Hadronic Vacuum Polarization in QCD and its Evaluation in the Euclidean

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See also next talk by David Greynat.

Theoretical Observation

QCD two-point functions of color singlet local operators, integrated over their euclidean momenta with appropriate weights, govern the hadronic contributions to many electromagnetic and weak interaction processes.

Example: Hadronic Vacuum Polarization two-point function (HVP)

$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iq\cdot x}\langle 0|T\left(J_\mu(x)J_\nu(0)\right)|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(Q^2 \equiv -q^2)\,,$$

$\frac{1}{2}(g_\mu - 2)_{\rm Hadrons} \equiv a_\mu^{\rm HVP}$ Euclidean Representation

$$\frac{1}{2}(g_\mu - 2)_{\rm Hadrons} \equiv a_\mu^{\rm HVP} = \frac{\alpha}{\pi}\int_0^1 dx(1-x)\left[-\Pi\left(\frac{x^2}{1-x}m_\mu^2\right)\right]\,, \quad Q^2 \equiv \frac{x^2}{1-x}m_\mu^2$$

Lautrup- de Rafael '69, also the representation used in lattice QCD (LQCD) *Blum '03*

Persistent discrepancy at the 3 to 4 σ level
between experimental determination of a_μ and its theoretical evaluation.
$a_\mu^{\rm HVP} = (6.926 \pm 0.033) \times 10^{-8}$ is the contribution with the largest error.

Mellin-Barnes Representation of $\Pi(Q^2)$

$\Pi(Q^2)$ obeys the (subtracted) Dispersion Relation:

$$\Pi(Q^2)=\int_{t_0=4m_{\pi^\pm}^2}^{\infty}\frac{dt}{t}\underbrace{\frac{-Q^2}{t+Q^2}}\frac{1}{\pi}\mathrm{Im}\Pi(t)\,,\quad Q^2=-q^2\geq 0\,.$$

Inserting

$$\frac{1}{1+\frac{Q^2}{t}}=\frac{1}{2\pi i}\int\limits_{c_s-i\infty}^{c_s+i\infty}ds\left(\frac{Q^2}{t}\right)^{-s}\Gamma(s)\Gamma(1-s)$$

there follows a Mellin-Barnes Representation of HVP in the Euclidean:

$$\Pi(Q^2)=-\frac{Q^2}{t_0}\frac{1}{2\pi i}\int\limits_{c_s-i\infty}^{c_s+i\infty}ds\left(\frac{Q^2}{t_0}\right)^{-s}\Gamma(s)\Gamma(1-s)\,\mathcal{M}(s)\,,\quad c_s\equiv\mathrm{Re}(s)\in]0,1[\,,$$

in terms of the Mellin Transform of the HVP Spectral Function

$$\mathcal{M}(s)=\int_{t_0}^{\infty}\frac{dt}{t}\left(\frac{t}{t_0}\right)^{s-1}\frac{1}{\pi}\mathrm{Im}\Pi(t)\,,\quad t_0=4m_{\pi^\pm}^2\,,\quad \mathrm{Re}\,s<1\,.$$

$$\mathcal{M}(s)\underset{s\to 1}{\sim}\left(\frac{\alpha}{\pi}\right)\left(\frac{2}{3}\right)N_c\,\frac{1}{3}\,\frac{1}{1-s}\,,\qquad \text{from pQCD.}$$

The Mellin-Barnes Representation of $\Pi(Q^2)$ is very useful for asymptotic expansions of $\Pi(Q^2)$ at Q^2 small (χPT) and Q^2 large (OPE).

See e.g. Friot-Greynat-de Rafael'08, Aguilar-Greynat-de Rafael'12, Friot-Greynat'12, ...

Mellin Moments and Lattice QCD (LQCD)

Determination of a few terms of the Taylor expansion of $\Pi(Q^2)$ at Q^2 small in LQCD, i.e. of a few derivatives of $\Pi(Q^2)$ at $Q^2=0$, equivalent to Mellin Moments at $n=0,1,2,\dots$

$$\mathcal{M}(-n)=\int_0^\infty \frac{dt}{t}\left(\frac{t_0}{t}\right)^{1+n}\frac{1}{\pi}\mathrm{Im}\Pi(t)=\frac{(-1)^{n+1}}{(n+1)!}(t_0)^{n+1}\left(\frac{\partial^{n+1}}{(\partial Q^2)^{n+1}}\Pi(Q^2)\right)_{Q^2=0}$$

