

## HVP Contribution to $\mu/e/\tau$ g-2 from 1st Principle at Physical Point

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CPPM-RPP 2017, Marseille, 24 April, 2017

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- Disconnected Part
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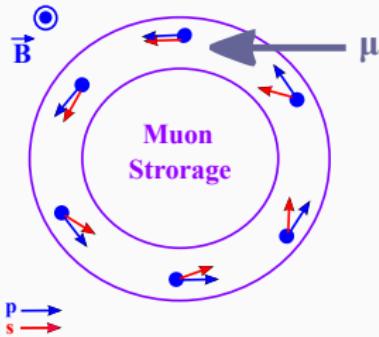
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# Muon Anomalous Magnetic Moment $a_\mu$



$$a_\mu \equiv \frac{g_\mu - 2}{2} , \quad (1)$$

$$\vec{M} = g_\mu \frac{e}{2m_\mu} \vec{S} . \quad (2)$$

$$a_\mu^{\text{exp.}} = a_\mu^{\text{SM}} ?$$

$a_\mu^{\text{exp.}}$  vs.  $a_\mu^{\text{SM}}$

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	Ref.
QED [5 loops]	$11658471.8951 \pm 0.0080$	[Aoyama et al '12]
HVP-LO (pheno.)	$692.3 \pm 4.2$	[Davier et al '11]
	$694.9 \pm 4.3$	[Hagiwara et al '11]
	$681.5 \pm 4.2$	[Benayoun et al '16]
HVP-NLO	$-9.84 \pm 0.07$	[Hagiwara et al '11] [Kurz et al '11]
HVP-NNLO	$1.24 \pm 0.01$	[Kurz et al '11]
HLbyL	$10.5 \pm 2.6$	[Prades et al '09]
Weak (2 loops)	$15.36 \pm 0.10$	[Gnendiger et al '13]
SM tot [0.42 ppm]	$11659180.2 \pm 4.9$	[Davier et al '11]
[0.43 ppm]	$11659182.8 \pm 5.0$	[Hagiwara et al '11]
[0.51 ppm]	$11659184.0 \pm 5.9$	[Aoyama et al '12]
Exp [0.54 ppm]	$11659208.9 \pm 6.3$	[Bennett et al '06]
Exp – SM	$28.7 \pm 8.0$	[Davier et al '11]
	$26.1 \pm 7.8$	[Hagiwara et al '11]
	$24.9 \pm 8.7$	[Aoyama et al '12]

FNAL E989 (2017): 0.14-ppm, J-PARC E34: 0.1-ppm

$a_\mu^{\text{exp.}}$  vs.  $a_\mu^{\text{SM}}$

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# Objective

Leading-Order (LO) Hadronic Vacuum Polarization (HVP) contribution to muon/electron/tau  $g - 2$ :

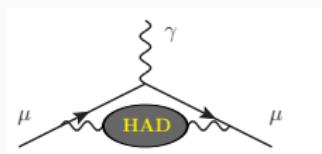
$$a_{\ell}^{\text{LO-HVP}, f} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 \omega(Q^2/m_\ell^2) \hat{\Pi}^f(Q^2), \quad (3)$$

$$\ell = e, \mu, \tau,$$

$$f = l(u, d), s, c, d(\text{disc}).$$

where,

$$\begin{aligned} \hat{\Pi}^f(Q^2) &= 4\pi^2 q_f^2 (\Pi(Q^2) - \Pi(0)) \\ &= 4\pi^2 \int_0^\infty dt t^2 \left[ 1 - \left( \frac{\sin(z/2)}{z/2} \right)^2 \right]_{z=Qt} \frac{1}{3} \sum_{i=1}^3 C_i^f(t), \end{aligned} \quad (4)$$



$$C_{\mu\nu}^{f=l,s,c}(t) = q_{f=l,s,c}^2 \int_0^\infty d^3x \underbrace{\langle j_\mu^f(x) j_\nu^f(0) \rangle}_{\text{conn}}, \quad (5)$$

$$C_{\mu\nu}^{f=d}(t) = q_{f=d}^2 \int_0^\infty d^3x \underbrace{\langle (\bar{l}\gamma_\mu l - \bar{s}\gamma_\mu s)(\bar{l}\gamma_\nu l - \bar{s}\gamma_\nu s) \rangle}_{\text{disc}}, \quad (6)$$

$$(q_l^2, q_s^2, q_c^2, q_d^2) = (5/9, -1/9, 4/9, 1/9). \quad (7)$$

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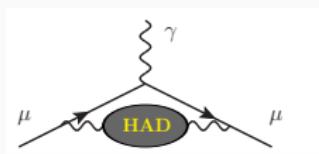
$$a_{\ell}^{\text{LO-HVP}, \textcolor{blue}{f}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 \omega(Q^2/m_{\ell}^2) \hat{\Pi}^{\textcolor{blue}{f}}(Q^2), \quad (3)$$

$$\ell = e, \mu, \tau,$$

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$$C_{\mu\nu}^{\textcolor{blue}{f}=l,s,c}(t) = q_{\textcolor{blue}{f}=l,s,c}^2 \int_0^\infty d^3x \underbrace{j_{\mu}^{\textcolor{blue}{f}}(x) j_{\nu}^{\textcolor{blue}{f}}(0)}_{|\text{conn}|}, \quad (5)$$

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$$(q_l^2, q_s^2, q_c^2, q_d^2) = (5/9, -1/9, 4/9, 1/9). \quad (7)$$

# Objective

Leading-Order (LO) Hadronic Vacuum Polarization (HVP) contribution to muon/electron/tau  $g - 2$ :

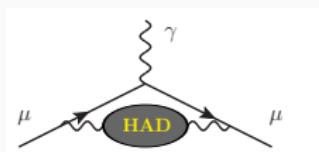
$$a_{\ell}^{\text{LO-HVP}, \textcolor{blue}{f}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 \omega(Q^2/m_{\ell}^2) \hat{\Pi}^{\textcolor{blue}{f}}(Q^2), \quad (3)$$

