First results about elliptic flow with a unified approach at RHIC energies

Sophys Gabriel

Supervisor: Klaus Werner at Subatech

RPP 2017 - 24 April 2017



ntroduction Unified Approach Anisotropic Flow Methods Conclusion

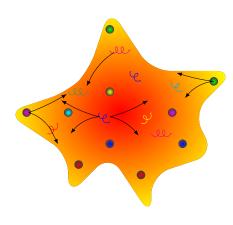
Overview

- 1 Introduction
- 2 Unified Approach
- **3** Anisotropic Flow
- 4 Methods
- **5** Conclusion



Introduction Unified Approach Anisotropic Flow Methods Conclusion

Context: QGP



- Quark-Gluon Plasma (QGP)
- Partons are deconfined
- QGP life-time: 10⁻²¹ s, size:
 10⁻¹⁵ m ⇒ cannot study directly the QGP
- Need theoretical models : **EPOS**



Introduction Unified Approach Anisotropic Flow Methods Conclusion

Introduction

Unified Approach: **EPOS**

- Energy conserving quantum mechanical multiple scattering approach
- based on Partons, parton ladders, strings
- Off-shell remnants
- Splitting of parton ladder



Introduction Unified Approach Anisotropic Flow Methods Conclusion

Introduction

Unified Approach: **EPOS**

- Energy conserving quantum mechanical multiple scattering approach
- based on Partons, parton ladders, strings
- Off-shell remnants
- Splitting of parton ladder

 $\label{eq:my-work} \mbox{My Work} \rightarrow \mbox{simplification of EPOS}$ At the end of Ph D \rightarrow open-diffusion of EPOS ?



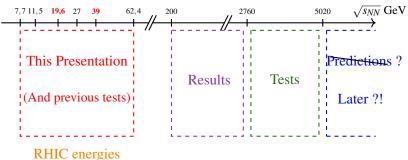
ntroduction Unified Approach Anisotropic Flow Methods Conclusion

Unified Approach

When do we use EPOS?

Monte Carlo Method

Model for very high energy.





EPOS : One Event

Unified Approach

How we construct one event?

Universal Model for all collisions

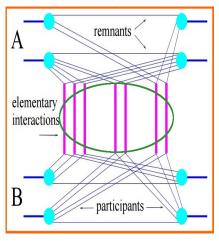
Same procedure applies, based on several stages:

- Initial Conditions
- 2 Core-Corona Approach
- 3 Viscous hydrodynamic expansion
- Statistical hadronization
- **6** Final state hadronic cascade



Unified Approach

Parton-Based-Gribov-Regge-Theory (PBGRT)



- Initial Conditions
- · Core-Corona Approach
- · Viscous hydrodynamic expansion
- · Statistical hadronization
- . Final state hadronic cascade
- Interaction between partons are: **Pomeron**: treated by Quantum Field Theory
- Energy conserved by participants and remnants partons

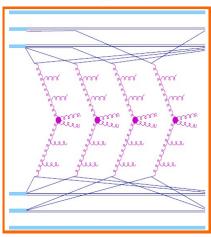
H. J. Drescher, M. Hladik, S. Ostapchenko, T. Pierog and K. Werner, Phys. Rept. 350, 93 (2001)



EPOS: Parton Based Gribov Regge Theory

Unified Approach

Parton-Based-Gribov-Regge-Theory (PBGRT)



- Initial Conditions
- · Core-Corona Approach
- · Viscous hydrodynamic expansion
- · Statistical hadronization
- · Final state hadronic cascade
- Interaction between partons are: Pomeron: treated by Quantum Field Theory
- Energy conserved by participants and remnants partons
- Pomerons become Partons Ladders
- Partons Ladders become Strings

H. J. Drescher, M. Hladik, S. Ostapchenko, T. Pierog and K. Werner, Phys. Rept. 350, 93 (2001)

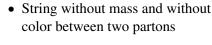


EPOS: Parton Based Gribov Regge Theory

Unified Approach

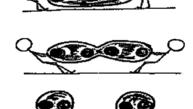
Lund Model: A phenomenological model of hadronization

- Initial Conditions
- · Core-Corona Approach
- · Viscous hydrodynamic expansion
- · Statistical hadronization
- Final state hadronic cascade





- When the potential is sufficient
 → one pair of quark-antiquark
 is created: Schwinger
 Mechanism
- Use at PYTHIA/JETSET





Unified Approach

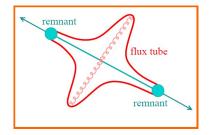
Core-Corona Approach

Initial Conditions

- · Core-Corona Approach
- · Viscous hydrodynamic expansion
- Statistical hadronization
- · Final state hadronic cascade

Using hydrodynamic \rightarrow the Core is treated fluid.