Integral Representation of a_μ^{HVP} in terms of the Mellin Transform $\mathcal{M}(s)$

$$a_\mu^{\mathrm{HVP}}=\left(\frac{\alpha}{\pi}\right)\frac{m_\mu^2}{t_0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{m_\mu^2}{t_0}\right)^{-s}\mathcal{F}(s)\,\underbrace{\mathcal{M}(s)}\,,\qquad \mathrm{Re\ c}\in\,]0,+1[$$

$$\mathcal{F}(s)=-\Gamma(3-2s)\,\Gamma(-3+s)\,\Gamma(1+s)$$

$$\mathcal{M}(s)=\underbrace{\int_{t_0}^\infty\frac{dt}{t}\left(\frac{t}{t_0}\right)^{s-1}\frac{1}{\pi}\mathrm{Im}\Pi(t)}_{\textit{Mellin Transform of the Spectral Function}}$$

In particular $a_\mu^{\mathrm{HVP}}\le\left(\frac{\alpha}{\pi}\right)\frac{m_\mu^2}{t_0}\mathcal{M}(0)$ (*J.S. Bell-E.deR.'69*)

This provides an interesting starting point towards an alternative evaluation of the HVP contribution to a_μ^{HVP} from first principles.

Ramanujan's Master Theorem and Marichev's Mellin Transforms

- Expansion for Q^2-small:

$$-\frac{t_0}{Q^2}\Pi(Q^2)_{Q^2\to 0} = \left\{\mathcal{M}(0) - \frac{Q^2}{t_0}\mathcal{M}(-1) + \left(\frac{Q^2}{t_0}\right)^2\mathcal{M}(-2) - \left(\frac{Q^2}{t_0}\right)^3\mathcal{M}(-3) + \cdots\right\},$$

- Ramanujan's Theorem:

$$\int_0^\infty d\left(\frac{Q^2}{m_\mu^2}\right)\left(\frac{Q^2}{m_\mu^2}\right)^{s-1}\left\{-\frac{t_0}{Q^2}\Pi(Q^2)\right\}_{Q^2\to 0} = \Gamma(s)\Gamma(1-s)\,\mathcal{M}(s)\,,$$

Guarantees the convergence of discrete Moments $\mathcal{M}(-n)$ to the full Mellin Transform $\mathcal{M}(s)$.

What is the Best Interpolating Procedure

when one only knows -numerically- a few moments?

Marichev's Class of Mellin Transforms and Interpolating Approach

$$\mathcal{M}(s) = C\prod_{i,j,k,l}\frac{\Gamma(a_i - s)\Gamma(c_j + s)}{\Gamma(b_k - s)\Gamma(d_l + s)}\,,$$

C, a_i, b_k, c_j, d_l real constants, s with only $\pm$ coefficient.
No poles , nor zeros, and monotonously deceasing for $s < 0$.

The inverse Mellin transform of a Marichev's class $\mathcal{M}(s)$-function
is a Generalized Hypergeometric Function

Ramanujan's Theorem and Marichev's Interpolation in QED

Ramanujan's Theorem in QED

- Lowest Order Vacuum Polarization in QED for a fermion of mass m:

$$-\frac{4m^2}{Q^2}\Pi^{\rm QED}(Q^2)\underset{Q^2\to 0}{\sim}\sum_{n=0}(-1)^n\left(\frac{Q^2}{4m^2}\right)^n\left\{\frac{\alpha}{2\pi}\frac{1}{n+1}\frac{\sqrt{\pi}}{2}\frac{\Gamma(3+n)}{\Gamma(\frac{7}{2}+n)}\right\}$$

- Then -*without doing the calculation*- Ramanujan's Theorem implies:

$$\mathcal{M}^{\rm QED}(s)\equiv\int_{4m^2}^{\infty}\frac{dt}{t}\left(\frac{t}{4m^2}\right)^{s-1}\frac{1}{\pi}{\rm Im}\Pi^{\rm QED}(t)=\frac{\alpha}{\pi}\frac{1}{3}\frac{1}{1-s}\frac{3\sqrt{\pi}}{4}\frac{\Gamma(3-s)}{\Gamma(\frac{7}{2}-s)}$$

