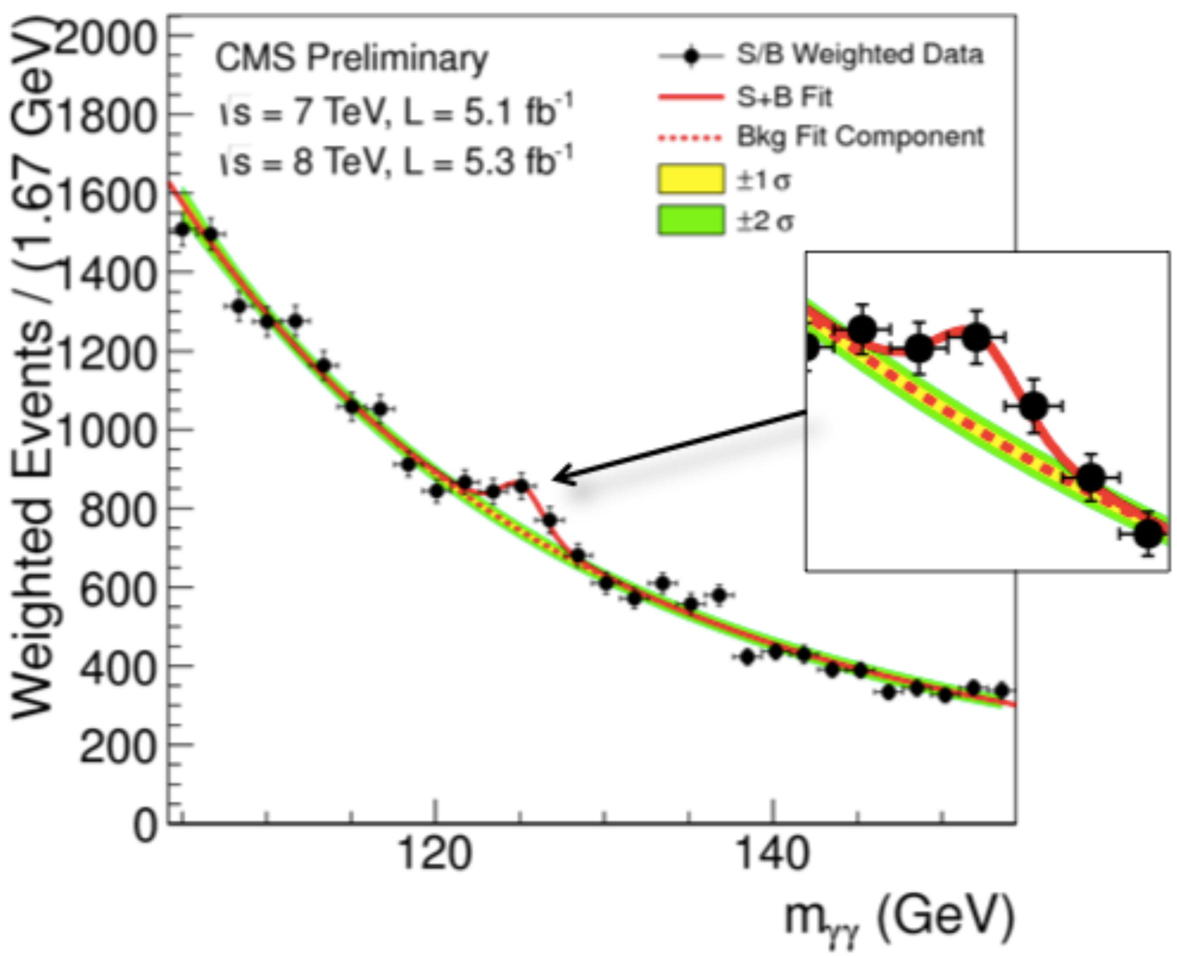
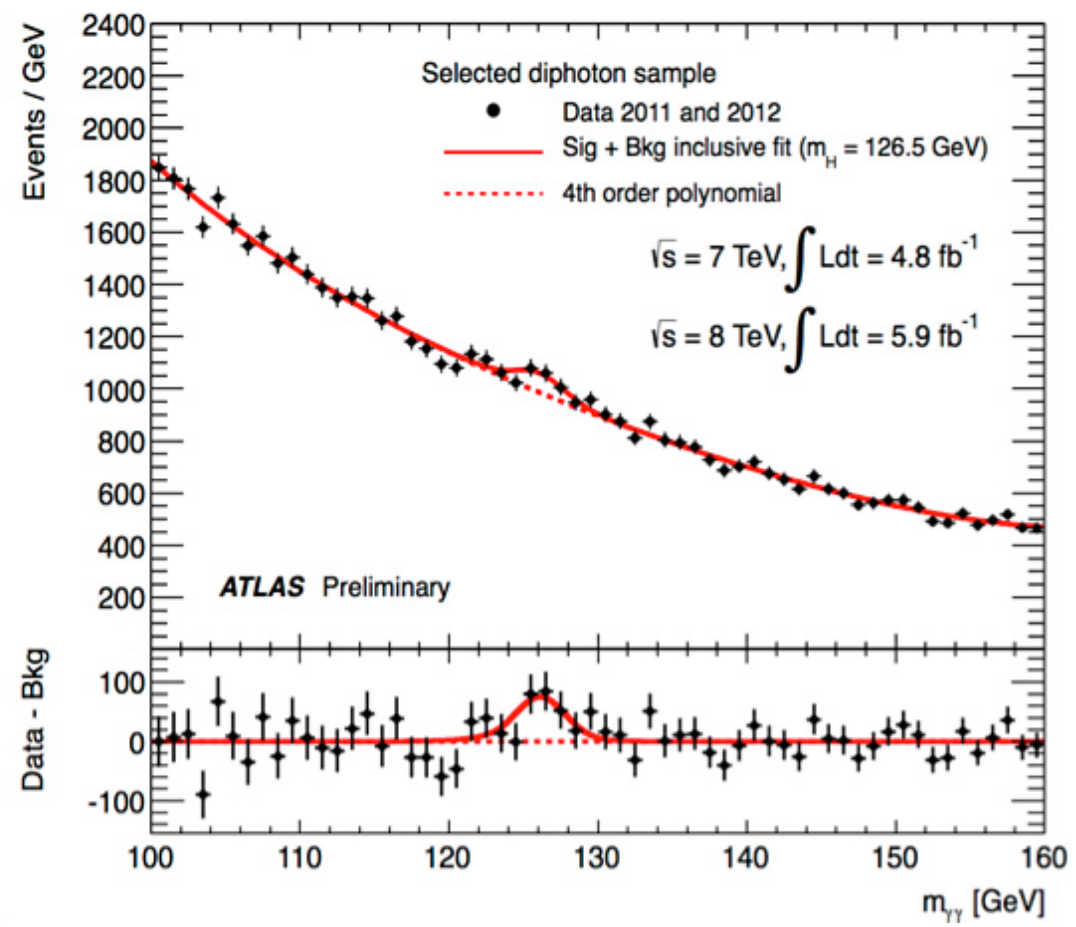