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where,

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$$C_{\mu\nu}^{\textcolor{blue}{f}=l,s,c}(t) = q_{\textcolor{blue}{f}=l,s,c}^2 \int_0^\infty d^3x \underbrace{\langle j_\mu^{\textcolor{blue}{f}}(x) j_\nu^{\textcolor{blue}{f}}(0) \rangle}_{\text{conn}}, \quad (5)$$

$$C_{\mu\nu}^{\textcolor{blue}{f}=d}(t) = q_{\textcolor{blue}{f}=d}^2 \int_0^\infty d^3x \underbrace{\langle (\bar{l}\gamma_\mu l - \bar{s}\gamma_\mu s)(\bar{l}\gamma_\nu l - \bar{s}\gamma_\nu s) \rangle}_{\text{disc}}, \quad (6)$$

$$(q_l^2, q_s^2, q_c^2, q_d^2) = (5/9, -1/9, 4/9, 1/9) . \quad (7)$$

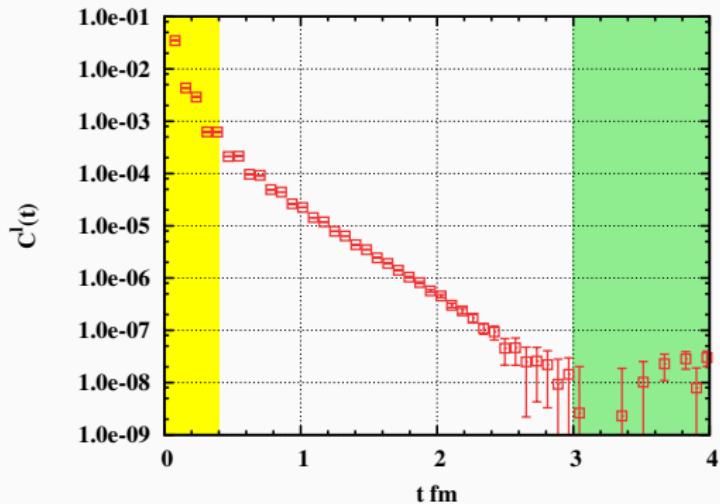
# Simulation Setup

$a[\text{fm}]$	$N_t$	$N_s$	#traj.	$M_\pi[\text{MeV}]$	$M_K[\text{MeV}]$	#SRC (l,s,c,d)
0.134	64	48	10000	$\sim 131$	$\sim 479$	(768, 64, 64, 9000)
0.118	96	56	15000	$\sim 132$	$\sim 483$	(768, 64, 64, 6000)
0.111	84	56	15000	$\sim 133$	$\sim 483$	(768, 64, 64, 6144)
0.095	96	64	25000	$\sim 133$	$\sim 488$	(768, 64, 64, 3600)
0.078	128	80	35000	$\sim 133$	$\sim 488$	(768, 64, 64, 6144)
0.064	144	96	04500	$\sim 133$	$\sim 490$	(768, 64, 64, -)

## State Of The Art

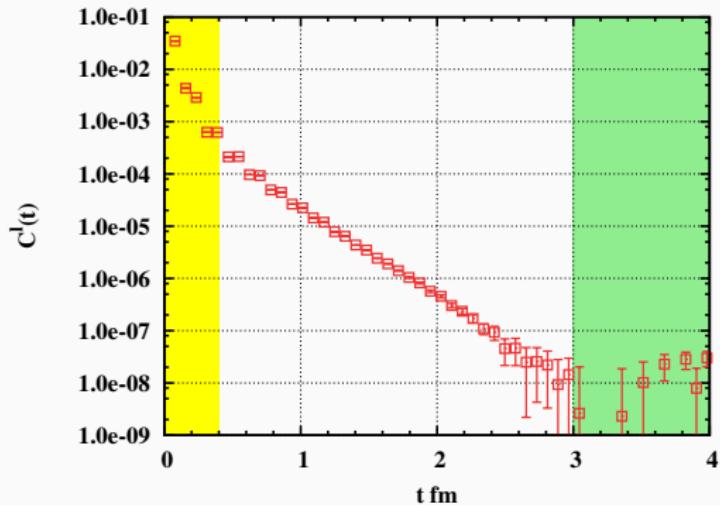
- Simulations at Physical Pion/Kaon Mass with 6 lattice spacings.
- $(L, T) \sim (6, 9 - 12)$  fm.
- $N_f = (2 + 1 + 1)$ : Two degenerate light quarks with strange and charm.
- Both Connected and Disconnected.

# Correlator: An Example



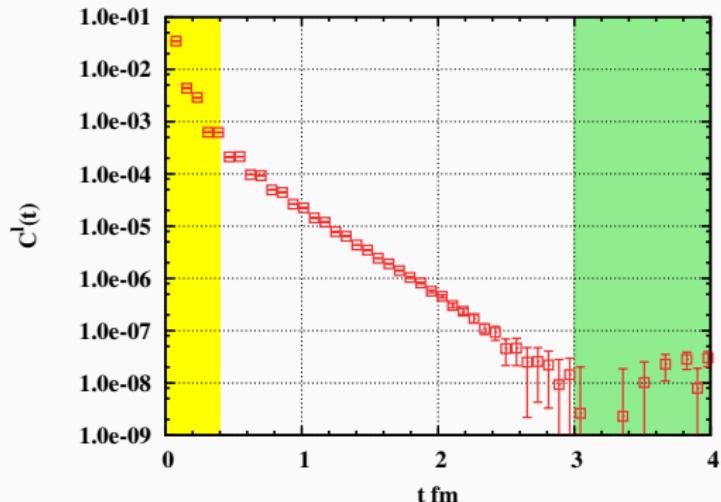
$$C'(t) = \frac{5}{9} \sum_{\vec{x}} \frac{1}{3} \sum_{i=1}^3 \langle j_i'(\vec{x}, t) j_i'(0) \rangle , \quad (8)$$