Marichev's Interpolation Approach in QED

- *First Marichev Interpolation [Matching at $s=1$]*

$$\mathcal{M}^{\rm QED}(s)\Rightarrow\mathcal{M}^{(1)}(s)=\frac{\alpha}{\pi}\frac{1}{3}\frac{1}{1-s}=\frac{\alpha}{\pi}\frac{1}{3}\frac{\Gamma(1-s)}{\Gamma(2-s)}$$

- *Second Marichev Interpolation [Matching at $s=1$ and $s=0$]*

$$\mathcal{M}^{\rm QED}(s)\Rightarrow\mathcal{M}^{(1)}_{(0)}(s)=\frac{\alpha}{\pi}\frac{1}{3}\frac{1}{1-s}\frac{\Gamma(b-1)}{\Gamma(b-s)}\,,\quad\Rightarrow b=\frac{9}{4}$$

- *Third Marichev Interpolation [Matching at $s=1$, $s=0$ and $s=2$]*

$$\mathcal{M}^{\rm QED}(s)\Rightarrow\mathcal{M}^{\rm QED}(s)\|^{(1),(2)}_{(0)}=\frac{\alpha}{\pi}\frac{1}{3}\frac{1}{1-s}\frac{\Gamma(c-1)}{\Gamma(c-s)}\frac{\Gamma(d-s)}{\Gamma(d-1)}\,,\quad\Rightarrow c=\frac{7}{2}\,,d=3$$

The Third Marichev Interpolation is already the *Exact Result* for $\mathcal{M}^{\rm QED}(s)$!!!

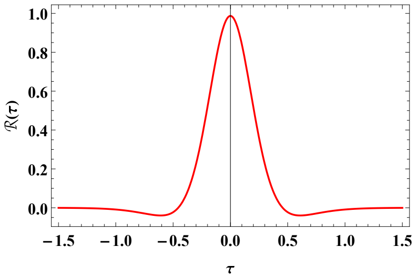
Application to $a^{\mathrm{QED}}(\mathrm{VP})$ (same mass for external and VP fermions)

$$a^{\mathrm{QED}}(\mathrm{VP}) = \left(\frac{\alpha}{\pi}\right)\frac{1}{4}\frac{1}{2\pi i}\int_{c_s-i\infty}^{c_s+i\infty} ds \left(\frac{1}{4}\right)^{-s}\mathcal{F}(s)\,\mathcal{M}^{\mathrm{QED}}(s)$$

With s within the *fundamental strip*, e.g. $s=\frac{1}{2}-i\tau$:

$$a^{\mathrm{QED}}(\mathrm{VP}) \doteq \left(\frac{\alpha}{\pi}\right)^2\frac{1}{4}\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\tau \underbrace{\left[\frac{1}{1+\tau^2}-\frac{10}{4+\tau^2}+\frac{40}{25+\tau^2}\right]\frac{\pi^2}{[\cosh(\pi\tau)]^2}}_{\Re(\tau)}$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \times 0.01568742185910 \quad \Leftrightarrow \left(\frac{\alpha}{\pi}\right)^2\left(\frac{119}{36}-\frac{\pi^2}{3}\right)$$



Tests with a Toy Model of the Hadronic Spectral Function (*L. Lellouch*)

- pQCD INPUT:

$$\mathcal{M}(1)=\mathcal{C}\,\frac{1}{1-s}\quad\text{and}\quad\mathcal{M}(2)=0\,,\quad\mathcal{C}\equiv\frac{\alpha}{\pi}\frac{1}{3}\,\mathrm{N_c}\left(\frac{2}{3}\right)$$

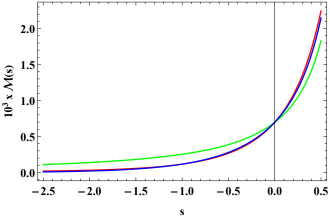
- *First Marichev Interpolation* [only $\mathcal{M}(0)$ is known]

$$\mathcal{M}_{(0)}^{(1,2)}(s)=\mathcal{C}\,\frac{1}{(1-s)\Gamma(2-s)}\,\frac{\Gamma\left[1+\mathcal{A}\,-s\right]}{\Gamma\left(\mathcal{A}\right)}\,,\quad\mathcal{A}=\frac{1}{\mathcal{C}}\mathcal{M}(0)$$

