Theory status of the muon $g-2$

Andreas Nyffeler

PRISMA Cluster of Excellence, Institut für Kernphysik,
Helmholtz-Institut Mainz
Johannes Gutenberg Universität Mainz, Germany
nyffeler@kph.uni-mainz.de

Current Trends in Flavor Physics
Institut Henri Poincaré, Paris, 29 - 31 March 2017

Outline

- Basics of the anomalous magnetic moment
- Muon $g-2$: QED, weak interactions, hadronic contributions
- Hadronic vacuum polarization (HVP)
- Hadronic light-by-light scattering (HLbL)
- New Physics contributions to the muon $g-2$
- Conclusions and Outlook

Basics of the anomalous magnetic moment

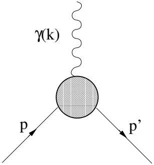
Electrostatic properties of charged particles:
Charge Q, Magnetic moment $\vec{\mu}$, Electric dipole moment $\vec{d}$

For a spin 1/2 particle:

$$\vec{\mu} = g \frac{e}{2m} \vec{s}, \qquad \underbrace{g = 2}_{\text{Dirac}}(1 + a), \qquad a = \frac{1}{2}(g - 2) : \text{anomalous magnetic moment}$$

Long interplay between experiment and theory: structure of fundamental forces

In Quantum Field Theory (with C,P invariance):



$$= \quad (-ie)\bar{\mathrm{u}}(p') \left[\gamma^\mu \underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu} k_\nu}{2m} \underbrace{F_2(k^2)}_{\text{Pauli}} \right] \mathrm{u}(p)$$

$$F_1(0) = 1 \quad \text{and} \quad F_2(0) = a$$

a_e: Test of QED. Most precise determination of $\alpha = e^2/4\pi$.
a_μ: Less precisely measured than a_e, but all sectors of Standard Model (SM),
i.e. QED, Weak and QCD (hadronic), contribute significantly.
Sensitive to possible contributions from New Physics. Often (but not always !):

$$a_\ell \sim \left(\frac{m_\ell}{m_{\text{NP}}}\right)^2 \Rightarrow \left(\frac{m_\mu}{m_e}\right)^2 \sim 43000 \text{ more sensitive than } a_e \text{ [exp. precision} \to \text{factor 19]}$$

Muon $g-2$: current status

Theory (Standard Model): $a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{had}}$

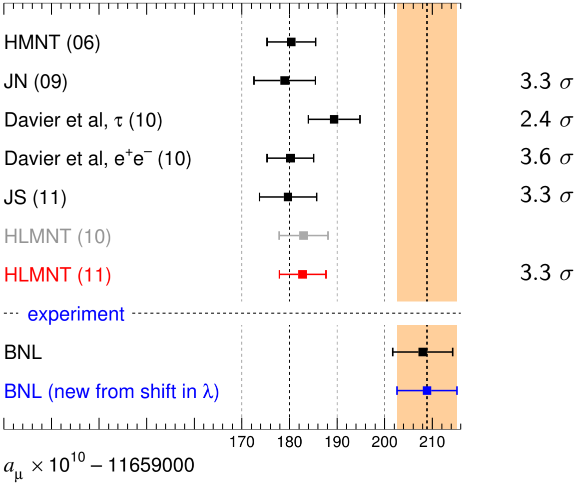
$$
\begin{aligned}
a_\mu^{\text{SM}} &= (116\,591\,780 \pm 53) \times 10^{-11} && \text{(various sources)} \\
a_\mu^{\text{exp}} &= (116\,592\,089 \pm 63) \times 10^{-11} && \text{(Bennett et al. (BNL) '06)} \\
a_\mu^{\text{exp}} - a_\mu^{\text{SM}} &= (309 \pm 82) \times 10^{-11} \qquad (3.8\,\sigma)
\end{aligned}
$$

Discrepancy a sign of New Physics ?

Largest source of error in SM prediction: hadronic uncertainties.

Need to be better controlled in order to fully profit from future $g-2$ experiments at Fermilab (E989) and J-PARC (E34) with $\delta a_\mu^{\text{exp}} = 16 \times 10^{-11}$ (0.14 ppm).

Muon $g-2$: other recent theoretical evaluations



Source: Hagiwara et al. '11. Note units of 10^{-10} !

Aoyama et al. '12: $a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = (249 \pm 87) \times 10^{-11}$ $[2.9\ \sigma]$

Benayoun et al. '15: $a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = (376.8 \pm 75.3) \times 10^{-11}$ $[5.0\ \sigma]$

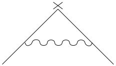
Jegerlehner '15: $a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = (310 \pm 82) \times 10^{-11}$ $[3.8\ \sigma]$

Muon $g-2$: Theory

In Standard Model: $a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{had}}$

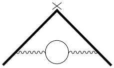
QED contributions

- At 1-loop: Schwinger's result '48 (a_μ dimensionless):



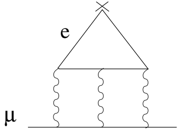
$$= \frac{\alpha}{2\pi}$$

- Diagrams with internal electron loops are enhanced.
  - At 2-loops: vacuum polarization from electron loops enhanced by QED short-distance logarithm



$$= \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + \mathcal{O}\left(\frac{m_e}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

  - At 3-loops: light-by-light scattering from electron loops enhanced by QED infrared logarithm [Aldins et al. '69, '70; Laporta, Remiddi '93]



$$+ \ldots \qquad = \left[\frac{2}{3}\pi^2 \ln\frac{m_\mu}{m_e} + \ldots\right]\left(\frac{\alpha}{\pi}\right)^3 = 20.947\ldots\left(\frac{\alpha}{\pi}\right)^3$$

- Loops with tau's suppressed (decoupling)

QED result up to 5 loops

Include contributions from all leptons (Schwinger '48; ...; Aoyama et al. '12):

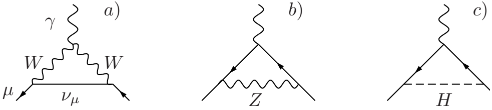
$$
\begin{aligned}
a_\mu^{\mathrm{QED}} &= 0.5 \times \left(\frac{\alpha}{\pi}\right) + 0.765\,857\,425 \underbrace{(17)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^2 \\
&\quad + 24.050\,509\,96 \underbrace{(32)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^3 + 130.8796 \underbrace{(63)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^4 \\
&\quad + 753.29 \underbrace{(1.04)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^5 \\
&= 116\,584\,718.853 \underbrace{(9)}_{m_\mu/m_{e,\tau}} \underbrace{(19)}_{c_4} \underbrace{(7)}_{c_5} \underbrace{(29)}_{\alpha(a_e)} [36] \times 10^{-11}
\end{aligned}
$$

- Up to 3-loop analytically known (Laporta, Remiddi '93).
- 4-loop: analytical results for electron and tau-loops (asymptotic expansions) by Steinhauser et al. '15 + '16.
- Earlier evaluation of 5-loop contribution yielded $c_5 = 662(20)$ (Kinoshita, Nio '06, numerical evaluation of 2958 diagrams, known or likely to be enhanced). New value is 4.5σ from this leading log estimate and 20 times more precise.
- Aoyama et al. '12: What about the 6-loop term ? Leading contribution from light-by-light scattering with electron loop and insertions of vacuum-polarization loops of electrons into each photon line $\Rightarrow a_\mu^{\mathrm{QED}}(\text{6-loops}) \sim 0.1 \times 10^{-11}$

Contributions from weak interaction

Numbers from recent reanalysis by Gnendiger et al. '13.

1-loop contributions [Jackiw, Weinberg '72; ...]:



$$a_\mu^{\text{weak, (1)}}(W) = \frac{\sqrt{2}G_\mu m_\mu^2}{16\pi^2}\frac{10}{3} + \mathcal{O}(m_\mu^2/M_W^2) = 388.70 \times 10^{-11}$$

$$a_\mu^{\text{weak, (1)}}(Z) = \frac{\sqrt{2}G_\mu m_\mu^2}{16\pi^2}\frac{(-1+4s_W^2)^2 - 5}{3} + \mathcal{O}(m_\mu^2/M_Z^2) = -193.89 \times 10^{-11}$$

Contribution from Higgs negligible: $a_\mu^{\text{weak, (1)}}(H) \leq 5 \times 10^{-14}$ for $m_H = 126$ GeV.

$$a_\mu^{\text{weak, (1)}} = (194.80 \pm 0.01) \times 10^{-11}$$

2-loop contributions (1678 diagrams) [Czarnecki et al. '95, '96; ...]:

$$a_\mu^{\text{weak, (2)}} = (-41.2 \pm 1.0) \times 10^{-11}, \quad \text{large since} \sim G_F m_\mu^2 \frac{\alpha}{\pi} \ln \frac{M_Z}{m_\mu}$$

Total weak contribution:

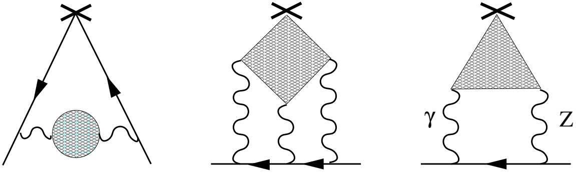
$$a_\mu^{\text{weak}} = (153.6 \pm 1.0) \times 10^{-11}$$

Under control ! With knowledge of $M_H = 125.6 \pm 1.5$ GeV, uncertainty now mostly hadronic $\pm 1.0 \times 10^{-11}$ (Peris et al. '95; Knecht et al. '02; Czarnecki et al. '03, '06).
3-loop effects via RG: $\pm 0.20 \times 10^{-11}$ (Degrassi, Giudice '98; Czarnecki et al. '03).