# Correlator: An Example



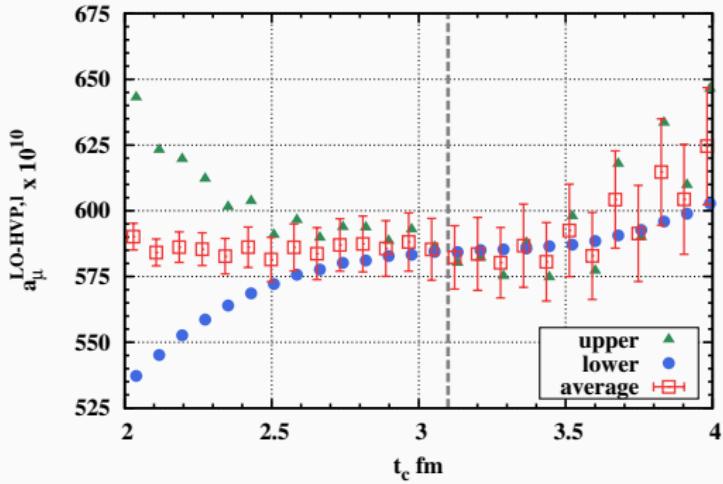
- $C'(t) \rightarrow C'_{\text{up}}(t, t_c)$  or  $C'_{\text{low}}(t, t_c)$ .
- For  $t \leq t_c$ ,  $C'_{\text{up}/\text{low}}(t, t_c) = C'(t)$ .
- For  $t > t_c$ ,  $C'_{\text{up}}(t, t_c) = C'(t_c) \frac{\cosh[2E_{2\pi}(T/2-t)]}{\cosh[2E_{2\pi}(T/2-t_c)]}$ ,  $E_{2\pi} = \sqrt{M_\pi^2 + (\frac{2\pi}{L})^2}$ ,  $C'_{\text{low}}(t, t_c) = 0.0$ .

# Correlator: An Example



- $C'(t) \rightarrow C'_{\text{up}}(t, t_c)$  or  $C'_{\text{low}}(t, t_c)$ .
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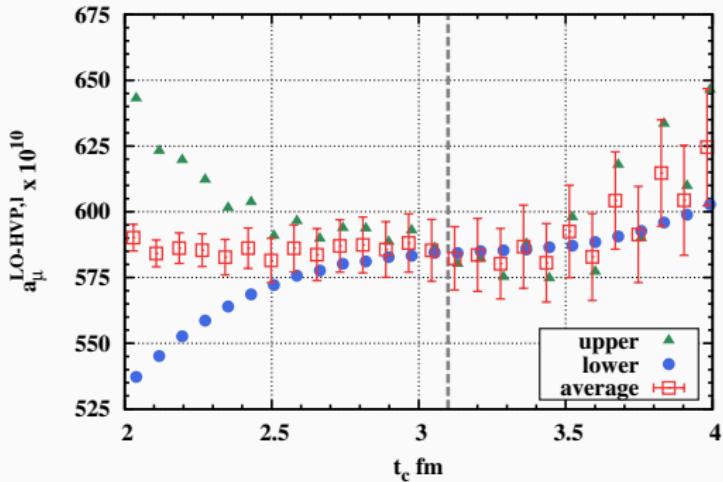
# IR-CUT ( $t_c$ ) Dependence of Light Component: $a_\ell^{\text{LO}-\text{HVP},l}$



$$a_{\ell, \text{up}/\text{low}}^{\text{LO}-\text{HVP},l}(t_c) = \sum_{t=0}^T W_\Pi(t, m_\ell) C_{\text{up}/\text{low}}^l(t, t_c) . \quad (9)$$

We adopt,  $a_\ell^{\text{LO}-\text{HVP},l} = 0.5(a_{\ell,\text{up}}^{\text{LO}-\text{HVP},l} + a_{\ell,\text{low}}^{\text{LO}-\text{HVP},l})|_{t_c=3.1\text{fm}}$

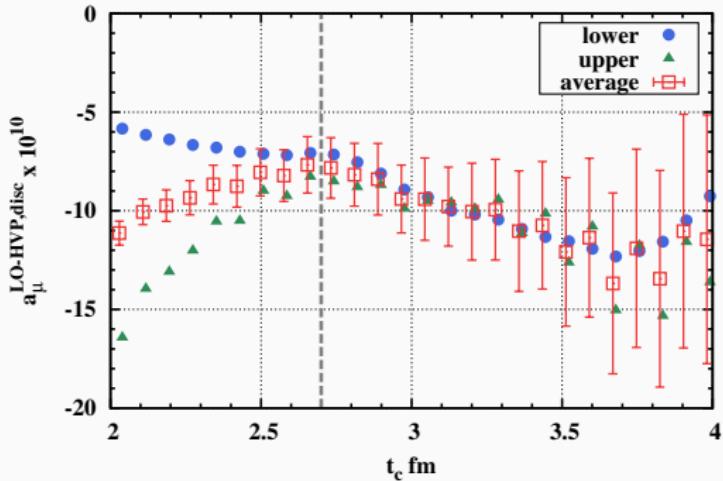
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# IR-CUT ( $t_c$ ) Dependence of Disc. Component: $a_\ell^{\text{LO}-\text{HVP},d}$



$$a_{\ell, \text{up}/\text{low}}^{\text{LO}-\text{HVP},d}(t_c) = \sum_{t=0}^T W_{\Pi}(t, m_\ell) C_{\text{up}/\text{low}}^d(t, t_c) . \quad (10)$$

We adopt,  $a_\ell^{\text{LO}-\text{HVP},d} = 0.5(a_{\ell,\text{up}}^{\text{LO}-\text{HVP},d} + a_{\ell,\text{low}}^{\text{LO}-\text{HVP},d})|_{t_c=2.7 \text{ fm}}$

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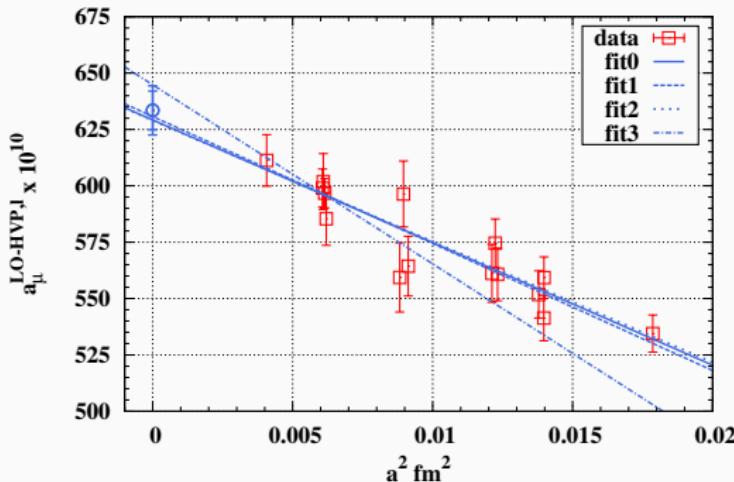
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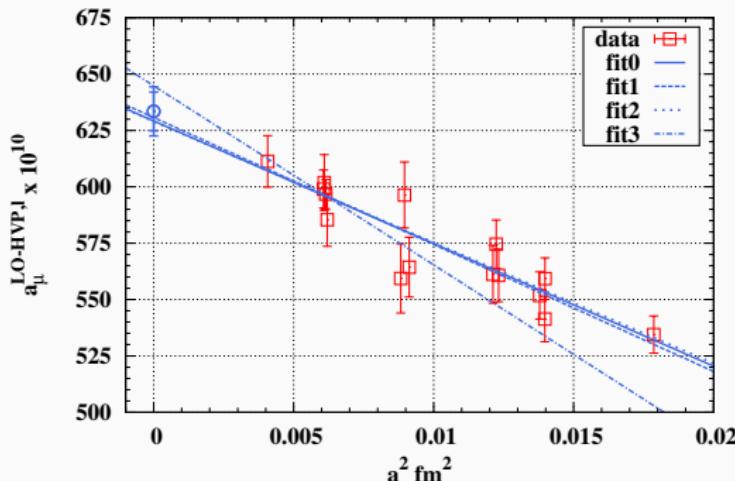
# Continuum Extrap. of Light Component: $a_\mu^{\text{LO}-\text{HVP},l}$