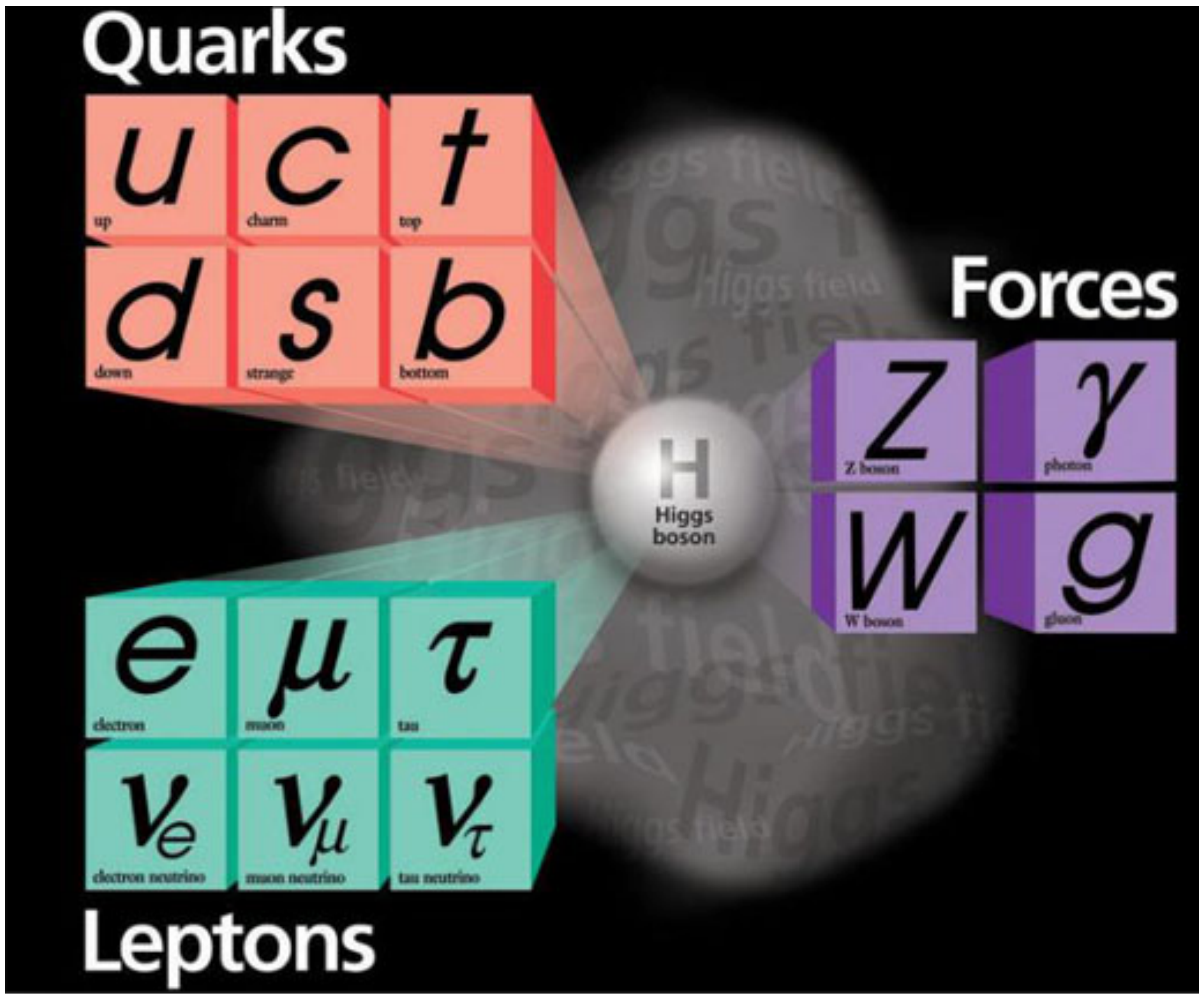
discovered in 2012