Hadronic contributions to the muon $g-2$

Largest source of uncertainty in theoretical prediction of a_μ !

Different types of contributions:



(a) (b) (c)

Light quark loop not well defined $\rightarrow$ Hadronic "blob"

(a) Hadronic vacuum polarization (HVP) $\mathcal{O}(\alpha^2), \mathcal{O}(\alpha^3), \mathcal{O}(\alpha^4)$

(b) Hadronic light-by-light scattering (HLbL) $\mathcal{O}(\alpha^3), \mathcal{O}(\alpha^4)$

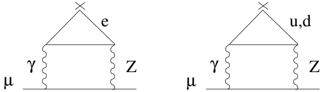
(c) 2-loop electroweak contributions $\mathcal{O}(\alpha G_F m_\mu^2)$

2-Loop EW

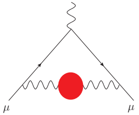
Small hadronic uncertainty from triangle diagrams.

Anomaly cancellation within each generation !

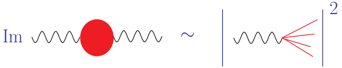
Cannot separate leptons and quarks !



Hadronic vacuum polarization

$a_\mu^{\text{HVP}} =$ 

Optical theorem (from unitarity; conservation of probability) for hadronic contribution $\to$ dispersion relation:



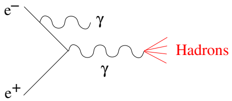
$\sigma(e^+e^- \to \gamma^* \to \text{hadrons})$

$$a_\mu^{\text{HVP}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s}\, K(s)\, R(s), \qquad R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

[Bouchiat, Michel '61; Durand '62; Brodsky, de Rafael '68; Gourdin, de Rafael '69]

$K(s)$ slowly varying, positive function $\Rightarrow$ a_μ^{HVP} positive. Data for hadronic cross section σ at low center-of-mass energies $\sqrt{s}$ important due to factor $1/s$: $\sim 70\%$ from $\pi\pi$ [$\rho(770)$] channel, $\sim 90\%$ from energy region below 1.8 GeV.

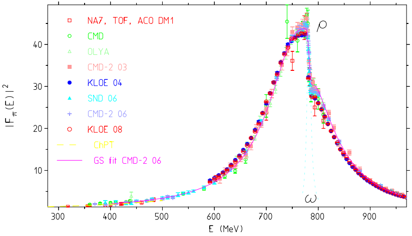
Other method instead of energy scan: Radiative return (initial state radiation) at colliders with fixed center-of-mass energy (DAΦNE, B-Factories, BEPC) [Binner et al. '99; Czyż et al. '00-'03]



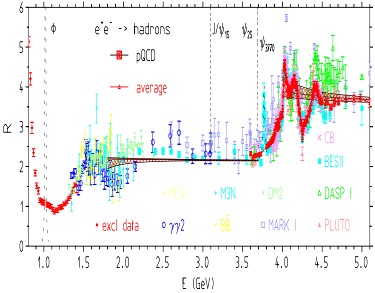
Measured hadronic cross-section

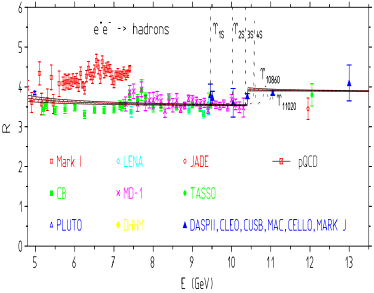
Pion form factor $|F_\pi(E)|^2$
($\pi\pi$-channel)

$$R(s) \quad = \quad \frac{1}{4}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{3}{2}} |F_\pi(s)|^2$$
$$(4m_\pi^2 < s < 9m_\pi^2)$$



R-ratio:

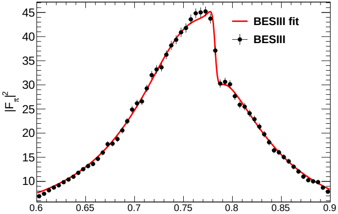




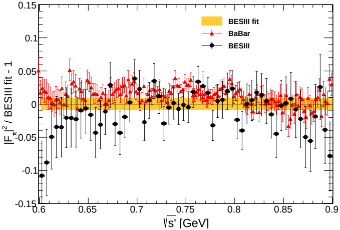
Jegerlehner, AN '09

New results on $e^+e^- \to \pi^+\pi^-$ from BESIII

Ablikim et al. (BESIII Collaboration) '16

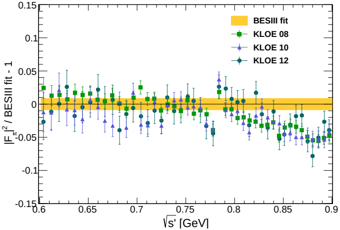


Fit of BESIII data with parametrization of pion form factor by Gounaris-Sakurai. Only statistical errors shown.



BaBar data higher than BESIII below ρ-mass, better agreement above.

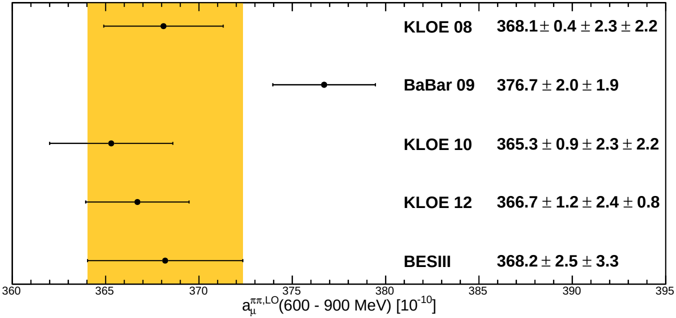
Statistical and systematic errors included in data points. Width of BESIII fit band shows systematic uncertainty only.



Good agreement with KLOE 08 and KLOE 12 up to mass range of $\rho - \omega$ interference, but disagreement with all three data sets at higher energy.

New results on $e^+e^- \to \pi^+\pi^-$ from BESIII (continued)

Comparison of value for $a_\mu^{\mathrm{HVP}}(600 - 900\ \mathrm{MeV})$ from the three experiments using radiative return method (initial state radiation):



Results from BESIII confirm KLOE, disagree with BaBar at level of $1 - 2\ \sigma$ (at least after integration in dispersion integral to get a_μ^{HVP}).

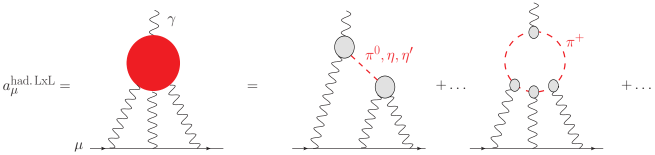
Hadronic vacuum polarization: some recent evaluations

Authors	Contribution to $a_\mu^{\text{HVP}} \times 10^{11}$
Davier et al. '11, '14 (e^+e^-) [$+\ \tau$]	6923 ± 42 [7030 ± 44]
Jegerlehner, Szafron '11 (e^+e^-) [$+\ \tau$]	6907.5 ± 47.2 [6909.6 ± 46.5]
Hagiwara et al. '11 (e^+e^-)	6949.1 ± 42.7
Benayoun at al. '15 ($e^+e^- + \tau$: BHLS improved)	6818.6 ± 32.0
Jegerlehner '15 (e^+e^-) [$+\ \tau$]	6885.7 ± 42.8 [6889.1 ± 35.2]
Davier '16 (e^+e^-)	6926 ± 33

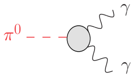
- Precision: $< 1\%$. Non-trivial because of radiative corrections (radiated photons).
- Even if values for a_μ^{HVP} after integration agree quite well, the systematic differences of a few % in the shape of the spectral functions from different experiments (BABAR, BESIII, CMD-2, KLOE, SND) indicate that we do not yet have a complete understanding.
- Use of τ data: additional sources of isospin violation ? Ghozzi, Jegerlehner '04; Benayoun et al. '08, '09; Wolfe, Maltman '09; Jegerlehner, Szafron '11 ($\rho - \gamma$-mixing), also included in Jegerlehner '15 and in BHLS-approach by Benayoun et al. '15 (additional BHLS model uncertainty can lead to maximal shift in central value of $^{+15}_{-27} \times 10^{-11}$).
- Lattice QCD: Various groups are working on it, precision at level of about 3-5% (systematics dominated), not yet competitive with phenomenological evaluations.