$$F(a_\mu^{\text{LO}-\text{HVP},l}, A, B, C_{M_\pi}, C_{M_K}) = a_\mu^{\text{LO}-\text{HVP},l} \frac{1 + A a^2 + \dots}{1 + B a^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K).$$

$$a_\mu^{\text{LO}-\text{HVP},l} = 633.44(8.61)(6.60), \quad \chi^2/\text{d.o.f.} = 8.8/12 \text{ (fit1 case).}$$

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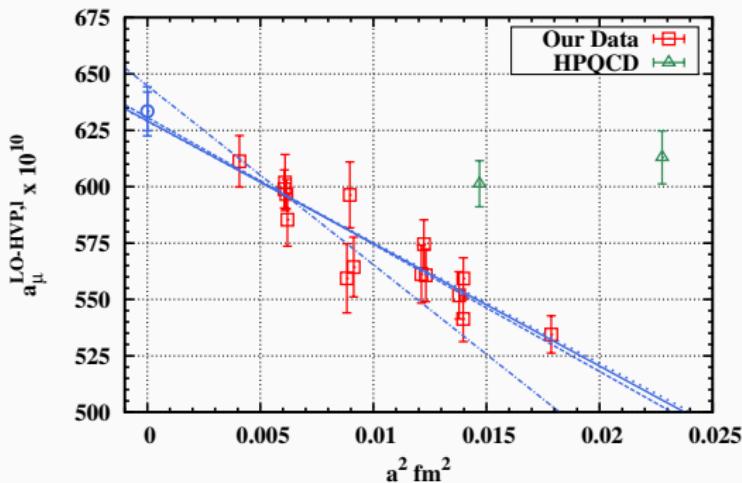


Figure: Red-squares = Our data. Green-triangles = HPQCD, 1403.1778.

$$a_\mu^{\text{LO}-\text{HVP},l} = 633.44(8.61)(6.60) , \quad \chi^2/\text{d.o.f.} = 8.8/12 \text{ (fit1 case).}$$

# Continuum Extrap. of Strange Component: $a_\mu^{\text{LO}-\text{HVP},s}$

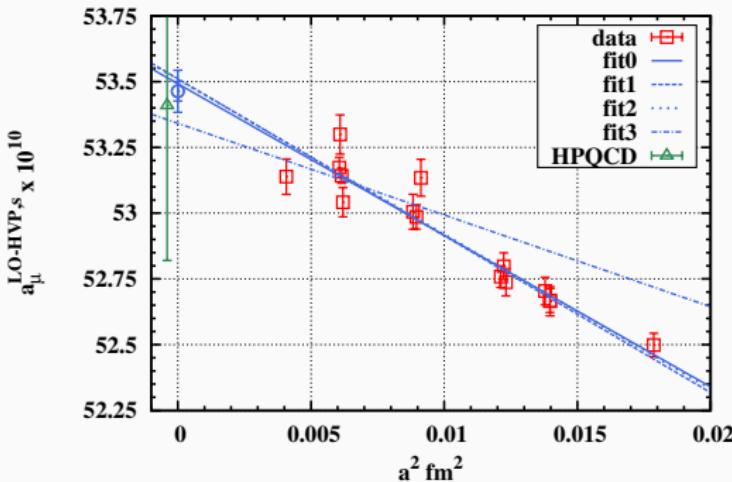


Figure: Red-squares = Our data. Green-triangles = HPQCD, 1403.1778.

$$F(a_\mu^{\text{LO}-\text{HVP},s}, A, B, C_{M_\pi}, C_{M_K}) = a_\mu^{\text{LO}-\text{HVP},s} \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K).$$

$$a_\mu^{\text{LO}-\text{HVP},s} = 53.46(0.04)(0.07) , \quad \chi^2/\text{d.o.f.} = 15.5/11 \text{ (fit1 case).}$$