Corona becomes Jet \Rightarrow Later Hadrons!



GDRE2012, Nantes, Jul 2012, Klaus WERNER, Subatech, Nantes

More Scattering \Rightarrow



EPOS: Core-Corona Approach & Hydrodynamical expansion

Unified Approach

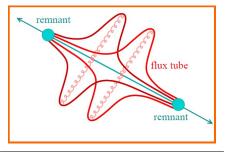
Core-Corona Approach

Initial Conditions

- · Core-Corona Approach
- · Viscous hydrodynamic expansion
- Statistical hadronization
- · Final state hadronic cascade

Using hydrodynamic \rightarrow the Core is treated fluid.

Corona becomes Jet \Rightarrow Later Hadrons!



GDRE2012, Nantes, Jul 2012, Klaus WERNER, Subatech, Nantes

More Scattering \Rightarrow



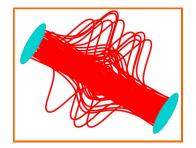
EPOS: Core-Corona Approach & Hydrodynamical expansion

Unified Approach

Core-Corona Approach

· Final state hadronic cascade Using hydrodynamic \rightarrow the Core is treated fluid.

Corona becomes $Jet \Rightarrow Later Hadrons !$



B. Guiot and K. Werner, J. Phys. Conf. Ser. 589 (2015) no.1



Initial Conditions

· Core-Corona Approach · Viscous hydrodynamic expansion · Statistical hadronization

EPOS: Core-Corona Approach & Hydrodynamical expansion

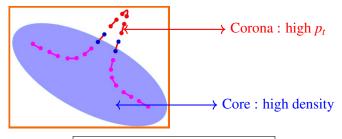
Unified Approach

Core-Corona Approach

- Initial Conditions
- · Core-Corona Approach
- · Viscous hydrodynamic expansion
- Statistical hadronization
- · Final state hadronic cascade

Using hydrodynamic \rightarrow the Core is treated fluid.

Corona becomes Jet \Rightarrow Later Hadrons!



B. Guiot and K. Werner, J. Phys. Conf. Ser. 589 (2015) no.1



Unified Approach

- Initial Conditions
- Core-Corona Approach
- Viscous hydrodynamic expansion
- Statistical hadronization
 Final state hadronic cascade

Core-Corona Approach

Using hydrodynamic \rightarrow the Core is treated as fluid.

Corona becomes Jet \Rightarrow Later Hadrons!

Hydrodynamical expansion

Core evolves with respect to the equation of relativistic viscous hydrodynamics

Local energy momentum:

$$\partial_{\mu}T^{\mu\nu}=0$$
 $\nu=0,\cdots,3$

and the conservation of net charges,

$$\partial N_k^{\mu} = 0, \qquad k = B, S, Q$$

with B, S and Q reffering to baryon number, strangeness and electric charge



EPOS: Statistical Hadronization & Hadronic cascade

Unified Approach

- Initial Conditions
- Core-Corona Approach
- Viscous hydrodynamic expansion
- Statistical hadronization
 Final state hadronic cascade

Statistical Hadronization

Core-Matter make hadronization Defined by a constant temperature T_H Procedure of Cooper-Frye

K. Werner, Iu. Karpenko, T. Pierog, M. Bleicher, K. Mikhailov, arXiv:1010.0400, Phys. Rev. C 83, 044915 (2011)

Hadronic Cascade

Hadron density still big \rightarrow hadron-hadron rescatterings Use **UrQMD Model**

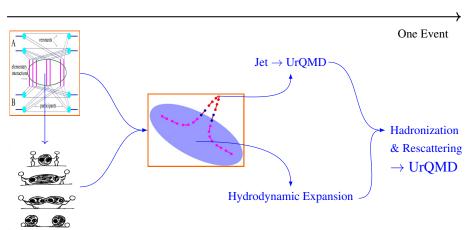
M. Bleicher et al., J. Phys. G25 (1999) 1859