Hadronic light-by-light scattering

HLbL in muon $g-2$ from strong interactions (QCD):



Coupling of photons to hadrons, e.g. π^0, via form factor:



Relevant scales ($\langle VVVV \rangle$ with offshell photons): $0-2$ GeV $\gg m_\mu$ (resonance region)

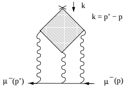
View before 2014: in contrast to HVP, no direct relation to experimental data
$\rightarrow$ size and even sign of contribution to a_μ unknown !

Approach: use hadronic model at low energies with exchanges and loops of resonances and some (dressed) "quark-loop" at high energies.

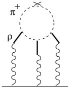
Constrain models using experimental data (processes of hadrons with photons: decays, form factors, scattering) and theory (ChPT at low energies; short-distance constraints from pQCD / OPE at high momenta).

Problems: Four-point function depends on several invariant momenta $\Rightarrow$ distinction between low and high energies not as easy as for two-point function in HVP.
Mixed regions: one loop momentum Q_1^2 large, the other Q_2^2 small and vice versa.

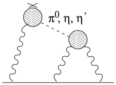
HLbL in muon $g-2$: summary of selected results (model calculations)



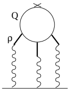
$=$



$+ \cdots +$



$+ \cdots +$ Exchange of other resonances ($f_0, a_1, f_2 \ldots$) $+$

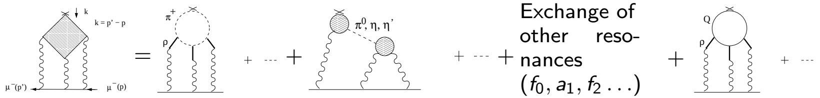


$+ \cdots$

de Rafael '94:

Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

HLbL in muon $g-2$: summary of selected results (model calculations)



de Rafael '94:

	p^4	p^6	p^8	p^8
Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Contribution to $a_\mu \times 10^{11}$:

BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) [f_0, a_1]	+2.3 [c-quark]
N,JN:	+116 (39)	-19 (13)	+99 (16)	+15 (7) [f_0, a_1]	+21 (3)
		ud.: -45	ud.: $+\infty$		ud.: +60

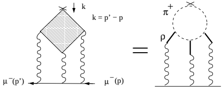
ud. = undressed, i.e. point vertices without form factors

Pseudoscalars: numerically dominant contribution (according to most models !).

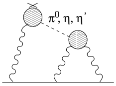
Recall (in units of 10^{-11}): $\delta a_\mu(\mathrm{HVP}) \approx 40$; $\delta a_\mu(\mathrm{exp\ [BNL]}) = 63$; $\delta a_\mu(\mathrm{future\ exp}) = 16$

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09 (compilation; "Glasgow consensus"); N,JN = AN '09; Jegerlehner, AN '09 (compilation)

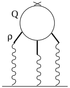
HLbL in muon $g-2$: summary of selected results (model calculations)



$+ \cdots +$



$+ \cdots +$ Exchange of other resonances $(f_0, a_1, f_2 \ldots)$ $+$



$+ \cdots$

de Rafael '94:

Chiral counting:		p^4	p^6	p^8	p^8
N_C-counting:		1	N_C	N_C	N_C

Contribution to $a_\mu \times 10^{11}$:

BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) $[f_0, a_1]$	+21 (3)
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) $[a_1]$	+10 (11)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) $[a_1]$	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) $[f_0, a_1]$	+2.3 [c-quark]
N,JN:	+116 (39)	-19 (13)	+99 (16)	+15 (7) $[f_0, a_1]$	+21 (3)
		ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

Pseudoscalars: numerically dominant contribution (according to most models !).

Recall (in units of 10^{-11}): $\delta a_\mu(\text{HVP}) \approx 40$; $\delta a_\mu(\text{exp [BNL]}) = 63$; $\delta a_\mu(\text{future exp}) = 16$

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09 (compilation; "Glasgow consensus"); N,JN = AN '09; Jegerlehner, AN '09 (compilation)

Recent reevaluations of axial vector contribution lead to much smaller estimates than in MV '04: $a_\mu^{\mathrm{HLbL;axial}} = (8 \pm 3) \times 10^{-11}$ (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15). Would shift central values of compilations downwards:

$a_\mu^{\mathrm{HLbL}} = (98 \pm 26) \times 10^{-11}$ (PdRV) and $a_\mu^{\mathrm{HLbL}} = (102 \pm 39) \times 10^{-11}$ (N, JN).

HLbL in muon $g - 2$

- Frequently used estimates:

$$a_\mu^{\rm HLbL} = (105 \pm 26) \times 10^{-11} \quad \text{(Prades, de Rafael, Vainshtein '09)}$$

("Glasgow consensus")

$$a_\mu^{\rm HLbL} = (116 \pm 39) \times 10^{-11} \quad \text{(AN '09; Jegerlehner, AN '09)}$$

Based almost on same input: calculations by various groups using different models for individual contributions. Error estimates are mostly guesses !

- Need much better understanding of complicated hadronic dynamics to get reliable error estimate of $\pm 20 \times 10^{-11}$ ($\delta a_\mu(\text{future exp}) = 16 \times 10^{-11}$).
- Recent new proposal: Colangelo et al. '14, '15; Pauk, Vanderhaeghen '14: use dispersion relations (DR) to connect contribution to HLbL from presumably numerically dominant light pseudoscalars to in principle measurable form factors and cross-sections:

$$\begin{aligned} \gamma^*\gamma^* &\to \pi^0, \eta, \eta' \\ \gamma^*\gamma^* &\to \pi^+\pi^-, \pi^0\pi^0 \end{aligned}$$

Could connect HLbL uncertainty to exp. measurement errors, like HVP. Note: no data yet with two off-shell photons !

- Future: HLbL from Lattice QCD (model-independent, first-principle). First steps and results: Blum et al. (RBC-UKQCD) '05, ..., '16, '17. Work ongoing by Mainz group: Green et al. '15; Asmussen et al. '16.

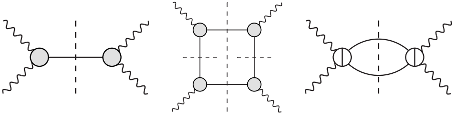
Data-driven approach to HLbL using dispersion relations

Strategy: Split contributions to HLbL into two parts:

- I: Data-driven evaluation using DR (hopefully numerically dominant):
  - (1) π^0, η, η' poles
  - (2) $\pi\pi$ intermediate state
- II: Model dependent evaluation (hopefully numerically subdominant):
  - (1) Axial vectors (3π-intermediate state), ...
  - (2) Quark-loop, matching with pQCD

Error goals: Part I: 10% precision (data driven), Part II: 30% precision.
To achieve overall error of about 20% ($\delta a_\mu^{\rm HLbL} = 20 \times 10^{-11}$).

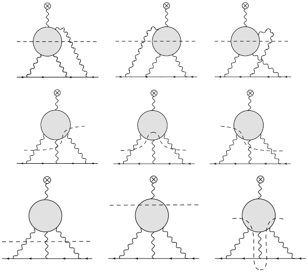
Colangelo et al. '14, '15:
Classify intermediate states in 4-point function. Then project onto $g-2$.



Colangelo et al. '17: pion-box contribution (middle diagram) using precise information on pion vector form factor and S-wave $\pi\pi$-rescattering effects from pion-pole in left-hand cut (LHC) (part of right diagram):

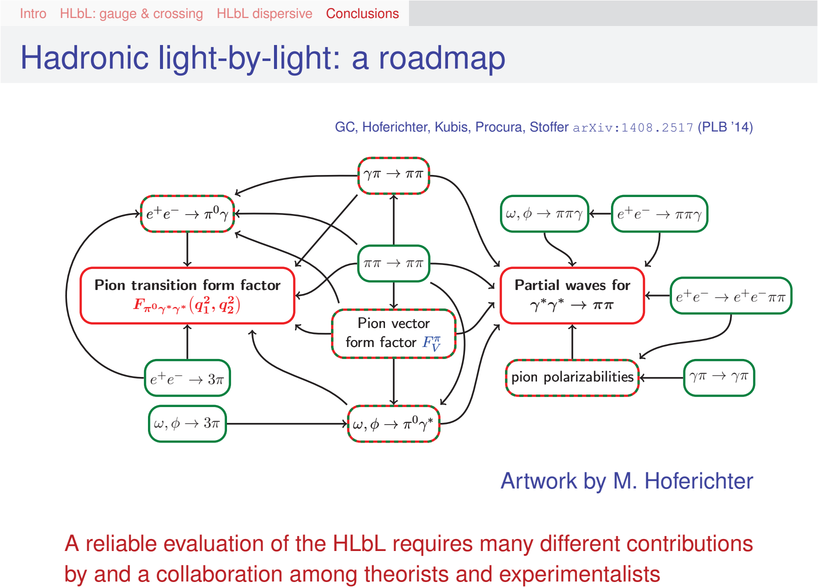
$$
\begin{aligned}
a_\mu^{\pi-\mathrm{box}} &= -15.9(2) \times 10^{-11} \\
a_{\mu,J=0}^{\pi\pi,\pi-\mathrm{pole\ LHC}} &= -8(1) \times 10^{-11} \\
\text{Sum of the two} &= -24(1) \times 10^{-11}
\end{aligned}
$$

Pauk, Vanderhaeghen '14:
DR directly for Pauli FF $F_2(k^2)$.