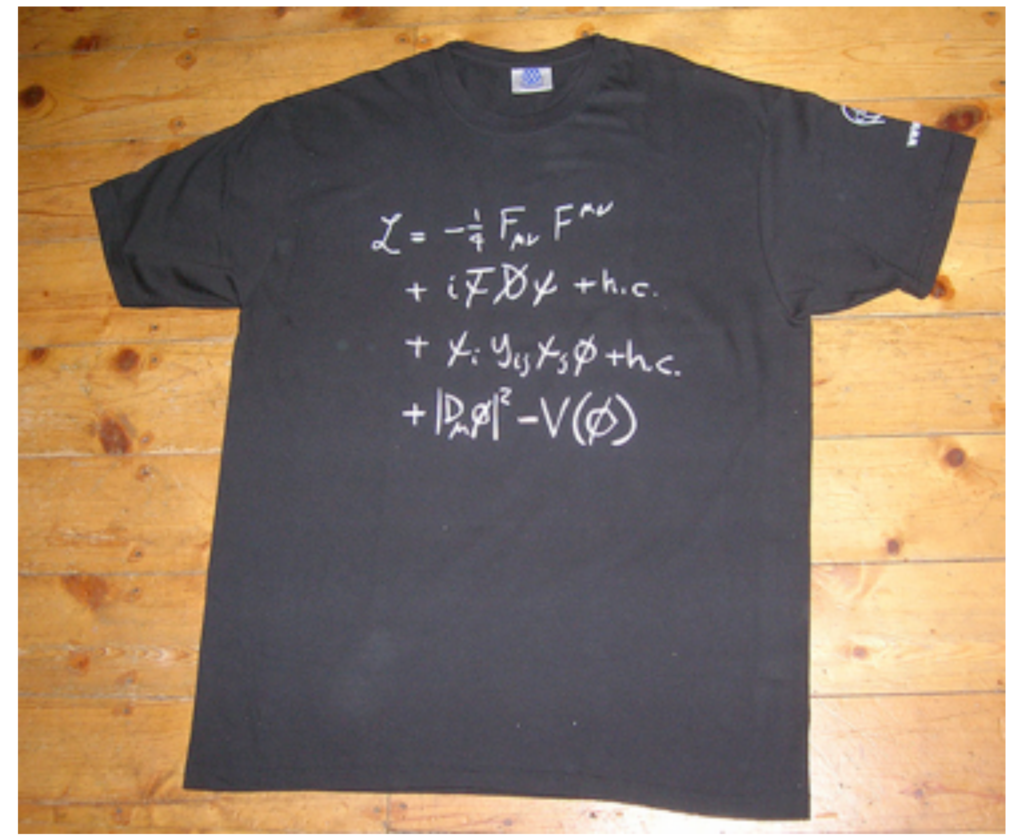




Periodic Table circa 2016 AD



The **Standard Model** (SM) for the strong, weak, and electromagnetic interactions



Compact
Easy to remember
Fits on a T-shirt

II. Some Basics

Overview

- Our goal (next chapter):

  Understand the SM at a slightly more detailed level

- Before, we review some basics helpful later for the understanding:

  - Units and scales in particle physics

  - The general theoretical framework

  - Symmetries

One page summary of the world

Gauge group

$\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$

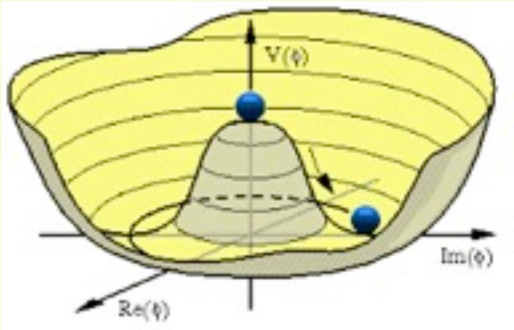
Particle content

MATTER				HIGGS		GAUGE	
$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3},\mathbf{2})_{1/3}$	$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$(\mathbf{1},\mathbf{2})_{-1}$	$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$	$(\mathbf{1},\mathbf{2})_1$	B	$(\mathbf{1},\mathbf{1})_0$
u_R^c	$(\overline{\mathbf{3}},\mathbf{1})_{-4/3}$	e_R^c	$(\mathbf{1},\mathbf{1})_2$			W	$(\mathbf{1},\mathbf{3})_0$
d_R^c	$(\overline{\mathbf{3}},\mathbf{1})_{2/3}$	ν_R^c	$(\mathbf{1},\mathbf{1})_0$			G	$(\mathbf{8},\mathbf{1})_0$

Lagrangian
(Lorentz + gauge + renormalizable)

$$\mathcal{L} = -\frac{1}{4}G^{\alpha}_{\mu\nu}G^{\alpha\mu\nu} + \ldots \overline{Q}_k \not{D} Q_k + \ldots (D_\mu H)^\dagger (D^\mu H) - \mu^2 H^\dagger H - \frac{\lambda}{4!}(H^\dagger H)^2 + \ldots Y_{k\ell}\overline{Q}_k H (u_R)_\ell$$

SSB



- $H \to H' + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$
- $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_Q$
- $B, W^3 \to \gamma, Z^0$ and $W^1_\mu, W^2_\mu \to W^+, W^-$
- Fermions acquire mass through Yukawa couplings to Higgs

Units and Scales

(Essential for the big picture/orders of magnitude estimates)

Units

- Use **natural units**:

$c = 1$ (SR), $\hbar = 1$ (QM), $\varepsilon_0 = 1$ (vacuum permittivity)

- $c = 1 = 3 \cdot 10^8$ m/s $\Rightarrow$ 1 s $= 3 \cdot 10^8$ m

  [time] = [length] ; [velocity] = pure number

- $E = m\,\gamma\,c^2 = m\,\gamma$ (Note: m is always the rest mass; $\gamma^{-2} = 1 - v^2/c^2$)

  [energy] = [mass] = [momentum]

- $\hbar = 1 = 1 \cdot 10^{-34}$ J s $\Rightarrow$ 1 s $= 10^{34}$ J^{-1} $= 0.15 \cdot 10^{22}$ MeV

  [time] = [length] = [energy]$^{-1}$

Scales

see PDG booklet

- Planck mass: $\sqrt{(\hbar c/G_N)} = \sqrt{(1/G_N)} \sim 1.2 \cdot 10^{19}$ GeV

- mass of a proton/neutron: $m_p \sim 1$ GeV

- proton/neutron radius: $r_p \sim 1$ fm $= 10^{-15}$ m $= 1$ fermi

  $\hbar c \sim 200$ MeV fm $= 1 \Rightarrow 1$ fermi $\sim (200 \text{ MeV})^{-1}$

- mass of an electron: $m_e \sim 0.5$ MeV

Scales

see PDG booklet

- Fine structure constant:

  $\alpha = e^2/(4\pi\ \varepsilon_0\ \hbar\ c) = e^2/(4\pi) = 1/137 \Rightarrow e = 0.3$

- Rydberg energy: $E_R = 1/2\ m_e\ c^2\ \alpha^2 = 1/2\ m_e\ \alpha^2 = 13.6\ eV$

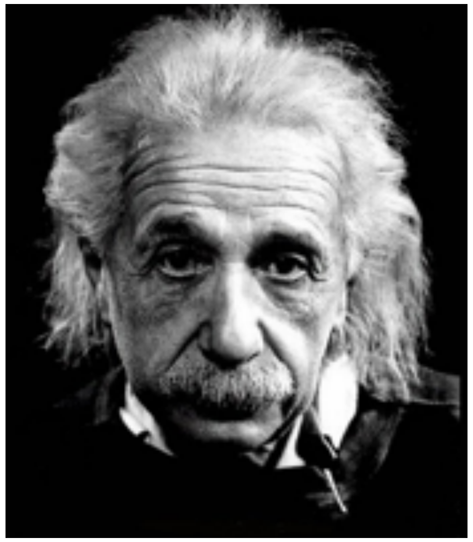
- Bohr radius: $a_B = \hbar/(m_e\ c\ \alpha) = 1/(m_e\ \alpha) \sim 0.5\ 10^{-10}\ m$