H. Petersen, J. Steinheimer, G. Burau, M. Bleicher and H. Stocker, Phys. Rev. C78 (2008) 044901



EPOS: Statistical Hadronization & Hadronic cascade

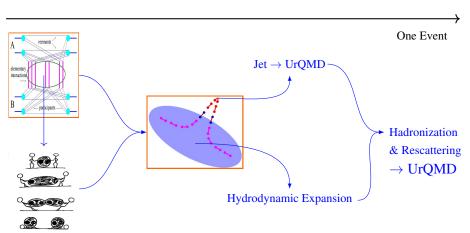
Unified Approach





EPOS: Statistical Hadronization & Hadronic cascade

Unified Approach



Let's stop talking about EPOS now



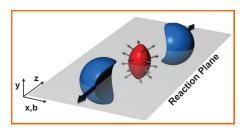
ntroduction Unified Approach **Anisotropic Flow** Methods Conclusion

Elliptic Flow

Anisotropic Flow

Direct evidence of flow: anisotropy in particle momentum distributions correlated with the reaction plane.

Azimuthal anisotropy characterizes expansion of highly-compressed matter



R. Snellings, New J. Phys. 13 (2011) 055008



ntroduction Unified Approach Anisotropic Flow Methods Conclusion

Elliptic Flow

Anisotropic Flow

A way of characterizing the various patterns of anisotropic flow is to use a Fourier expansion:

$$E\frac{d^3N}{d^3\mathbf{p}} = \frac{1}{2\pi} \frac{d^2}{p_t dp_t dy} \left(1 + 2\sum_{n=1}^{\infty} \mathbf{v_n} \cos\left[n(\phi - \psi_{RP})\right] \right)$$

 $E: energy\ of\ the\ particle\ ;\ p: momentum\ ;\ pt: transverse\ momentum\ ;\ \varphi: azimuthal\ angle\ ;\ y: \\ rapidity\ ;\ \psi_{RP}: reaction\ plane\ angle.$



14 / 25

Elliptic Flow

Anisotropic Flow

A way of characterizing the various patterns of anisotropic flow is to use a Fourier expansion:

$$E\frac{d^3N}{d^3\mathbf{p}} = \frac{1}{2\pi} \frac{d^2}{p_t dp_t dy} \left(1 + 2\sum_{n=1}^{\infty} \mathbf{v_n} \cos\left[n(\phi - \psi_{RP})\right] \right)$$

E : energy of the particle ; p : momentum ; pt : transverse momentum ; φ : azimuthal angle ; y : rapidity ; ψ_{RP} : reaction plane angle.

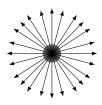
Anisotropic Flow: (n=1: Directed Flow, n=2: Elliptic Flow)

$$v_n(pt, y) = \langle \cos [n (\phi(pt, y) - \psi_{RP})] \rangle$$



Anisotropic Flow

Anisotropy \neq Isotropy



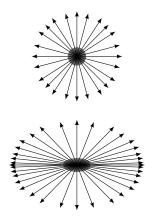
 Elementary Collisions : Isotropy of particles production

 $v_2 = 0$: Elliptic Flow



Anisotropic Flow

Anisotropy \neq Isotropy



• Elementary Collisions : **Isotropy** of particles production

 $v_2 = 0$: Elliptic Flow

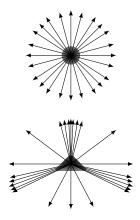
A-A Collisions: Anisotropy of particles production



 $v_2 > 0$

Anisotropic Flow

Anisotropy \neq Isotropy



• Elementary Collisions : **Isotropy** of particles production

 $v_2 = 0$: Elliptic Flow

A-A Collisions: Anisotropy of particles production

$$v_2 > 0$$

- $v_3 > 0$
- Something more than elementary processus : **QGP**?