HLbL sum rules to get constraints from data on models: Pascalutsa, Vanderhaeghen '10; Pascalutsa, Pauk, Vanderhaeghen '12; Danilkin, Vanderhaeghen '17

Data-driven approach to HLbL using dispersion relations (continued)

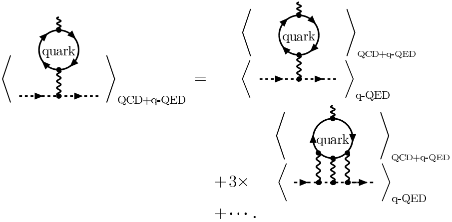


From talk by Colangelo at Radio Monte Carlo Meeting, Frascati, May 2016

HLbL in muon $g-2$ from Lattice QCD: RBC-UKQCD approach

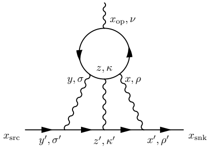
Blum, Hayakawa, et al. '05, ..., '15:

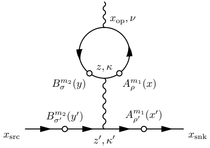
- Put QCD + (quenched) QED on the lattice (cf. talk by Andreas Jüttner).
- QED treated non-perturbatively $\Rightarrow$ all orders in α
- Need to subtract lower order non-HLbL contribution $\Rightarrow$ very noisy on the lattice. First signal for $F_2(q^2)$ for $q^2 \geq 0.11$ GeV^2 only in '15.



Jin et al. '15, '16, '17:

- Step by step improvement of method to reduce statistical error by one or two orders of magnitude and remove some systematic errors.
- Perturbative expansion in QED to deal only with HLbL contribution (no subtraction needed).
- Exact propagator on lattice between z, z'. Stochastic photon propagators between x, x' and y, y'.
- Calculate $a_\mu^{\mathrm{HLbL}} = F_2(q^2 = 0)$ via moment method in position-space (no extrapolation to $q^2 = 0$ needed).

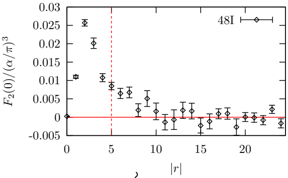


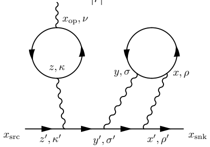


HLbL in muon $g-2$ from Lattice QCD: RBC-UKQCD approach (cont.)

Jin et al. '16, '17

- Later used exact expression for all photon propagators. Treat $r = x - y$ stochastically by sampling points x, y. Found empirically: short-distance contribution at small $|r|$ dominates.
- Take all points with $|r| \leq r_{\max} \sim 4-6$ (in lattice units, corresponds to about 0.6 fm). Sample with empirical weight above $r_{\max}$.
- Test: Reproduce result for QED with muon loop after extrapolation to $a = 0$ and $L = \infty$.
- Calculate leading quark-disconnected diagrams (dHLbL).





Results (for $m_\pi = m_{\pi,\text{phys}}$, lattice spacing $a^{-1} = 1.73$ GeV, $L = 5.5$ fm):

$$
\begin{aligned}
a_\mu^{\text{cHLbL}} &= (116.0 \pm 9.6) \times 10^{-11} \\
a_\mu^{\text{dHLbL}} &= (-62.5 \pm 8.0) \times 10^{-11} \\
a_\mu^{\text{HLbL}} &= (53.5 \pm 13.5) \times 10^{-11}
\end{aligned}
$$

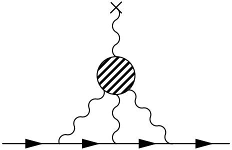
Beware ! Statistical error only ! Missing systematic effects:

- Expect large finite-volume effects from QED $\sim 1/L^2$: put small QCD box into larger QED box ?
- Expect large finite-lattice-spacing effects.
- Omitted subleading quark-disconnected diagrams (10% effect ?).

HLbL in muon $g-2$ from Lattice QCD: Mainz approach

Independently developed approach (Asmussen, Green, Meyer, AN)

Talk by Asmussen at German Physical Society (DPG) Meeting, March 2015,
Green et al., Lattice 2015 (arXiv:1510.08384),
Asmussen, Green, Meyer, AN, Lattice 2016 (arXiv:1609.08454), work in progress.



- QCD blob: lattice regularization
- Everything else: position-space perturbation theory in Euclidean formulation

Similarities to approach by RBC-UKQCD '15, '16, '17:

- Position space (most natural for lattice QCD)
- Perturbative treatment of the QED part
- Get directly $a_\mu^{\mathrm{HLbL}} = F_2(k^2 = 0)$ as spatial moment

Differences (strengths of our approach):

- Semi-analytical calculation
- QED part computed in continuum and in infinite volume
- No power law effects $1/L^2$ in the volume

Challenges:

- Need to calculate a QCD four-point function on the lattice
- Numerical efficiency not yet shown

HLbL in muon $g-2$ in position space

Project on anomalous magnetic moment (Euclidean space):

$$a_\mu^{\mathrm{HLbL}} = F_2(0) = \frac{-i}{48m} \mathrm{Tr}\{[\gamma_\rho, \gamma_\sigma](-i\not{p} + m)\Gamma_{\rho\sigma}(p,p)(-i\not{p} + m)\}$$

with on-shell muon momentum $p = im\hat{\epsilon}$ ($p^2 = -m^2$; $\hat{\epsilon}$: unit vector).

Vertex function in terms of position-space functions:

$$\begin{aligned}
\Gamma_{\rho\sigma}(p,p) &= -\, e^6 \int_{x,y} K_{\mu\nu\lambda}(x,y,p)\, \widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \\
K_{\mu\nu\lambda}(x,y,p) &= \gamma_\mu(i\not{p} + \not{\partial}^{(x)} - m)\gamma_\nu(i\not{p} + \not{\partial}^{(x)} + \not{\partial}^{(y)} - m)\gamma_\lambda\, \mathcal{I}(\hat{\epsilon},x,y) \\
\mathcal{I}(\hat{\epsilon},x,y) &= \int_{q,k,\text{IR-reg}} \frac{1}{q^2k^2(q+k)^2}\, \frac{1}{(p-q)^2+m^2}\, \frac{1}{(p-q-k)^2+m^2}\, e^{-i(q\cdot x + k\cdot y)} \\
&= \int_{u,\text{IR-reg}} G_0(u-y)J(\hat{\epsilon},u)J(\hat{\epsilon},x-u) \\
J(\hat{\epsilon},u) &= \int_v G_0(v+u)e^{-m\hat{\epsilon}\cdot v}G_m(v) \\
\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) &= \int_z iz_\rho \,\langle j_\mu(x)j_\nu(y)j_\sigma(z)j_\lambda(0)\rangle
\end{aligned}$$

- $G_0(x)$, $G_m(x)$: massless and massive propagators in position space.
- $\mathcal{I}$ is logarithmically infrared divergent for $p^2 = -m^2 \Rightarrow$ introduce IR regulator.
- In a_μ^{HLbL} only terms with derivatives remain and $K_{\mu\nu\lambda}$ is infrared finite.

HLbL master formula in position space

Evaluate Dirac trace in projector on a_μ, average over direction of muon momentum, perform angular average using Gegenbauer polynomials (hyperspherical approach):

$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \int_y \int_x \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$$

After contracting the Lorentz indices the integration reduces to a 3-dimensional integral over $x^2, y^2, x\cdot y$.

QCD four-point function

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int_z z_\rho \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0)\rangle$$

QED kernel function $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$

- Weights the QCD four-point function in position space.
- Tensor decomposition leads to 6 weight functions that depend on the 3 variables $x^2, y^2, x\cdot y$.
- We have computed these weight functions on a grid, once and for all, and stored on disk.

Numerical test: Pion-pole contribution to a_μ^{HLbL}

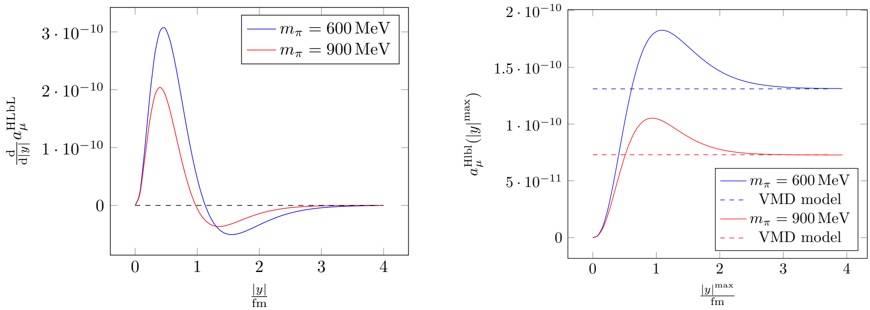
VMD model for pion transition form factor for illustration. Result for arbitrary pion mass can be easily obtained from 3-dimensional momentum-space representation (Jegerlehner + AN '09).