Theorist's prejudice

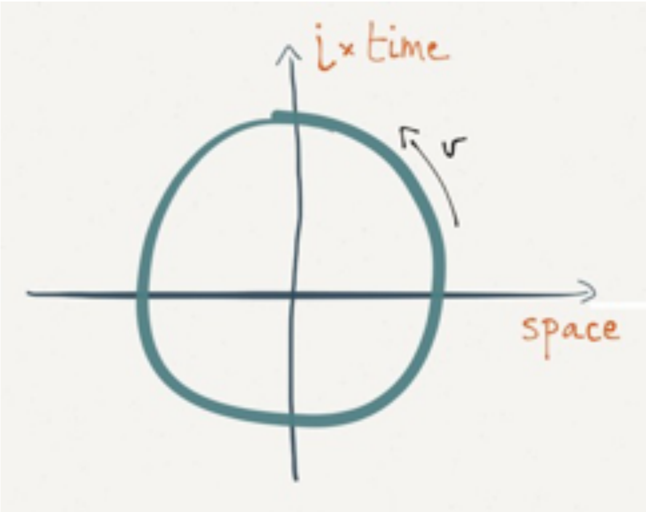
- Everything that is not forbidden is realized!
  - Not forbidden (by symmetries) but not observed = problem!
- The only 'allowed' numbers are 0, 1, infinity
  (this is very ignorant, of course!)
  - 0: forbidden because of symmetry
  - 1: natural number
  - infinity: to be redefined
  - small but non-zero couplings = problem ('unnatural')
  - large finite couplings (>>1) = non-perturbative

The general theoretical framework

Special relativity (SR)



- All inertial observers see the same physics:
  - same light speed c
  - Lorentz symmetries = Space-time "rotations"



$$x^\mu = (t, \vec{x})$$
$$x^2 = \eta_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu = \text{invariant}$$
$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

- Energy-momentum relation: p = (E,**p**), $p^2 = m^2 = E^2$ - $\mathbf{p}^2$

Special relativity (SR)

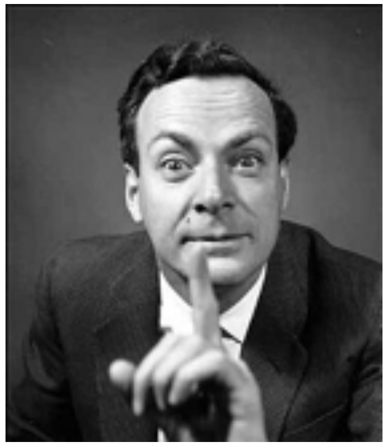
- Lorentz group $O(1,3) = \{\Lambda \mid \Lambda^T \eta \Lambda = \eta\}$
- Proper Lorentz group $SO(1,3) = \{\Lambda \mid \Lambda^T \eta \Lambda = \eta, \det \Lambda = 1\}$
- Proper orthochronous Lorentz group $SO_+(1,3)$: $\Lambda_{00} \geq 1$
  Called the Lorentz group in the following
- Poincaré group = Inhomogeneous Lorenz group = $ISO_+(1,3)$

  $SO_+(1,3)$ and space-time Translations

Quantum Mechanics (QM)

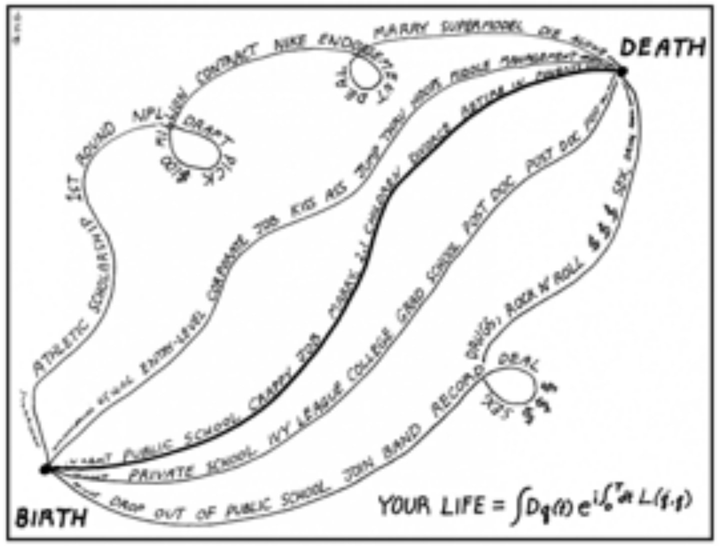
- Determinism is not fundamental: $\Delta x^{\mu} \times \Delta p_{\nu} \geq (\hbar/2)\delta^{\mu}_{\nu}$
  - Nature is random → probability rules
  - The vacuum is not void, it fluctuates!
- Classical physics emerges from constructive interference of probability amplitudes:

Feynman's path integral:



$$A = \int [dq] \exp(iS[q(t), \dot{q}(t)])$$

a rational for the least action principle



The Path Integral Formulation of Your Life

Quantum Field Theory (QFT)

- The general theoretical framework in particle physics is Quantum Field Theory

- Weinberg I:

  QFT is the only way to reconcile quantum mechanics with special relativity

  “QFT = QM + SR”

Quantum Field Theory (QFT)

- **QM**: It's the same quantum mechanics as we know it!
- **SR**:
  - Relativistic wave equations are not sufficient!
    We need to change **number** and **types** of particles in particle reactions
  - Need **fields** and **quantize** them ("quantum fields")