# Continuum Extrap. of Charm Component: $a_\mu^{\text{LO}-\text{HVP},c}$

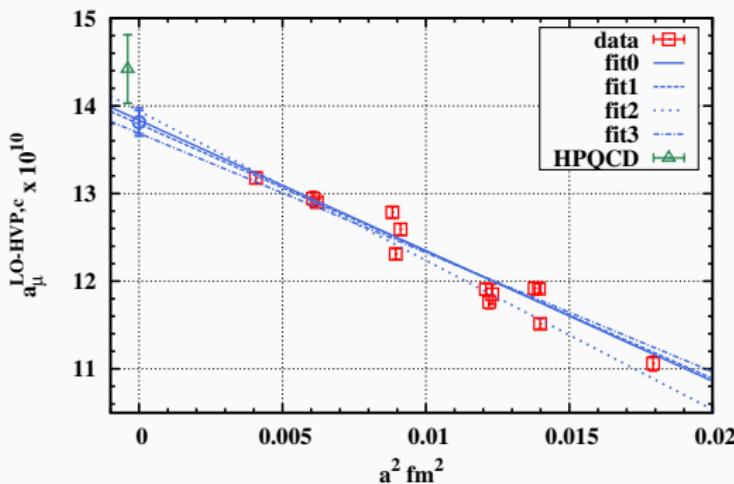
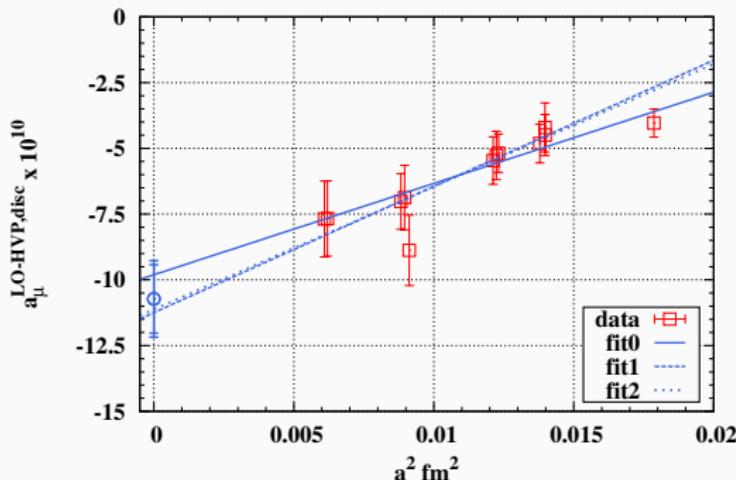


Figure: Red-squares = Our data. Green-triangles = HPQCD, 1403.1778.

$$F(a_\mu^{\text{LO}-\text{HVP},c}, A, B, C_{M_\pi}, C_{M_K}) = a_\mu^{\text{LO}-\text{HVP},c} \frac{1 + A a^2 + \dots}{1 + B a^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K).$$

$$a_\mu^{\text{LO}-\text{HVP},c} = 13.81(0.13)(0.09), \quad \chi^2/\text{d.o.f.} = 24.9/9 \text{ (fit1 case).}$$

## Continuum Extrap. of Disc. Component: $a_\mu^{\text{LO}-\text{HVP},d}$



$$F(a_\mu^{\text{LO}-\text{HVP},d}, A, B, C_{M_\pi}, C_{M_K}) = a_\mu^{\text{LO}-\text{HVP},d} \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K).$$

$$a_\mu^{\text{LO}-\text{HVP},d} = -10.73(1.30)(0.65) , \quad \chi^2/\text{d.o.f.} = 2.9/9 \text{ (fit1 case).}$$

# Short Summary on $a_\mu^{\text{LO}-\text{HVP},f}$ (Preliminary)

component ( $f$ )	$a_\mu^{\text{LO}-\text{HVP},f} \times 10^{10}$
light	633.44(8.61)(6.60)
strange	53.46(0.04)(0.07)
charm	13.81(0.13)(0.09)
disconnected	-10.73(1.30)(0.65)
$I = 0$	119.90(3.97)(2.95)
$I = 1$	570.09(7.75)(5.94)
<b>total</b>	<b>689.99(8.71)(6.64)</b>

TOTAL ERROR: 1.6 % for  $a_\mu^{\text{LO}-\text{HVP}}$

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# Corrections: FV, Perturv, and Isospin Breaking

- **Finite Volume Correction:** (c.f. Aubin et.al., PRD (2016).)

$$(a_\mu^{\text{LO-HVP},I}(\text{16fm}) - a_\mu^{\text{LO-HVP},I}(\text{6fm}))|_{\text{XPT}} \times 10^{10} \quad (11)$$

$$= -7.46(3.73) , \quad (1.08\%) . \quad (12)$$

- **Perturbative Correction:**

$$a_\mu^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left( \int_0^{Q_{max}} + \int_{Q_{max}}^{\infty} \right) \frac{dQ}{m_\mu} \omega\left(\frac{Q^2}{m_\mu^2}\right) \hat{\Pi}(Q^2) , \quad (13)$$

$$= a_\mu^{\text{LO-HVP}}(Q \leq Q_{max}) + a_\mu^{\text{LO-HVP}}(Q > Q_{max}) \quad (14)$$

$$a_\mu^{\text{LO-HVP}}(Q > Q_{max}) \xrightarrow{Q_{max}=\sqrt{2}\text{GeV}} 2.1613(36)(50) , (0.3\%) . \quad (15)$$

- **Isospin Breaking Correction (muon):**

- $\rho - \omega$  ( $2.80 \pm 0.19$ ) and  $\rho - \gamma$  ( $-2.71 \pm 0.27$ ) mixing,
- final state radiation ( $3.86 \pm 0.39$ ) with 10% error added,
- $\pi^0\gamma$  ( $4.42 \pm 0.19$ ),  $\eta\gamma$  ( $0.64 \pm 0.02$ ) contributions.
- Above effects are largely cancelled by the mass shift:  $M_\pi^{\text{iso}} \rightarrow M_{\pi^\pm}$ .

Total: 1.3% uncertainty (Ref: 1612.02364 (BMW-Collaboration)).

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$$= -7.46(3.73) , \quad (1.08\%) . \quad (12)$$

- **Perturative Correction:**

$$a_\mu^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left( \int_0^{Q_{max}} + \int_{Q_{max}}^{\infty} \right) \frac{dQ}{m_\mu} \omega\left(\frac{Q^2}{m_\mu^2}\right) \hat{\Pi}(Q^2) , \quad (13)$$

$$= a_\mu^{\text{LO-HVP}}(Q \leq Q_{max}) + a_\mu^{\text{LO-HVP}}(Q > Q_{max}) \quad (14)$$

$$a_\mu^{\text{LO-HVP}}(Q > Q_{max}) \xrightarrow{Q_{max}=\sqrt{2}\text{GeV}} 2.1613(36)(50) , (0.3\%) . \quad (15)$$

- **Isospin Breaking Correction (muon):**

- $\rho - \omega$  ( $2.80 \pm 0.19$ ) and  $\rho - \gamma$  ( $-2.71 \pm 0.27$ ) mixing,
- final state radiation ( $3.86 \pm 0.39$ ) with 10% error added,
- $\pi^0\gamma$  ( $4.42 \pm 0.19$ ),  $\eta\gamma$  ( $0.64 \pm 0.02$ ) contributions.
- Above effects are largely cancelled by the mass shift:  $M_\pi^{\text{iso}} \rightarrow M_{\pi^\pm}$ .

Total: 1.3% uncertainty (Ref: 1612.02364 (BMW-Collaboration)).