3-dim. integration in position space:

- $\int_y \to 2\pi^2 \int_0^\infty \mathrm{d}|y| |y|^3$
- $\int_x \to 4\pi \int_0^\infty \mathrm{d}|x|\, |x|^3 \int_0^\pi \mathrm{d}\beta \sin^2\beta$ (cutoff for x integration: $|x|^{\max} = 4.05$ fm)

Integrand after integration over $|x|, \beta$:

Result for $a_\mu^{\mathrm{HLbL}}(|y|^{\max})$:



- All 6 weight functions contribute to final result, some only at the percent level.
- $|y|^{\max} \gtrsim 2 - 3$ fm needed even for $m_\pi = 600 - 900$ MeV.
- For the physical pion mass, one needs to go to very large values of $|x|$ and $|y|$, i.e. very large lattice volumes, to reproduce known result of $5.7 \cdot 10^{-10}$.

Muon $g-2$: current status

Contribution	$a_\mu \times 10^{11}$	Reference
QED (leptons)	116 584 718.853 $\pm$ 0.036	Aoyama et al. '12
Electroweak	153.6 $\pm$ 1.0	Gnendiger et al. '13
HVP: LO	6889.1 $\pm$ 35.2	Jegerlehner '15
NLO	-99.2 $\pm$ 1.0	Jegerlehner '15
NNLO	12.4 $\pm$ 0.1	Kurz et al. '14
HLbL	102 $\pm$ 39	Jegerlehner '15 (JN '09)
NLO	3 $\pm$ 2	Colangelo et al. '14
Theory (SM)	116 591 780 $\pm$ 53	
Experiment	116 592 089 $\pm$ 63	Bennett et al. '06
Experiment - Theory	309 $\pm$ 82	3.8 σ

Discrepancy a sign of New Physics ?

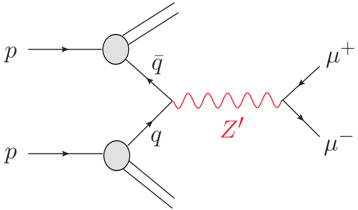
Hadronic uncertainties need to be better controlled in order to fully profit from future $g-2$ experiments at Fermilab (E989) and J-PARC (E34) with $\delta a_\mu = 16 \times 10^{-11}$.

Way forward for HVP seems clear: more precise measurements for $\sigma(e^+e^- \to \text{hadrons})$. Not so obvious how to improve HLbL.

Tests of the Standard Model and search for New Physics

Search for New Physics with two complementary approaches:

1. High Energy Physics:
   e.g. Large Hadron Collider (LHC) at CERN
   Direct production of new particles
   e.g. heavy Z' $\Rightarrow$ resonance peak in invariant mass distribution of $\mu^+\mu^-$ at $M_{Z'}$.

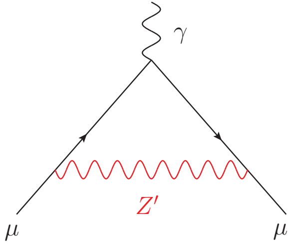


2. Precision physics:
   e.g. anomalous magnetic moments a_e, a_μ
   Indirect effects of virtual particles in quantum corrections
   $\Rightarrow$ Deviations from precise predictions in SM

$$\text{For } M_{Z'} \gg m_\ell : \quad a_\ell \sim \left(\frac{m_\ell}{M_{Z'}}\right)^2$$

Note: there are also non-decoupling contributions of heavy New Physics !
Another example: new light vector meson ("dark photon") with $M_{\gamma'} \sim (10 - 100)$ MeV.

a_e, a_μ allow to exclude some models of New Physics or to constrain their parameter space.



New Physics contributions to the muon $g-2$

Define:

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (309 \pm 82) \times 10^{-11}$$

Absolute size of discrepancy is actually unexpectedly large, compared to weak contribution (although there is some cancellation there):

$$\begin{aligned} a_\mu^{\text{weak}} &= a_\mu^{\text{weak, (1)}}(W) + a_\mu^{\text{weak, (1)}}(Z) + a_\mu^{\text{weak, (2)}} \\ &= (389 - 194 - 41) \times 10^{-11} \\ &= 154 \times 10^{-11} \end{aligned}$$

Assume that New Physics contribution with $M_{\text{NP}} \gg m_\mu$ decouples:

$$a_\mu^{\text{NP}} = \mathcal{C}\,\frac{m_\mu^2}{M_{\text{NP}}^2}$$

where naturally $\mathcal{C} = \frac{\alpha}{\pi}$, like from a one-loop QED diagram, but with new particles. Typical New Physics scales required to satisfy $a_\mu^{\text{NP}} = \Delta a_\mu$:

$\mathcal{C}$	1	$\frac{\alpha}{\pi}$	$\left(\frac{\alpha}{\pi}\right)^2$
M_{NP}	$1.9^{+0.3}_{-0.2}$ TeV	92^{+15}_{-10} GeV	4^{+1}_{-1} GeV

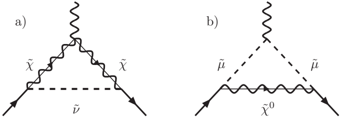
Therefore, for New Physics model with particles in $250-300$ GeV mass range and electroweak-size couplings $\mathcal{O}(\alpha)$, we need some additional enhancement factor, like large $\tan\beta$ in the MSSM, to explain the discrepancy Δa_μ.

a_μ: Supersymmetry

Supersymmetry for large $\tan\beta, \mu > 0$:

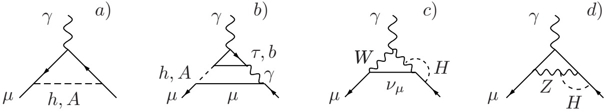
$$a_\mu^{\text{SUSY}} \approx 123 \times 10^{-11} \left(\frac{100 \text{ GeV}}{M_{\text{SUSY}}}\right)^2 \tan\beta$$

(Czarnecki, Marciano, '01)



Explains $\Delta a_\mu = 309 \times 10^{-11}$ if $M_{\text{SUSY}} \approx (89 - 399)$ GeV $\quad (2 < \tan\beta < 40)$.

In some regions of parameter space, large 2-loop contributions (2HDM):



Barr-Zee diagram (b) yields enhanced contribution, which can exceed 1-loop result.
Enhancement factor m_b^2/m_μ^2 compensates suppression by α/π
$((\alpha/\pi) \times (m_b^2/m_\mu^2) \sim 4 > 1)$.

a_μ and Supersymmetry after first LHC run

- LHC so far only sensitive to strongly interacting supersymmetric particles, like squarks and gluinos (ruled out below about 1 TeV).
- Muon $g-2$ and SUSY searches at LHC only lead to tension in constrained MSSM (CMSSM) or NUHM1 / NUHM2 (non-universal contributions to Higgs masses).
- More general SUSY models (e.g. pMSSM10 = phenomenological MSSM with 10 soft SUSY-breaking parameters) with light neutralinos, charginos and sleptons, can still explain muon $g-2$ discrepancy and evade bounds from LHC.

a_e, a_μ: Dark photon

In some dark matter scenarios, there is a relatively light, but massive "dark photon" A'_μ that couples to the SM through mixing with the photon:

$$\mathcal{L}_{\mathrm{mix}} = \frac{\varepsilon}{2} F^{\mu\nu} F'_{\mu\nu}$$

$\Rightarrow$ A'_μ couples to ordinary charged particles with strength $\varepsilon \cdot e$.
$\Rightarrow$ additional contribution of dark photon with mass $m_{\gamma'}$ to the $g-2$ of a lepton (electron, muon) (Pospelov '09):

$$\begin{aligned} a_\ell^{\text{dark photon}} &= \frac{\alpha}{2\pi}\varepsilon^2 \int_0^1 dx \frac{2x(1-x)^2}{\left[(1-x)^2 + \frac{m_{\gamma'}^2}{m_\ell^2} x\right]} \\ &= \frac{\alpha}{2\pi}\varepsilon^2 \times \begin{cases} 1 & \text{for } m_\ell \gg m_{\gamma'} \\ \frac{2m_\ell^2}{3m_{\gamma'}^2} & \text{for } m_\ell \ll m_{\gamma'} \end{cases} \end{aligned}$$

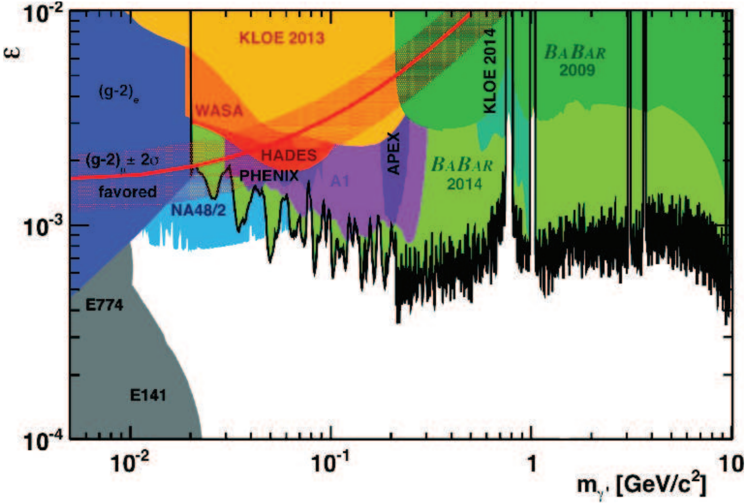
For values $\varepsilon \sim (1-2) \times 10^{-3}$ and $m_{\gamma'} \sim (10-100)$ MeV, the dark photon could explain the discrepancy $\Delta a_\mu \sim 300 \times 10^{-11}$.