Particles = Excitations (quanta) of fields

Symmetries I
(Lie groups, Lie algebras)

Symmetries are described by Groups

A group $(G, \odot)$ is a set of elements G together with an operation $\odot : G \times G \to G$ which satifies the following axioms:

- Associativity: $\forall a, b, c \in G : (a \odot b) \odot c = a \odot (b \odot c)$
- Neutral element: $\exists e \in G : \forall a \in G : e \odot a = a \odot e = a$
- Inverse element: $\forall a \in G : \exists a^{-1} \in G : a^{-1} \odot a = a \odot a^{-1} = e$

The group is called <u>commutative</u> or <u>Abelian</u> if also the following axiom is satisfied:

- Commutativity: $\forall a.b \in G : a \odot b = b \odot a$

Lie groups (simplified)

A Lie group is a group with the property that it depends differentiably on the parameters that define it.

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- The number of (essential) parameters is called the dimension of the group.
- Choose the parametrization such that $g(\vec{0}) = e$.

Lie groups (simplified)

A Lie group is a group with the property that it depends differentiably on the parameters that define it.

- The number of (essential) parameters is called the dimension of the group.
- Choose the parametrization such that $g(\vec{0}) = e$.

Example:
Rotation $R(\phi) \in \mathrm{SO}(3)$ by an angle ϕ around the z-axis:

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Generators of a Lie group

Be $D(\vec{\alpha})$ an element of a n-dimensional Lie-group G, $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

We can do a Taylor expansion around $\vec{\alpha} = \vec{0}$ with $D(\vec{0}) = e$:

$$\begin{aligned} D(\vec{\alpha}) &= D(\vec{0}) + \sum_a \frac{\partial}{\partial \alpha_a} D(\vec{\alpha})_{|\vec{\alpha}=0} \alpha_a + \ldots \\ &= e + i \sum_a \alpha_a T^a + \ldots \end{aligned}$$

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The T^a $(a = 1, \ldots, n)$ are the generators of the Lie group:

$$T^a := -i \left[\frac{\partial}{\partial \alpha_a} D(\vec{\alpha}) \right]_{|\vec{\alpha}=0}$$

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&= e + i \sum_a \alpha_a T^a + \ldots
\end{aligned}
$$

The T^a $(a = 1, \ldots, n)$ are the generators of the Lie group:

$$
T^a := -i \left[\frac{\partial}{\partial \alpha_a} D(\vec{\alpha}) \right]_{|\vec{\alpha}=0}
$$

The group element for general $\vec{\alpha}$ can be recovered
by exponentiation:

$$
D(\vec{\alpha}) = \lim_{k \to \infty} (e + \sum_a \frac{i \alpha_a T^a}{k})^k = e^{i \sum_a \alpha_a T^a}
$$

Lie algebra

- The generators T^a form a **basis** of a **Lie algebra**

Def.: A **Lie algebra g** is a vector space together with a skew-symmetric bilinear map [,]: g x g → g (called the Lie bracket) which satisfies the Jacobi identity

Lie algebra

- The generators T^a form a **basis** of a **Lie algebra**
- $[T^a, T^b] = i f^{ab}{}_c T^c$ (Einstein convention)
- The $f^{ab}{}_c$ are called **structure constants**

Def.: A **Lie algebra g** is a vector space together with a skew-symmetric bilinear map [,]: g x g → g (called the Lie bracket) which satisfies the Jacobi identity

Lie algebra

- The generators T^a form a **basis** of a **Lie algebra**
- $[T^a, T^b] = i f^{ab}{}_c T^c$ (Einstein convention)
- The $f^{ab}{}_c$ are called **structure constants**
- Any group element connected to the neutral element can be generated using the generators:

  $\mathbf{g = \exp(i\, c_a T^a)}$ (Einstein convention)

Def.: A **Lie algebra g** is a vector space together with a skew-symmetric bilinear map [,]: g x g → g (called the Lie bracket) which satisfies the Jacobi identity

Rank

- Rank = Number of simultanesouly diagonalizable generators
- Rank = Number of good quantum numbers
- Rank = Dimension of the Cartan subalgebra

- Rank[SU(2)] = 1, Rank[SU(3)] = 2, Rank[$ISO_+(1,3)$] = 2
- Rank[$G_1 x G_2$] = Rank[G_1] + Rank[G_2]
- Rank[G_{SM}] = Rank[SU(3)xSU(2)xU(1)] = 2 + 1 + 1 = 4

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Symmetries II
(Representations)

Representations of a group

- Def.: A linear representation of a group G on a vector space V is a group homomorphism $D: G \to GL(V)$.