Event Plane Method

Elliptic Flow

Eta-Sub: Event Plane Method



Elliptic Flow

Event Plane Method

Eta-Sub: Event Plane Method

Event Flow vector (projection of azimuthal angle):

$$Q_{n,x} = \sum_{i} w_{i} \cos(n\phi_{i}) = Q_{n} \cos(n\Psi_{n})$$
$$Q_{n,y} = \sum_{i} w_{i} \sin(n\phi_{i}) = Q_{n} \sin(n\Psi_{n})$$

The sum goes over all particles i used in the event plane calculation. ϕ_i and w_i are the lab azimuthal angle and weight for particle

Where Ψ_n the event plane angle :

$$\Psi_n = \frac{1}{n} \tan^{-1} \left(\frac{\sum_i w_i \sin(n\phi_i)}{\sum_i w_i \cos(n\phi_i)} \right)$$



Event Plane Method

Elliptic Flow

Eta-Sub: Event Plane Method

$$v_n^{\text{obs}}(p_T, y) = \langle \cos[n(\phi_i - \Psi_n)] \rangle$$

Average over all particles in all events with their azimuthal angle ϕ_i in a given rapidity and p_T momentum space.



Event Plane Method

Elliptic Flow

Eta-Sub: Event Plane Method

$$v_n^{\text{obs}}(p_T, y) = \langle \cos[n(\phi_i - \Psi_n)] \rangle$$

Average over all particles in all events with their azimuthal angle ϕ_i in a given rapidity and p_T momentum space.

$$\mathscr{R} = \langle \cos[n(\Psi_n - \Psi_{RP})] \rangle$$

average over all events



Event Plane Method

Elliptic Flow

Eta-Sub: Event Plane Method

$$v_n^{\text{obs}}(p_T, y) = \langle \cos[n(\phi_i - \Psi_n)] \rangle$$

Average over all particles in all events with their azimuthal angle ϕ_i in a given rapidity and p_T momentum space.

Eta-sub method : two planes defined by **negative** (A) and **positive** (B) pseudorapidity with \approx equal multiplicity :

$$\mathscr{R}_{n,sub} = \sqrt{\langle \cos[n(\Psi_n^A - \Psi_n^B)] \rangle}$$



Event Plane Method

Elliptic Flow

Eta-Sub: Event Plane Method

$$v_n^{\text{obs}}(p_T, y) = \langle \cos[n(\phi_i - \Psi_n)] \rangle$$

Average over all particles in all events with their azimuthal angle ϕ_i in a given rapidity and p_T momentum space.

Eta-sub method : two planes defined by **negative** (A) and **positive** (B) pseudorapidity with \approx equal multiplicity :

$$\mathscr{R}_{n,sub} = \sqrt{\langle \cos[n(\Psi_n^A - \Psi_n^B)] \rangle}$$

The final flow coefficients are : $v_n = \frac{v_n^{\text{obs}}}{\mathcal{R}_{n.sub}}$



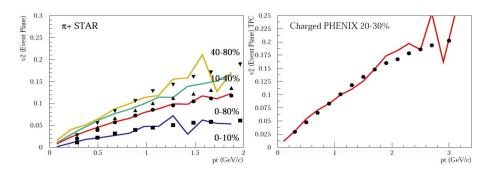
Results with EPOS

Results : v_2 vs p_t



Results : v_2 vs p_t

Results with EPOS



At energy collisions $\sqrt{s_{NN}} = 19.6 \text{ GeV}$ First good results but work in progress!



ntroduction Unified Approach Anisotropic Flow **Methods** Conclusion

The flow and the Cumulant Method

Cumulant Method

Q-Cumulant Method: Reference Flow

Q-Cumulant \rightarrow Recent Method to calculate cumulants \rightarrow one loop over data

A. Bilandzic, R. Snellings, and S. Voloshin Phys. Rev. C 83, 044913 - Published 26 April 2011

Flow vector :
$$Q_n = \sum_{i=1}^{M} e^{in\phi_i}$$



The flow and the Cumulant Method

Cumulant Method

Q-Cumulant Method: Reference Flow

Q-Cumulant \rightarrow Recent Method to calculate cumulants \rightarrow one loop over data

A. Bilandzic, R. Snellings, and S. Voloshin Phys. Rev. C 83, 044913 - Published 26 April 2011

Flow vector :
$$Q_n = \sum_{i=1}^{M} e^{in\phi_i}$$

Azimuthal particles Correlations between 2 or 4 references particles (REP)

$$\langle 2 \rangle \equiv \langle e^{in(\phi_1 - \phi_2)} \rangle = \frac{|Q_n|^2 - M}{M(M-1)}$$

$$\langle 4 \rangle \equiv \langle e^{in(\phi_1 + \phi_2 - \phi_3 - \phi_4)} \rangle \propto |Q_n|^4 + |Q_{2n}|^2 - 2 \times \mathcal{R}|Q_{2n}Q_n^*Q_n^*|Q_n^*|$$