Various searches for the dark photon have been performed, are under way or are planned at BABAR, Jefferson Lab, KLOE, MAMI and other experiments.

For a recent overview, see: *Dark Sectors and New, Light, Weakly-Coupled Particles* (Snowmass 2013), Essig et al., arXiv:1311.0029 [hep-ph].

Status of dark photon searches

Essentially all of the parameter space in the $(m_{\gamma'}, \varepsilon)$-plane to explain the muon $g-2$ discrepancy has now been ruled out.



From: F. Curciarello, FCCP15, Capri, September 2015

Different conclusions if dark photon decays (mostly) invisibly !

Conclusions and Outlook

- Over many decades, the (anomalous) magnetic moments of the electron and the muon have played a crucial role in atomic and elementary particle physics.
- Gained important insights into the structure of the fundamental interactions and matter in the universe (quantum field theory).
- a_μ: Test of Standard Model, potential window to New Physics.
- Current situation:

  $$a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = (309 \pm 82) \times 10^{-11} \qquad [3.8\ \sigma]$$

  Hadronic effects ? Sign of New Physics ?
- Two new planned $g-2$ experiments at Fermilab (E989) and J-PARC (E34) with goal of $\delta a_\mu^{\mathrm{exp}} = 16 \times 10^{-11}$ (factor 4 improvement)
- Theory needs to match this precision !
- Concerted effort needed of experiments (measuring processes with hadrons and photons), phenomenology (modelling and data-driven using dispersion relations) and lattice QCD to improve HVP and HLbL estimates with reliable uncertainties.

Theory Initiative for the Muon $g-2$

- Tasks:
  1. Organize workshops (about once a year) to survey and summarize the status of theoretical calculations of hadronic contributions (HVP, HLbL) to the muon $g-2$. Encourage participation from all theorists and phenomenologists who are working on such calculations.
  2. Form working groups on different topics (HVP, HLbL) and methods (Dispersive, Lattice, Models). Ongoing work between the workshops.
  3. Produce reports, authored by the participants of the workshops and working groups, on the current status of relevant theoretical work, including a scientific assessment of each work / method (cf. FLAG reports). Hopefully the reports can provide up-to-date values for HVP and HLbL with reliable uncertainties. Publication coordinated with annoucements of new experimental results on muon $g-2$.
- 1st Workshop: June 3-6, 2017 at Fermilab

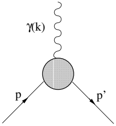
  `https://indico.fnal.gov/conferenceDisplay.py?confId=13795`
- 9 member Steering Committee:
  - 2 from future $g-2$ experiments: Lee Roberts (Fermilab E989 experiment), Tsutomu Mibe (J-PARC E34 experiment)
  - 5 from theory / phenomenology: Gilberto Colangelo, Michel Davier, Simon Eidelman, Andreas Nyffeler, Thomas Teubner
  - 2 from Lattice QCD: Aida El-Khadra, Christoph Lehner

Backup slides

Anomalous magnetic moment in quantum field theory

Quantized spin 1/2 particle interacting with external, classical electromagnetic field

4 form factors in vertex function
(momentum transfer $k = p' - p$, not assuming parity or charge conjugation invariance)



$$
\begin{aligned}
&\equiv \quad i\langle p', s'|j^\mu(0)|p, s\rangle \\
&= \quad (-ie)\bar{\mathrm{u}}(p', s')\Bigg[\gamma^\mu \underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu}k_\nu}{2m}\underbrace{F_2(k^2)}_{\text{Pauli}} \\
&\qquad + \gamma^5\frac{\sigma^{\mu\nu}k_\nu}{2m}F_3(k^2) + \gamma^5\left(k^2\gamma^\mu - \not{k}\; k^\mu\right)F_4(k^2)\Bigg]\mathrm{u}(p, s)
\end{aligned}
$$

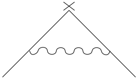
$\not{k} = \gamma^\mu k_\mu$. Real form factors for spacelike $k^2 \leq 0$. Non-relativistic, static limit:

$$
\begin{aligned}
F_1(0) &= 1 \qquad \text{(renormalization of charge } e\text{)} \\
\mu &= \frac{e}{2m}(F_1(0) + F_2(0)) \qquad \text{(magnetic moment)} \\
a &= F_2(0) \qquad \text{(anomalous magnetic moment)} \\
d &= -\frac{e}{2m}F_3(0) \qquad \text{(electric dipole moment, violates P and CP)} \\
F_4(0) &= \text{anapole moment (violates P)}
\end{aligned}
$$

Some theoretical comments on the $g-2$

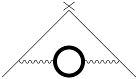
- Anomalous magnetic moment is finite and calculable
  Corresponds to effective interaction Lagrangian of mass dimension 5:
  $$\mathcal{L}_{\text{eff}}^{\text{AMM}} = -\frac{e_\ell a_\ell}{4m_\ell}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)F_{\mu\nu}(x)$$
  (mass dimension 6 in SM with $SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}$ invariant operator)
  $a_\ell = F_2(0)$ can be calculated unambiguously in renormalizable QFT, since there is no counterterm to absorb potential ultraviolet divergence.

- Anomalous magnetic moments are dimensionless
  To lowest order in perturbation theory in quantum electrodynamics (QED):



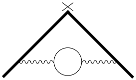
$$= a_e = a_\mu = \frac{\alpha}{2\pi}$$ [Schwinger '48]

- Loops with different masses $\Rightarrow a_e \neq a_\mu$
  - Internal large masses decouple (not always !):



$$= \left[\frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2 + \mathcal{O}\left(\frac{m_e^4}{m_\mu^4}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

  - Internal small masses give rise to large log's of mass ratios:



$$= \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + \mathcal{O}\left(\frac{m_e}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

HLbL scattering: selected results for $a_\mu^{\mathrm{HLbL}} \times 10^{11}$

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops +subl. N_C	–	–	–	0 ± 10	–	–	–
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3 (c-quark)	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

- Pseudoscalar-exchanges dominate numerically. Other contributions not negligible. Cancellation between π, K-loops and quark loops !
- Note that recent reevaluations of axial vector contribution lead to much smaller estimates than in MV: $a_\mu^{\mathrm{HLbL;axial}} = (8 \pm 3) \times 10^{-11}$ (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15). This would shift central values of compilations downwards: $a_\mu^{\mathrm{HLbL}} = (98 \pm 26) \times 10^{-11}$ (PdRV) and $a_\mu^{\mathrm{HLbL}} = (102 \pm 39) \times 10^{-11}$ (N, JN).
- PdRV: Analyzed results obtained by different groups with various models and suggested new estimates for some contributions (shifted central values, enlarged errors). Do not consider dressed light quark loops as separate contribution. Added all errors in quadrature !
- N, JN: New evaluation of pseudoscalar exchange contribution imposing new short-distance constraint on off-shell form factors. Took over most values from BPP, except axial vectors from MV. Added all errors linearly.

Model calculations of HLbL: recent developments

- Most calculations for neutral pion and all light pseudoscalars agree at level of 15%, but full range of estimates (central values) much larger:

$$a_\mu^{\mathrm{HLbL};\pi^0} = (50 - 80) \times 10^{-11} = (65 \pm 15) \times 10^{-11} \quad (\pm 23\%)$$
$$a_\mu^{\mathrm{HLbL;P}} = (59 - 114) \times 10^{-11} = (87 \pm 27) \times 10^{-11} \quad (\pm 31\%)$$

- New estimates for axial vectors (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15):

$$a_\mu^{\mathrm{HLbL;axial}} = (6 - 8) \times 10^{-11}$$

  Substantially smaller than in MV '04 !

- First estimate for tensor mesons (Pauk, Vanderhaeghen '14):

$$a_\mu^{\mathrm{HLbL;tensor}} = 1 \times 10^{-11}$$

- Open problem: Dressed pion-loop
  Potentially important effect from pion polarizability and a_1 resonance
  (Engel, Patel, Ramsey-Musolf '12; Engel '13; Engel, Ramsey-Musolf '13):

$$a_\mu^{\mathrm{HLbL};\pi-\mathrm{loop}} = -(11 - 71) \times 10^{-11}$$

  Maybe large negative contribution, in contrast to BPP '96, HKS '96.
  Not confirmed by recent reanalysis by Bijnens, Relefors '15, '16. Essentially get again old central value from BPP, but smaller error estimate:

$$a_\mu^{\mathrm{HLbL};\pi-\mathrm{loop}} = (-20 \pm 5) \times 10^{-11}$$

- Open problem: Dressed quark-loop
  Dyson-Schwinger equation approach (Fischer, Goecke, Williams '11, '13):

$$a_\mu^{\mathrm{HLbL;quark-loop}} = 107 \times 10^{-11} \qquad \text{(still incomplete !)}$$

  Large contribution, no damping seen, in contrast to BPP '96, HKS '96.