- Remarks:

  - $g \mapsto D(g)$, where $D(g)$ is a linear operator acting on V

  - The operators $D(g)$ preserve the group structure:
    $D(g_1\, g_2) = D(g_1)\, D(g_2)$, $D(e)$ = identity operator

  - V is called the base space, $\dim V$ = dimension of the representation

Representations of a group

- A representation (D,V) is <u>reducible</u> if a non-trivial subspace $U \subset V$ exists which is invariant with respect to D:

  $\forall g \in G: \forall \mathbf{u} \in U: D(g)\mathbf{u} \in U$

- A representation (D,V) is <u>irreducible</u> if it is not reducible

- A representation (D,V) is <u>completely reducible</u> if all $D(g)$ can be written in block diagonal form (with suitable base choice)

Representations of a Lie algebra

- Def.: A linear representation of a Lie algebra A on a vector space V is an algebra homomorphism $D: A \to End(V)$.

- Remarks:

  - $t \mapsto T = D(t)$, where T is a linear operator acting on V

  - The operators D(t) preserve the algebra structure:
    $[t^a, t^b] = i f^{ab}{}_c\, t^c \to [T^a, T^b] = i f^{ab}{}_c\, T^c$

  - A representation for the Lie algebra induces a representation for the Lie group

Tensor product

Composite systems are described mathematically by the **tensor product of representations**

- Tensor products of irreps are in general reducible!
- They are a direct sum of irreps: Clebsch-Gordan decomposition
- Examples:
  - System of two spin-1/2 electrons
  - Mesons: quark-anti-quark systems, Baryons: systems of three quarks

Symmetries III
(Space-time symmetry)

Space-time symmetry

- The minimal symmetry of a (relativistic) QFT is the **Poincaré symmetry**
- **Observables** should not change under Poincaré transformations of
  - Space-time coordinates $x = (t,\mathbf{x})$
  - Fields $\phi(x)$
  - States of the Hilbert space $|\mathbf{p}, \ldots\rangle$
- Need to know how the group elements are **represented** as operators acting on these objects (space-time, fields, states)
- At the classical level **Poincaré invariant Lagrangians** is all we need

Poincaré algebra I

- Poincaré group = Lorentz group $SO_+(1,3)$ + Translations
- Lorentz group has 6 generators: $J_{\mu\nu} = - J_{\nu\mu}$

  Lorentz algebra: $[J_{\mu\nu} , J_{\rho\sigma}]= -i (\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} - [\mu \leftrightarrow \nu])$

- Poincaré group has 10=6+4 generators: $J_{\mu\nu} , P_\mu$

  Poincaré algebra:
  $[P_\mu,P_\nu]=0,[J_{\mu\nu},P_\lambda]=i(\eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu)$,Lorentz algebra

Poincaré algebra II

- Poincaré group has 10=6+4 generators: $J_{\mu\nu}$, P_μ
  - 3 Rotations → angular momentum $J_i=1/2\ \varepsilon_{ijk} J_{jk}$
    $[J_i,J_j] = i\ \varepsilon_{ijk}\ J_k$
  - 3 Boosts → $K_i = J_{0i}$
    $[K_i,K_j] = -i\ \varepsilon_{ijk}\ J_k$; $[J_i,K_j] = i\ \varepsilon_{ijk}\ K_k$
  - 4 Translations → energy/momentum P_μ
    $[J_i,P_j] = i\ \varepsilon_{ijk}\ P_k$, $[K_i,P_j] = -i\ \delta_{ij}\ P_0$, $[P_0,J_i] = 0$, $[P_0,K_i] = i\ P_i$

Tensor representations of so(1,3) (integer spin)

- All physical quantities can be classified according to their transformation properties under the Lorentz group

- Representations characterized by two invariants: **mass**, **spin** (Casimir operators P^2, W^2)

- Physical particles are irreps of the Poincaré group:

ϕ = scalar, V_μ = vector, $T_{\mu\nu}$ = tensor, ...

s=0 s=1 s=2

Spinor representations of so(1,3) (half integer spin)

- so(1,3) ~ sl(2,**C**) ~ $su(2)_L \times su(2)_R$

  $J_m^+ := J_m + i K_m$, $J_m^- := J_m - i K_m$: $[J_m^+, J_n^-] = 0$, $[J_i^+, J_j^+] = i\, \varepsilon_{ijk} J_k^+$, $[J_i^-, J_j^-] = i\, \varepsilon_{ijk} J_k^-$

- $su(2)_{L,R}$ labelled by $j_{L,R}$ = 0, 1/2, 1, 3/2, 2, ...
  - (j_L, j_R) = (0,0) scalar
  - (1/2,0) left-handed Weyl spinor; (0,1/2) right-handed Weyl spinor
  - (1/2,1/2) vector
- Dirac spinor = (1/2,0) + (0,1/2) is reducible (not fundamental)
  Note: (1/2,0) and (0,1/2) can have different interactions

Representation of so(1,3) on fields

- A field $\phi(x)$ is a function of the coordinates
- Lorentz transformation: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$, $\phi \to \phi'$
- Scalar field: $\phi'(x') = \phi(x)$

  At the same time $\phi'(x') = \exp(i\ 1/2\ \omega_{\mu\nu}\, J^{\mu\nu})\ \phi(x)$