Differential Flow

Definitions of vectors p and q:

For particles labeled as POI:

For particles labeled as **both** POI and REP:

$$p_n \equiv \sum_{i=1}^{m_p} e^{in\psi_i}$$

$$q_n \equiv \sum_{i=1}^{m_q} e^{in\psi_i}$$

Average of two- and four-particles azimuthal correlations:

$$\langle 2' \rangle = \frac{\mathscr{R}[p_n Q_n^*] - m_q}{m_p M - m_q} \qquad \langle 4' \rangle \propto \mathscr{R}[p_n Q_n Q_n^* Q_n^*] + \mathscr{R}[q_n Q_n^*] \dots$$



Approach of Q-Cumulant

Procedure to create cumulants by directs calculations:

- **①** Decompose azimuthal correlations as expressions like $|Q_n|^2$, $|Q_n|^4$... in terms of $\langle 2 \rangle$, $\langle 4 \rangle$...
- **2** Solved system of coupled equations for multi-particle scattering in same harmonic $\langle 2 \rangle, \langle 4 \rangle$... *results at previous slides*
- **3** Create $\langle\langle 2\rangle\rangle, \langle\langle 4\rangle\rangle$, average on all events, taking in account weigths of event
- **4** Create Cumulants with terms of $\langle \langle 2 \rangle \rangle$, $\langle \langle 4 \rangle \rangle$ etc ...



Reference Flow

Cumulants for reference flow:

$$c_n\{2\} = \langle \langle 2 \rangle \rangle$$

$$c_n\{4\} = \langle \langle 4 \rangle \rangle - 2 \times \langle \langle 2 \rangle \rangle^2$$

Reference flow or integrated flow:

$$v_n{2} = \sqrt{c_n{2}}$$

 $v_n{4} = \sqrt[4]{-c_n{4}}$



Reference Flow

Cumulants for reference flow:

$$c_n\{2\} = \langle \langle 2 \rangle \rangle$$

$$c_n\{4\} = \langle \langle 4 \rangle \rangle - 2 \times \langle \langle 2 \rangle \rangle^2$$

Cumulants for differential flow:

$$d_n\{2\} = \langle \langle 2' \rangle \rangle$$

$$d_n\{4\} = \langle \langle 4' \rangle \rangle - 2 \times \langle \langle 2' \rangle \rangle \langle \langle 2 \rangle \rangle$$

Reference flow or integrated flow:

$$v_n{2} = \sqrt{c_n{2}}$$

 $v_n{4} = \sqrt[4]{-c_n{4}}$

Differential flow:

$$v'_n\{2\} = d_n\{2\} / \sqrt{c_n\{2\}}$$
$$v'_n\{4\} = -d_n\{4\} / (-c_n\{4\})^{3/4}$$

 $A.\ Bilandzic,\ R.\ Snellings,\ and\ S.\ Voloshin\ Phys.\ Rev.\ C\ 83,\ 044913-Published\ 26\ April\ 2011$



Results with EPOS

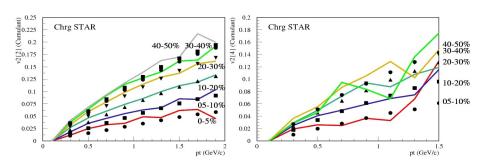
Results : v_2 vs p_t



Introduction Unified Approach Anisotropic Flow **Methods** Conclusion

Results with EPOS

Results : v_2 vs p_t



At energy collisions $\sqrt{s_{NN}} = 39$ GeV First good results but work in progress!



ntroduction Unified Approach Anisotropic Flow Methods **Conclusion**

Conclusion

EPOS was created for **very high energy** with an application for multiple scattering

EPOS has good results at number of observables \rightarrow **very high energy**



Conclusion

Conclusion

EPOS was created for very high energy with an application for multiple scattering

EPOS has good results at number of observables \rightarrow very high energy

But: First Study at "low" energies ($\sqrt{s} = 7.7$ GeV at $\sqrt{s} = 62.4$ GeV) for v_2

We obtain good results!