$a_\mu^{\mathrm{HLbL;P}}, \mathrm{P} = \pi^0, \eta, \eta'$: impact of precision of form factor measurements

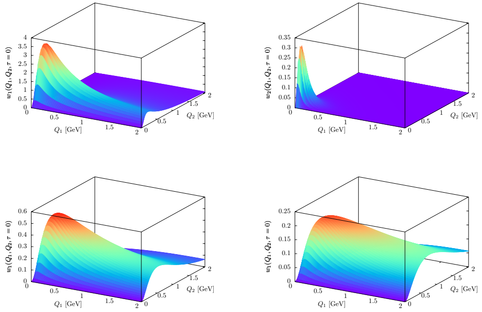
AN '16

In Jegerlehner, AN '09, a 3-dimensional integral representation for the pseudoscalar-pole contribution was derived. Schematically:

$$a_\mu^{\mathrm{HLbL;P}} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sum_i w_i(Q_1, Q_2, \tau)\, f_i(Q_1, Q_2, \tau)$$

with universal weight functions w_i (for Euclidean (space-like) momenta: $Q_1 \cdot Q_2 = |Q_1||Q_2|\tau, \tau = \cos\theta$). Dependence on form factors resides in the f_i.

Weight functions w_i:



Top: weight functions $w_{1,2}(Q_1, Q_2, \tau)$ for π^0 with $\theta = 90^\circ (\tau = 0)$.

Bottom: weight functions $w_1(Q_1, Q_2, \tau)$ for η (left) and η' (right).

- Relevant momentum regions below 1 GeV for π^0, below 1.5 GeV for η, η'.
- Analysis of current and future measurement precision of single-virtual $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q^2, 0)$ and double-virtual transition form factor $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$, based on Monte Carlo study for BESIII by Denig, Redmer, Wasser (Mainz).
- Data-driven precision for HLbL pseudoscalar-pole contribution that could be achieved in a few years:

$$\delta a_\mu^{\mathrm{HLbL};\pi^0} / a_\mu^{\mathrm{HLbL};\pi^0} = 14\%$$

$$\delta a_\mu^{\mathrm{HLbL};\eta} / a_\mu^{\mathrm{HLbL};\eta} = 23\%$$

$$\delta a_\mu^{\mathrm{HLbL};\eta'} / a_\mu^{\mathrm{HLbL};\eta'} = 15\%$$

Vertex function for HLbL in momentum space

Project on anomalous magnetic moment (Euclidean space):

$$a_\mu^{\mathrm{HLbL}} = F_2(0) = \frac{-i}{48m} \mathrm{Tr}\{[\gamma_\rho, \gamma_\sigma](-i\not{p} + m)\Gamma_{\rho\sigma}(p,p)(-i\not{p} + m)\}$$

with on-shell muon momentum $p = im\hat{\epsilon}$ ($p^2 = -m^2$; $\hat{\epsilon}$: unit vector).

$$\begin{aligned}\Gamma_{\rho\sigma}(p', p) = -\, e^6 \int_{q_1,q_2} & \tfrac{1}{q_1^2 q_2^2 (q_1+q_2-k)^2} \tfrac{1}{(p'-q_1)^2+m^2} \tfrac{1}{(p'-q_1-q_2)^2+m^2} \\ & \times \gamma_\mu (i\not{p}' - i\not{q}_1 - m)\gamma_\nu (i\not{p} - i\not{q}_1 - i\not{q}_2 - m)\gamma_\lambda \\ & \times \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)\end{aligned}$$

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \int_{x_1,x_2,x_3} e^{-i(q_1 x_1 + q_2 x_2 + q_3 x_3)} \langle j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) j_\sigma(0)\rangle$$

Where we used the following relation derived from the Ward identities to extract one factor of k to get $F_2(k^2)$ (Kinoshita *et al.* '70):

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2) = -k_\sigma \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)$$

Notation: $\int_q \equiv \int \frac{\mathrm{d}^4 q}{(2\pi)^4}$, $\int_x \equiv \int \mathrm{d}^4 x$

Evaluating $\mathcal{I}(\hat{\epsilon}, x, y)$

$$\mathcal{I}(\hat{\epsilon}, x, y) = \int_{u,\text{IR-reg}} G_0(u - y) J(\hat{\epsilon}, u) J(\hat{\epsilon}, x - u)$$

$$\begin{aligned} J(\hat{\epsilon}, u) &= \int_v G_0(v + u) e^{-m\hat{\epsilon}\cdot v} G_m(v) \\ &= \sum_{n\geq 0} z_n(u^2) U_n(\hat{\epsilon}\cdot\hat{u}) \end{aligned}$$

Last line: expansion in terms of Chebyshev polynomials of the second kind U_n (special case of the Gegenbauer polynomials)

$z_n =$ linear combination of products of two modified Bessel functions

Propagators in position space:

$$\begin{aligned} G_0(x) &= \frac{1}{4\pi^2 x^2} \\ G_m(x) &= \frac{m}{4\pi^2 |x|} K_1(m|x|) \quad (K_1 \text{ is a modified Bessel function}) \end{aligned}$$

Averaging over direction of muon momentum $p = im\hat{\epsilon}$

Evaluating Dirac trace in projector, one obtains an expression of the form:

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int_y \int_x \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon}, x, y) i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x, y)$$

Exploit invariance of a_μ under $O(4)$ rotations of the muon momentum and average kernel $\mathcal{L}$ over direction $\hat{\epsilon}$ (Barbieri + Remiddi '75):

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y) = \frac{1}{2\pi^2} \int \mathrm{d}\Omega_{\hat{\epsilon}} \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon}, x, y) \equiv \langle \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon}, x, y) \rangle_{\hat{\epsilon}}$$

Angular average can be performed analytically by using orthogonality property of Chebyshev (Gegenbauer) polynomials that appear in QED kernel $\mathcal{L}$ via $\mathcal{I}$ and J (hyperspherical approach):

$$\langle U_n(\hat{\epsilon} \cdot \hat{x}) U_m(\hat{\epsilon} \cdot \hat{y}) \rangle_{\hat{\epsilon}} = \frac{\delta_{nm}}{n+1} U_n(\hat{x} \cdot \hat{y})$$

Tensor decomposition of QED kernel and weight functions

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{A=I,II,III} \mathcal{G}^{A}_{\delta\rho\sigma\mu\alpha\nu\beta\lambda} T^{A}_{\alpha\beta\delta}(x,y)$$

$$\mathcal{G}^{I,II,III}_{\delta\rho\sigma\mu\alpha\nu\beta\lambda} = \text{sums of products of Kronecker deltas (from Dirac trace)}$$

$$T^{I}_{\alpha\beta\delta}(x,y) = \partial^{(x)}_{\alpha}(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta}) V_{\delta}(x,y)$$

$$T^{II}_{\alpha\beta\delta}(x,y) = m\partial^{(x)}_{\alpha}\left(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta}S(x,y)\right)$$

$$T^{III}_{\alpha\beta\delta}(x,y) = m(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})\left(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta}S(x,y)\right)$$

Scalar: $S(x,y) = \langle \mathcal{I} \rangle_{\hat{\epsilon}}$ (IR regulated)

Vector: $V_{\delta}(x,y) = \langle \hat{\epsilon}_{\delta}\mathcal{I} \rangle_{\hat{\epsilon}}$

Tensor: $T_{\beta\delta}(x,y) = \langle (\hat{\epsilon}_{\beta}\hat{\epsilon}_{\delta} - \frac{1}{4}\delta_{\beta\delta})\mathcal{I} \rangle_{\hat{\epsilon}}$

$$S(x,y) = g^{(0)}$$

$$V_{\delta}(x,y) = x_{\delta} g^{(1)} + y_{\delta} g^{(2)}$$

$$T_{\alpha\beta}(x,y) = \left(x_{\alpha}x_{\beta} - \frac{x^2}{4}\delta_{\alpha\beta}\right) l^{(1)} + \left(y_{\alpha}y_{\beta} - \frac{y^2}{4}\delta_{\alpha\beta}\right) l^{(2)} + \left(x_{\alpha}y_{\beta} + y_{\alpha}x_{\beta} - \frac{x\cdot y}{2}\delta_{\alpha\beta}\right) l^{(3)}$$

where the 6 weight functions depend on $x^2, y^2, x\cdot y$.