  Comparison allows to find a concrete expression for $J^{\mu\nu}$:
  $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$ with $S^{\mu\nu}=0$, $L^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu$ where $P^\mu = i\ \partial^\mu$
- Similar procedure for Weyl, Dirac, Vector fields, ...
  and for the full Poincaré group

Symmetries IV
(Unitary symmetries)

Internal symmetries

- Coleman-Mandula theorem:

  The <u>most general</u> symmetry of a relativistic QFT:

  Space-time symmetry x Internal symmetry (**direct product**)

- Algebra: **direct sum** of space-time generators and internal symmetry generators
  - 3 rotations
  - 3 boosts
  - 4 translations
  - generators T^a of internal symmetry

SU(n)

- Group: $SU(n) = \{U \in M_n(\mathbb{C}) \mid U^\dagger U = \mathbf{1}_n, \det U = 1\}$
- Algebra: $su(n) = \{t \in M_n(\mathbb{C}) \mid tr(t) = 0, t^\dagger = t\}$
- $\dim SU(n) = \dim su(n) = n^2-1$
- $\text{rank } su(n) = n-1$
- Important representations (D,V):
  - The fundamental representation: **n** (V is an n-dimensional vector space)
  - The anti-fundamental representation: **n***
  - The adjoint representation: $V = su(n)$, dimension of adjoint representation = n^2-1

SU(2)

- $\dim SU(2) = \dim su(2) = 2^2\text{-}1 = 3$
- $\text{rank } su(2) = 2\text{-}1 = 1$
- Algebra: $[t_k,t_l]=i\ \varepsilon_{klm}\ t_m$
- The fundamental representation: **2**
  $T_i = 1/2\ \sigma_i$ (i=1,2,3), σ_i Pauli matrices
- irreps: Basis states $|j,j_z>$, $j=0,1/2,1,3/2,2,\ldots$; $j_z=-j, -j+1, \ldots, j-1, j$

SU(3)

- dim SU(3) = dim su(3) = 3^2-1 = 8

- rank su(3) = 3-1 = 2

- Algebra: $[t_a,t_b]=i\, f_{abc}\, t_c$

- The fundamental representation: **3**
  $T_i = 1/2\ \lambda_i$ (i=1,2,3), λ_i Gell-Mann matrices

- The structure constants can be calculated using the generators in the fundamental irrep: $f_{abc} = -2i\, Tr([Ta,Tb]Tc)$

- irreps: labeled by 2 integer numbers (rank = 2)

Glossary of Group Theory: I. Basics

- Group
  - discrete, continuous, Abelian, non-Abelian
  - subgroup = subset which is a group
  - invariant subgroup = normal subgroup
  - simple group = has no *proper* invariant subgroups
- Lie group: continuous group which depends differentiably on its parameters
  - dimension = number of essential parameters
- Lie algebra
  - generators = basis of the Lie algebra; elements of the tangent space T_eG
  - dimension = number of linearly independent generators
  - structure constants = specifiy the algebra (basis dependent)
  - subalgebra = subset which is an algebra
  - ideal = invariant subalgebra
  - simple algebra = has no *proper* ideals (smallest building block; irreducible)
  - semi-simple algebra = direct sum of simple algebras

Glossary of Group Theory: II. Representations

- Representations
  - of groups
  - of algebras
  - equivalent, unitary, reducible, entirely reducible
  - irreducible representations (irreps)
  - fundamental representation
  - adjoint representation
- Direct sum of two representations
- Tensor product of two representations
  - Clebsch-Gordan decomposition
  - Clebsch-Gordan coefficients
- Quadratic Casimir operator
- Dynkin index

Glossary of Group Theory: III. Cartan-Weyl

- Cartan-Weyl analysis of simple Lie algebras: G = H ⊕ E
  - H = Cartan subalgebra = maximal Abelian subalgebra of G
  - rank G = dimension of Cartan subalgebra = number of simultaneously diagonalisable operators
  - E = space of ladder operators
  - Root vector (labels the ladder operators)
    - positive roots = if first non-zero component positive (basis dependent)
    - simple roots = positive root which is *not* a linear combination of other positive roots with positive coefficients
  - Weight vector (quantum numbers of the physical states)
    - heighest weight

Glossary of Group Theory: IV. Dynkin

- Dynkin diagrams
  - complete classification of all simple Lie algebras by Dynkin
  - Dynkin diagrams ↔ simple roots → roots → ladder operators
  - Dynkin diagrams ↔ simple roots → roots → geometrical interpretation of commutation relations
- Cartan matrix
  - Simple Lie algebra ↔ root system ↔ simple roots ↔ Dynkin diagrams ↔ Cartan matrix
- Dynkin lables (of a weight vector)
- Dynkin diagrams + Dynkin labels ⇒ recover whole algebra structure
  - analysis of any irrep of any simple Lie algebra (non-trivial in other notations)
  - tensor products
  - subgroup structure, branching rules