- *Second Marichev Interpolation* [$\mathcal{M}(0)$ and $\mathcal{M}(-1)$ are known]

$$\mathcal{M}_{(0,-1)}^{(1,2)}(s)=\mathcal{C}\,\frac{1}{1-s}\,\frac{1}{\Gamma(2-s)}\,\frac{\Gamma(e-s)}{\Gamma(e-1)}\,\frac{\Gamma(f-1)}{\Gamma(f-s)}$$

$$\mathcal{A}=\frac{1}{\mathcal{C}}\mathcal{M}(0)\,,\quad\mathcal{R}=4\,\frac{\mathcal{M}(-1)}{\mathcal{M}(0)}\,,\quad f=\frac{1-\mathcal{A}}{\mathcal{R}-\mathcal{A}}\,,\quad\text{and}\quad\mathrm{e}=\mathcal{R}\mathrm{f}$$



- First Marichev reproduces $a_\mu^{(\mathrm{HVP})}(\mathrm{Toy\ Model})$ at the 6.6% level (*green*)
- Second Marichev reproduces $a_\mu^{(\mathrm{HVP})}(\mathrm{Toy\ Model})$ at the 0.6% level (*blue* - *toy model red*)

Application to LQCD determination of $\mathcal{M}(0)$ and $\mathcal{M}(-1)$ (*BMWc'17*)

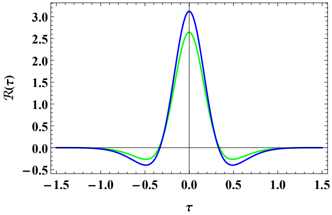
With charm subtracted:

$$\mathcal{M}(0)_{\rm BMWc} = (0.704 \pm 0.021) \times 10^{-3} \quad \text{and} \quad \mathcal{M}(-1)_{\rm BMWc} = (0.101 \pm 0.007) \times 10^{-3}$$

Integral Representation of $a_\mu^{\rm HVP}$ in terms of Marichev Mellin Transforms

$$a_\mu^{\rm HVP}(\textit{Second}) = \left(\frac{\alpha}{\pi}\right) \frac{m_\mu^2}{t_0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau \left(\frac{m_\mu^2}{t_0}\right)^{-\left(\frac{1}{2}-i\tau\right)} \mathcal{F}\left(\frac{1}{2}-i\tau\right) \mathcal{M}_{(0,-1)}^{(1,2)}\left(\frac{1}{2}-i\tau\right)$$

$$\mathcal{R}(\tau) = \frac{1}{\mathcal{C}} \left(\frac{m_\mu^2}{t_0}\right)^{-\left(\frac{1}{2}-i\tau\right)} \mathcal{F}\left(\tfrac{1}{2}-i\tau\right) \left[\mathcal{M}_{(0,-1)}^{(1,2)}\left(\tfrac{1}{2}-i\tau\right)\right] \left[\mathcal{M}_{(0)}^{(1,2)}\left(\tfrac{1}{2}-i\tau\right)\right]$$



The predicted values using these Marichev Interpolations are:

$$a_\mu^{\rm HVP}(\textit{First}) = (6.23 \pm 0.18) \times 10^{-8} \quad a_\mu^{\rm HVP}(\textit{Second}) = (6.81 \pm 0.30) \times 10^{-8}$$

Conclusions

- We conclude that with a precise determination of $\mathcal{M}(0)$ i.e. *with a precise determination of just the slope of the HVP function at the origin from LQCD*, one can already obtain a result for $a_\mu^{\rm HVP}$ which provides a first rough test of the determinations using experimental data.
- We wish to emphasize that the method we propose, besides the eventual determination of $\mathcal{M}(0)$, only uses as other input two well known properties of QCD: *asymptotic freedom* and the fact that in the chiral limit there is *no* $1/Q^2$ *term in the OPE of* $\Pi(Q^2)$.
- The *Second Marichev Interpolation* of the Mellin Transform of the hadronic spectral function results in a much more accurate determination. It includes as an input the determinations of the *first two moments* $\mathcal{M}(0)$ *and* $\mathcal{M}(-1)$ *accessible to LQCD*.
- The test with the *Toy Model* above results in a determination of $a_\mu^{\rm HVP}$ with an *accuracy of* 0.6% which is very encouraging.
- The application to the determination of the $\mathcal{M}(0)$ and $\mathcal{M}(-1)$ moments from LQCD (BMWc'16) points towards a *very promising future* in this direction.