# Corrections: FV, Perturv, and Isospin Breaking

- Finite Volume Correction: (c.f. Aubin et.al., PRD (2016).)

$$(a_\mu^{\text{LO-HVP},I}(\textcolor{red}{16\,fm}) - a_\mu^{\text{LO-HVP},I}(\textcolor{red}{6\,fm}))|_{\text{XPT}} \times 10^{10} \quad (11)$$

$$= -7.46(3.73) , \quad (1.08\%) . \quad (12)$$

- Perturulative Correction:

$$a_\mu^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left( \int_0^{Q_{max}} + \int_{Q_{max}}^{\infty} \right) \frac{dQ}{m_\mu} \omega\left(\frac{Q^2}{m_\mu^2}\right) \hat{\Pi}(Q^2) , \quad (13)$$

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Total: 1.3% uncertainty (Ref: 1612.02364 (BMW-Collaboration)).

# Short Summary on $a_\mu^{\text{LO}-\text{HVP},f}$ (Preliminary)

component ( $f$ )	$a_\mu^{\text{LO}-\text{HVP},f} \times 10^{10}$
light	633.44(8.61)(6.60)
strange	53.46(0.04)(0.07)
charm	13.81(0.13)(0.09)
disconnected	-10.73(1.30)(0.65)
$I = 0$	119.90(3.97)(2.95)
$I = 1$	570.09(7.75)(5.94)
<b>total</b>	<b>689.99(8.71)(6.64)</b>
<b>tot. + crr.</b>	<b>699.61(8.71)(6.64)(3.73)(9.09)</b>

HPQCD '16	666(6)(12)
Davier et al '11	692.3(4.2)
Hagiwara et al '11	694.9(4.3)
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## Summary Taboe: $g - 2$ for $e$ and $\tau$ (Prelim.)

component ( $f$ )	$a_e^{\text{LO-HVP}} \times 10^{14}$	$a_\tau^{\text{LO-HVP}} \times 10^8$
light	170.08(3.05)(2.04)	140.24(0.61)(0.79)
strange	13.56(0.01)(0.02)	15.59(0.01)(0.01)
charm	3.31(0.01)(0.02)	5.91(0.01)(0.03)
disconnected	-3.11(0.43)(0.19)	-1.47(0.10)(0.08)
$I = 0$	30.77(1.40)(0.91)	34.06(0.28)(0.36)
$I = 1$	153.07(2.74)(1.84)	126.22(0.55)(0.71)
<b>total</b>	<b>183.84(3.08)(2.05)</b>	<b>160.28(0.62)(0.80)</b>
<b>tot. + crr.</b>	<b>186.41(3.08)(2.05)(1.28)(-)</b>	<b>336.49(67)(87)(41)(-)</b>

### Three Remarks

- Burger et.al.'(15):  $a_e^{\text{LO-HVP}} = 178.2(6.4)(8.6)$ ,  $a_\tau^{\text{LO-HVP}} = 341(8)(6)$   
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## 1 Introduction

## 2 Result: Muon $g - 2$ with LO-HVP

- Connected Part
- Disconnected Part
- Short Summary on  $a_\mu^{\text{LO-HVP}, f}$

## 3 Discussions

- Corrections: FV, Perturv, and Isospin Breaking
- The  $g - 2$  for  $e$  and  $\tau$

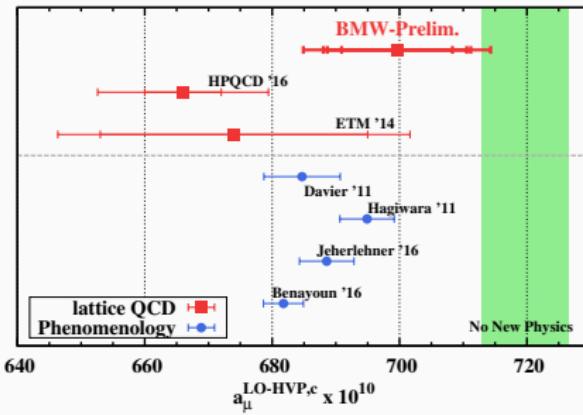
## 4 Summary of Summary and Perspective

# Summary of Summary and Perspective

The 1st principle determination of  $g - 2$

1. for All Leptons,
2. at Physical Point,
3. with Full Systematics,
4. at a Few % Level Uncertainty.

$$a_{\mu}^{\text{LO-HVP}} \times 10^{10} = 699.61(8.71)(6.64)(3.73)(9.09)$$



## Future Perspective

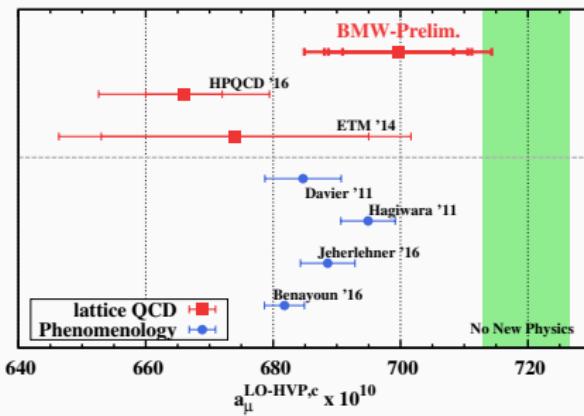
- To investigate FV by lattice QCD data without recourse to XPT.
- To take account of the isospin breaking effects and electromagnetism in lattice QCD simulations themselves.

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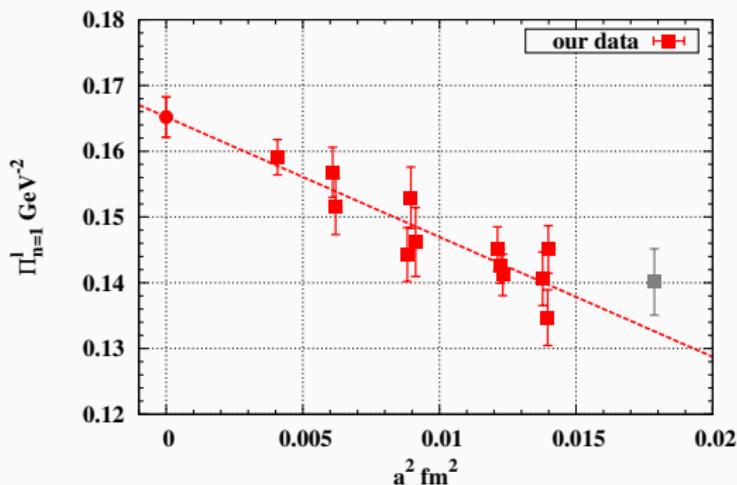
## Future Perspective