Example: Weight function $g^{(2)}(x^2, x\cdot y, y^2)$

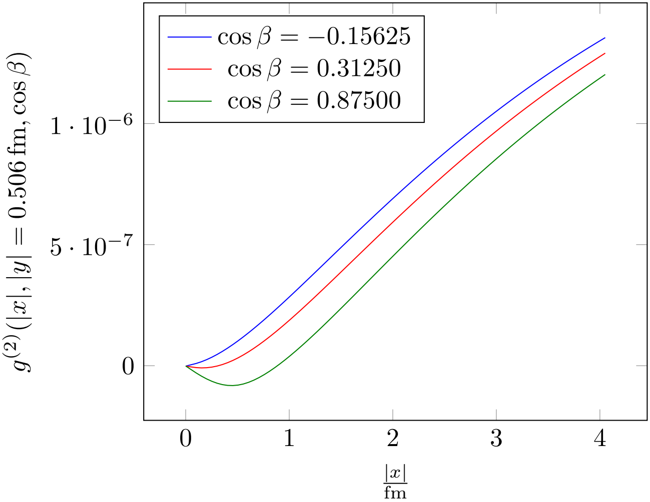
$$
\begin{aligned}
g^{(2)}(x^2, x\cdot y, y^2) &= \frac{1}{8\pi y^2 |x| \sin^3\beta} \int_0^\infty du\, u^2 \int_0^\pi d\phi_1 \\
&\times \left[2\sin\beta + \left(\frac{y^2+u^2}{2|u||y|} - \cos\beta\cos\phi_1 \right) \frac{\log\chi}{\sin\phi_1} \right] \\
&\times \sum_{n=0}^{\infty} \Bigg\{ z_n(|u|) z_{n+1}(|x-u|) \left[|x-u|\cos\phi_1 \frac{U_n}{n+1} + (|u|\cos\phi_1 - |x|) \frac{U_{n+1}}{n+2} \right] \\
&\quad + z_{n+1}(|u|) z_n(|x-u|) \left[(|u|\cos\phi_1 - |x|) \frac{U_n}{n+1} + |x-u|\cos\phi_1 \frac{U_{n+1}}{n+2} \right] \Bigg\}
\end{aligned}
$$

where

$$
\begin{aligned}
x\cdot y &= |x||y|\cos\beta\,, \quad |x-u| = \sqrt{|x|^2 + |u|^2 - 2|x||u|\cos\phi_1} \\
\chi &= \frac{y^2+u^2-2|u||y|\cos(\beta-\phi_1)}{y^2+u^2-2|u||y|\cos(\beta+\phi_1)}\,, \quad U_n = U_n\left(\frac{|x|\cos\phi_1 - |u|}{|u-x|}\right) \\
z_n &= \text{linear combination of products of two modified Bessel functions.}
\end{aligned}
$$

Example: Weight function $g^{(2)}(x^2, x \cdot y, y^2)$ (continued)

For $|y| = 0.506\ \mathrm{fm}$:



- Computed all 6 weight functions on grid to about 5 digits precision.
- Stored once and for all on disk.

Electron $g-2$: Theory

Main contribution in Standard Model (SM) from mass-independent Feynman diagrams in QED with electrons in internal lines (perturbative series in α):

$$
\begin{aligned}
a_e^{\mathrm{SM}} \quad = \quad & \sum_{n=1}^{5} c_n \left(\frac{\alpha}{\pi}\right)^n \\
& +2.7478(2) \times 10^{-12} \quad [\text{Loops in QED with } \mu, \tau] \\
& +0.0297(5) \times 10^{-12} \quad [\text{weak interactions}] \\
& +1.706(15) \times 10^{-12} \quad [\text{strong interactions / hadrons}]
\end{aligned}
$$

The numbers are from Aoyama et al. '15.

QED: mass-independent contributions to a_e

- α: 1-loop, 1 Feynman diagram; Schwinger '48:

  $c_1 = \frac{1}{2}$

- α^2: 2-loops, 7 Feynman diagrams; Petermann '57, Sommerfield '57:

  $c_2 = \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 + \frac{3}{4}\zeta(3) = -0.32847896557919378\ldots$

- α^3: 3-loops, 72 Feynman diagrams; ..., Laporta, Remiddi '96:

$$\begin{aligned} c_3 &= \frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2\ln 2 + \frac{139}{18}\zeta(3) - \frac{239}{2160}\pi^4 \\ &\quad + \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left\{\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2\ln^2 2\right\} \\ &= 1.181241456587\ldots \end{aligned}$$

- α^4: 4-loops, 891 Feynman diagrams; Kinoshita et al. '99, ..., Aoyama et al. '08; '12, '15:

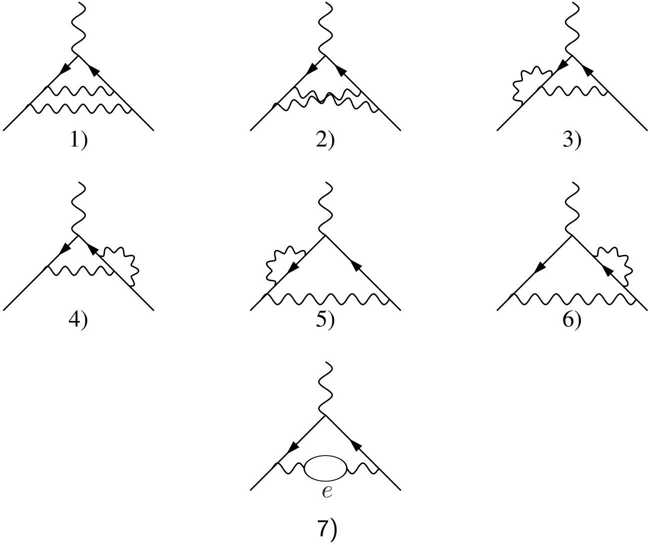
  $c_4 = -1.91298(84)$ (numerical evaluation)

- α^5: 5-loops, 12672 Feynman diagrams; Aoyama et al. '05, ..., '12, '15:

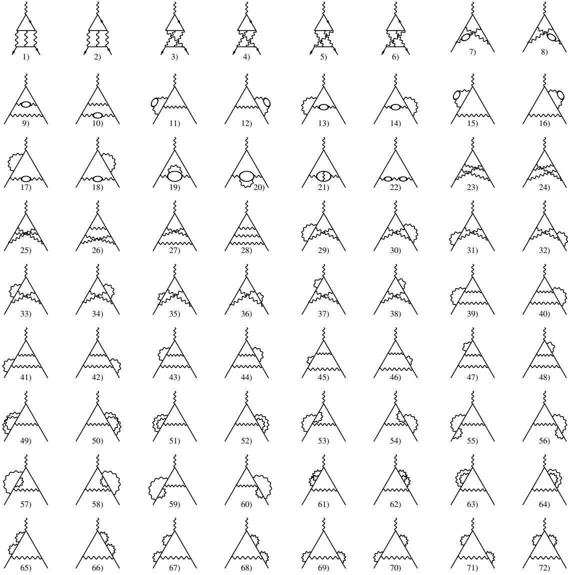
  $c_5 = 7.795(336)$ (numerical evaluation)

  Replaces earlier rough estimate $c_5 = 0.0 \pm 4.6$.
  Result removes biggest theoretical uncertainty in a_e !

Mass-independent 2-loop Feynman diagrams in a_e



Mass-independent 3-loop Feynman diagrams in a_e



Determination of fine-structure constant α from $g-2$ of electron

- Recent measurement of α via recoil-velocity of Rubidium atoms in atom interferometer (Bouchendira et al. '11 and recent CODATA input):

  $$\alpha^{-1}(\mathrm{Rb}) = 137.035\ 999\ 049(90) \quad [0.66\mathrm{ppb}]$$

  This leads to (Aoyama et al. '15):

  $$a_e^{\mathrm{SM}}(\mathrm{Rb}) = 1\ 159\ 652\ 181.643\ \underbrace{(25)}_{c_4}\ \underbrace{(23)}_{c_5}\ \underbrace{(16)}_{\mathrm{had}}\ \underbrace{(763)}_{\alpha(\mathrm{Rb})}\ [764] \times 10^{-12} \quad [0.67\mathrm{ppb}]$$

  $$\Rightarrow a_e^{\mathrm{exp}} - a_e^{\mathrm{SM}}(\mathrm{Rb}) = -0.91(0.82) \times 10^{-12} \quad [\text{Error from } \alpha(\mathrm{Rb}) \text{ dominates !}]$$

  $\rightarrow$ Test of QED !

- Use a_e^{exp} to determine α from series expansion in QED (contributions from weak and strong interactions under control !). Assume: Standard Model "correct", no New Physics (Aoyama et al. '15):

  $$\alpha^{-1}(a_e) = 137.035\ 999\ 1570\ \underbrace{(29)}_{c_4}\ \underbrace{(27)}_{c_5}\ \underbrace{(18)}_{\mathrm{had+EW}}\ \underbrace{(331)}_{a_e^{\mathrm{exp}}}\ [334] \quad [0.25\mathrm{ppb}]$$

  The uncertainty from theory has been improved considerably by Aoyama et al. '12, '15, the experimental uncertainty in a_e^{exp} is now the limiting factor.

- Today the most precise determination of the fine-structure constant α, a fundamental parameter of the Standard Model.