Maybe EPOS can also reproduce observables at less high energy

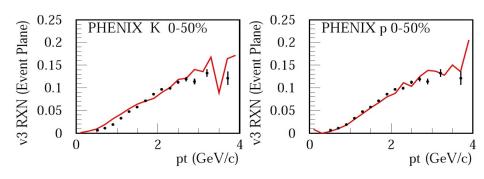
Next step: Physics Investigation of result, implementing scalar product method and comparison with $\sqrt{s_{NN}} = 5.02$ Tev



Thanks you !!!



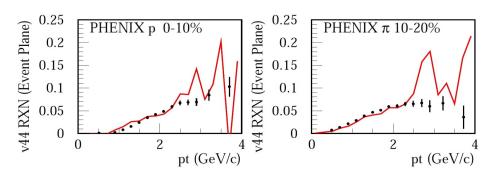






Results

v4





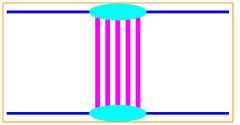
Gribov-Regge Theory and Pomeron

Effective Field Theory

Elementary interaction → Pomeron exchange

Pomeron : Quantum numbers of vacuum

Vladimir Gribov in ≈ 1960



Elastic Amplitude : $T(s,t) \approx is^{\alpha_0 + \alpha' t}$



Hydrodynamic equations

Based on the four-momenta of string segments, we compute the energy momentum tensor and the flavor flow vector at some position x (at $\tau = \tau_0$) as:

$$T^{\mu\nu} = \sum_{i} \frac{\delta p_{i}^{\mu} \delta p_{i}^{\nu}}{\delta p_{i}^{0}} g(x - x_{i})$$

$$N_q^{\mu}(x) = \sum_i \frac{\delta p_i^{\mu}}{\delta p_i^0} q_i g(x - x_i)$$

where q = u,d,s

arXiv:1312.1233v1 [nucl-th] 4 Dec 2013



Unified Approach

Initial Conditions : PBGRT to initialize collisions, Pomerons treated by partons ladders after by Lund string

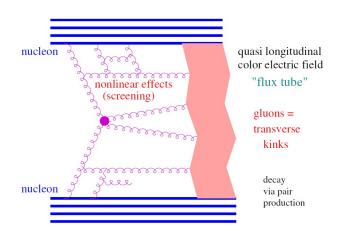
Core-Corona Approach Determine Jet and Quark Gluon Plasma

Viscous hydrodynamic expansion for the "core-matter"

Statistical Hadronization Procedure of Cooper-Frye

Final state hadronic Cascade Rescattering of products particle for the transport : UrQMD Model







Initial conditions. A Gribov-Regge multiple scattering approach is employed, where the elementary object (by definition called Pomeron) is a DGLAP parton ladder, using in addition a CGC motivated saturation scale for each Pomeron. The parton ladders are treated as classical relativistic (kinky) strings.

Core-corona approach. At some early proper time τ_0 , one separates fluid (core) and escaping hadrons, including jet hadrons (corona), based on the momenta and the density of string segments. The corresponding energy-momentum tensor of the core part is transformed into an equilibrium one, needed to start the hydrodynamical evolution. This is based on the hypothesis that equilibration happens rapidly and affects essentially the space components of the energy-momentum tensor.



Viscous hydrodynamic expansion. Starting from the initial proper time τ_0 , the core part of the system evolves according to the equations of relativistic viscous hydrodynamics. A cross-over equation-of-state is used, compatible with lattice QCD.

Statistical hadronization. The "core-matter" hadronizes on some hypersurface defined by a constant temperature T_H , where a so-called Cooper-Frye procedure is employed, using equilibrium hadron distributions.

Final state hadronic cascade. After hadronization, the hadron density is still big enough to allow hadron-hadron rescatterings. For this purpose, we use the UrQMD model.