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## 5 Backups

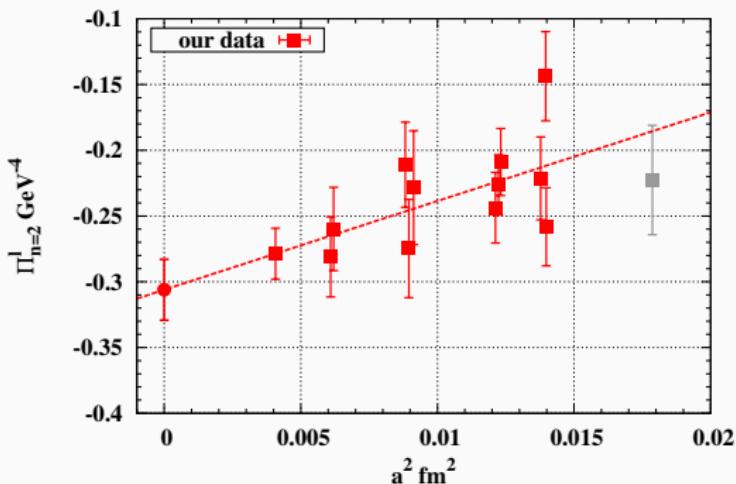
# Continuum Extrapolation of $\Pi_{n=1}^I$



$$F(C_\Pi^{(2)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi^{(2)}}{a^2} \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\Pi_{n=1}^I|_{a^2 \rightarrow 0} = 0.1652(31), \quad \chi^2/\text{d.o.f.} = 24.3/20$$

# Continuum Extrapolation of $\Pi_{n=2}^I$



$$F(C_\Pi^{(4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi^{(4)}}{a^4} \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\Pi_{n=2}^I|_{a^2 \rightarrow 0} = -0.306(23) .$$

# Continuum Extrapolation of $\Pi_{n=1,2}^s$

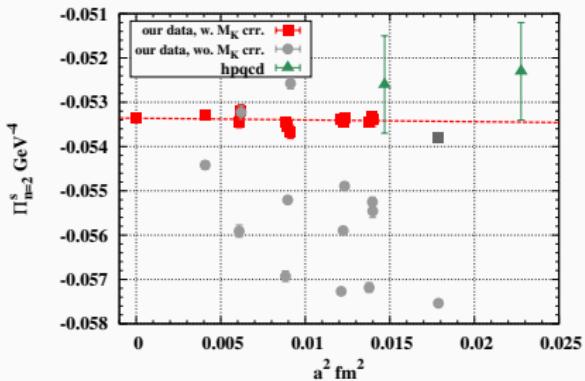
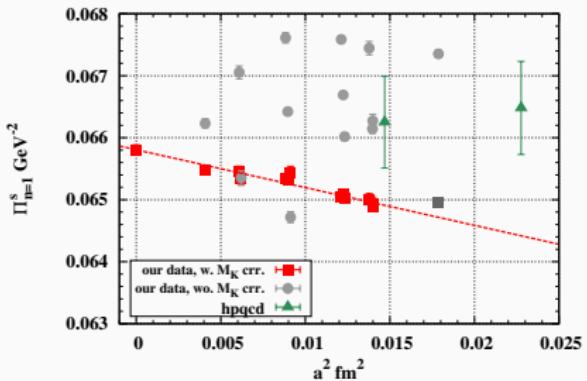
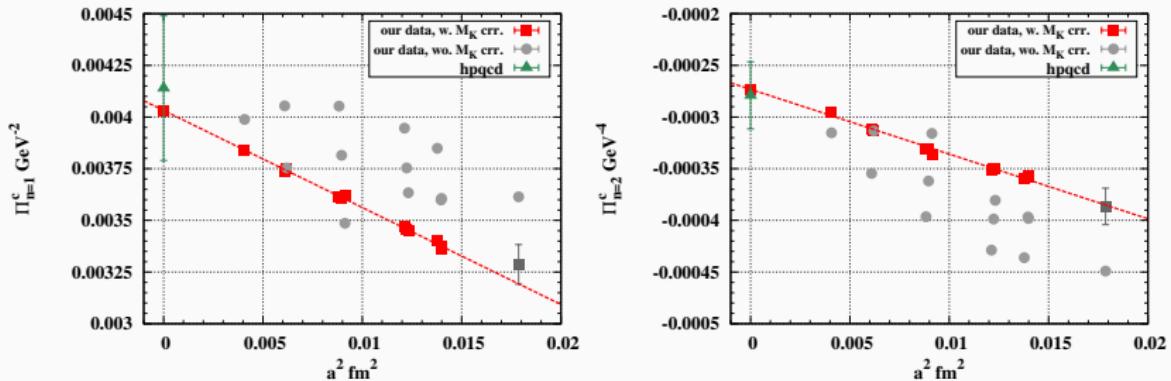


Figure: Red-squares = Our data. Green-triangles = HPQCD, 1403.1778.

$$F(C_\Pi^{(2,4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi}{a^{2,4}} \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\Pi_{n=1}^s|_{a^2 \rightarrow 0} = 0.0658(1) , \quad \Pi_{n=2}^s|_{a^2 \rightarrow 0} = -0.0534(2) , \quad \chi^2/\text{d.o.f.} = 20.9/18$$