We will present a "realistic" treatment of the hydrodynamic evolution of ultrarelativistic heavy ion collisions, based on the following features:

- initial conditions obtained from a flux tube approach (EPOS),
 - compatible with the string model used since many years for elementary collisions (electron-positron, proton proton),
 - and the color glass condensate picture;
- consideration of the possibility to have a (moderate) initial collective transverse flow:
- > event-by-event procedure,
 - taking into the account the highly irregular space structure of single events.
 - leading to so-called ridge structures in two-particle correlations;
- core-corona separation. considering the fact that only a part of the matter thermalizes;



Data Comparison

RHIC at few energies: Q-Cumulant

Inclusive charged hadron elliptic flow in Au + Au collisions at

$$\sqrt{s_{NN}} = 7.7 - 39 \text{ GeV}$$

L. Adamczyk et al. [STAR Collaboration], Phys. Rev. C 86 (2012) 054908

Measurements of the elliptic flow, v_2 , of charged hadrons in Au+Au collisions at $\sqrt{s_{NN}} = 7.7$, 19.6, 27, 39 GeV are presented.

Here, we compare at 39 GeV GeV

We use the same method of STAR to compare Data and Results



Data Comparison

RHIC at few energies: Event Plane

Elliptic flow of identified hadrons in Au+Au collisions at

$$\sqrt{s_{NN}} = 7.7 - 62.4 \text{ GeV}$$

L. Adamczyk et al. [STAR Collaboration], Phys. Rev. C 88 (2013) 014902

Measurements of the elliptic flow, v_2 , of identified hadrons $(\pi^{\pm}, K^{\pm}, K_s^0, p, \overline{p}, \phi, \Lambda, \overline{\Lambda}, \Xi^-, \overline{\Xi}^+, \Omega^-, \overline{\Omega}^+)$ in Au+Au collisions at $\sqrt{s_{NN}} = 7.7, 11.5, 19.6, 27, 39$ and 62.4 GeV are presented.

Here, we compare at 19.6 GeV

We use the same method of STAR to compare Data and Results



Q-Cumulant Method: Reference Flow

Q-Cumulant \rightarrow Recent Method to calculate cumulants \rightarrow one loop over data

Correlations between 2 or 4 references particles (REP)



Q-Cumulant Method: Reference Flow

Q-Cumulant \rightarrow Recent Method to calculate cumulants \rightarrow one loop over data

Correlations between 2 or 4 references particles (REP)

Flow vector :
$$Q_n = \sum_{i=1}^M e^{in\phi_i}$$



Q-Cumulant Method: Reference Flow

Q-Cumulant \rightarrow Recent Method to calculate cumulants \rightarrow one loop over data

Correlations between 2 or 4 references particles (REP)

Flow vector :
$$Q_n = \sum_{i=1}^M e^{in\phi_i}$$

$$\langle 2 \rangle = \frac{|Q_n|^2 - M}{M(M-1)}$$

$$\langle 4 \rangle = \frac{|Q_n|^4 + |Q_{2n}|^2 - 2 \times \mathcal{R} [Q_{2n} Q_n^* Q_n^*]}{M(M-1)(M-2)(M-3)} - 2 \frac{2(M-2) \times |Q_n|^2 - M(M-3)}{M(M-1)(M-2)(M-3)}$$



Event Weight

Event Average:

$$\begin{split} \langle \langle 2 \rangle \rangle &= \frac{\sum\limits_{events} (W_{\langle 2 \rangle})_i \langle 2 \rangle_i}{\sum\limits_{events} (W_{\langle 2 \rangle})_i} \qquad \langle \langle 4 \rangle \rangle = \frac{\sum\limits_{events} (W_{\langle 4 \rangle})_i \langle 4 \rangle_i}{\sum\limits_{events} (W_{\langle 4 \rangle})_i} \\ \langle \langle 2' \rangle \rangle &= \frac{\sum\limits_{events} (w_{\langle 2' \rangle})_i \langle 2' \rangle_i}{\sum\limits_{events} (w_{\langle 2' \rangle})_i} \qquad \langle \langle 4' \rangle \rangle = \frac{\sum\limits_{events} (w_{\langle 4' \rangle})_i \langle 4' \rangle_i}{\sum\limits_{events} (w_{\langle 4' \rangle})_i} \end{split}$$

Definition of weights:

$$W_2 = M(M-1)$$
 $W_4 = M(M-1)(M-2)(M-3)$
 $w_{2'} = m_p M - m_q$ $w_{4'} = (m_p M - 3m_q)(M-1)(M-2)$