# Continuum Extrapolation of $\Pi_{n=1,2}^c$

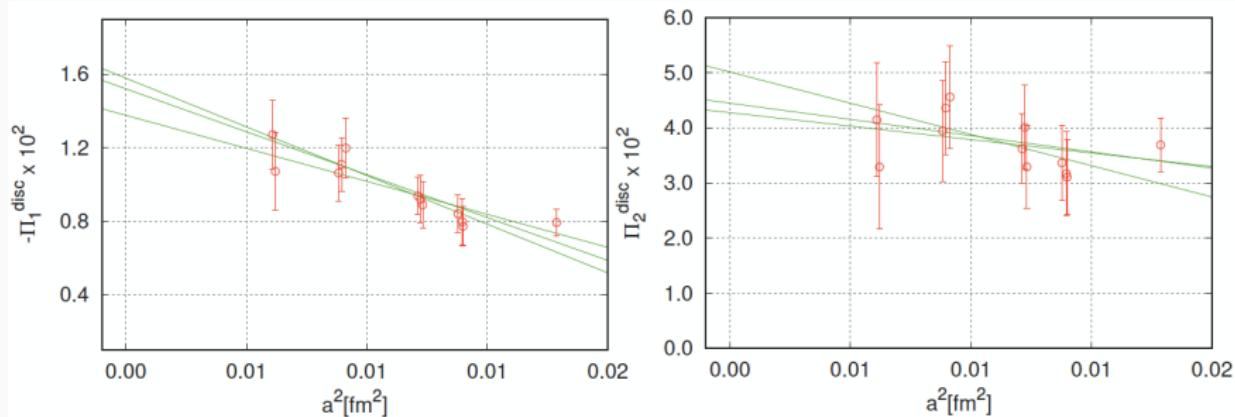


**Figure:** Red-squares = Our data. Green-triangles = HPQCD, 1208.2855.

$$F(C_\Pi^{(2,4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi}{a^{2,4}} \frac{1 + A a^2 + \dots}{1 + B a^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\Pi_{n=1}^c|_{a^2 \rightarrow 0} = 0.00403(2) , \quad \Pi_{n=2}^c|_{a^2 \rightarrow 0} = -2.73(2) \times 10^{-4} .$$

# Disconnected Contributions



**Figure:** From Presentaion of T.Kawanai in Lattice 2016.

$$\Pi_1^{\text{disc}} = -1.5(2)(1) \times 10^{-3} \text{ GeV}^{-2}, \quad (16)$$

$$\Pi_2^{\text{disc}} = -4.6(1.0)(0.4) \times 10^{-3} \text{ GeV}^{-4}. \quad (17)$$

## Summary Table of Moments (Preliminary)

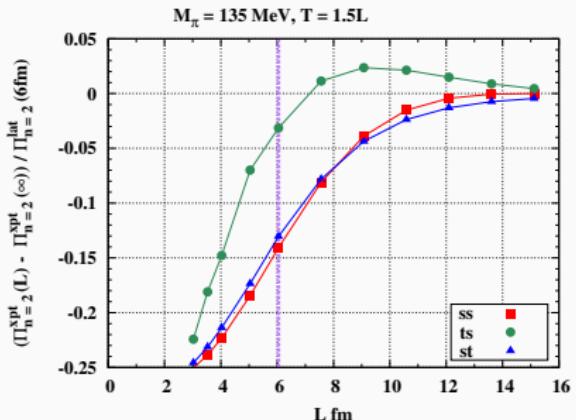
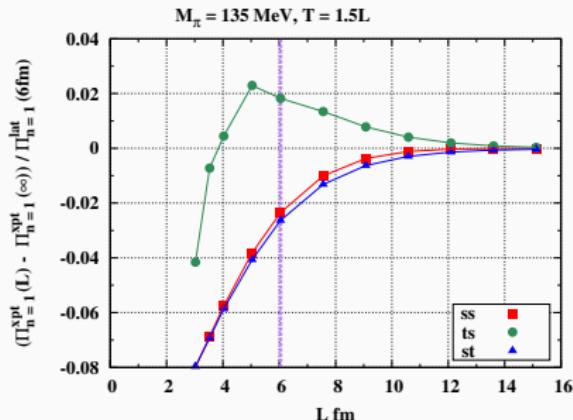
	$\Pi_1[\text{GeV}^{-2}]$	$\Pi_2[\text{GeV}^{-4}]$
light	0.1657(16)(18)	-0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$I = 0$	0.0166(2)(2)	-0.017(1)(1)
$I = 1$	0.0828(8)(9)	-0.148(5)(2)
<b>total</b>	<b>0.0995(9)(10)</b>	<b>-0.166(6)(3)</b>

**Table:** Preliminary results on the first two moments of the HVP function.

TOTAL ERROR: 1.4% for  $\Pi_1$ , and 4.0% for  $\Pi_2$ .

# FV via Box Asymmetry, XPT Estimate for Various $L$ I

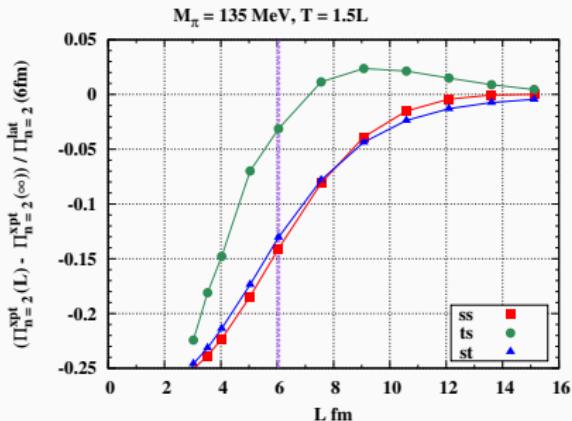
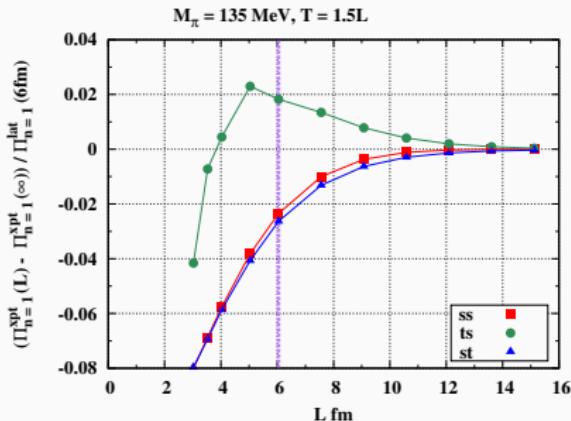
c.f. Aubin et.al., PRD (2016).



$$\begin{aligned} \Delta_{n=1,2}^i(L) &= [\Pi_{n=1,2}^{\text{xpt},i}(L) - \Pi_{n=1,2}^{\text{xpt}}(\infty)] , \quad i = \text{ss}, \text{ts}, \text{st} , \\ \frac{\Delta_n^i(L=6\text{fm})}{\Pi_n^{\text{lat},i}} &\sim \begin{cases} 2\% & (\text{for the 1st moment, } n=1) , \\ 10\% & (\text{for the 2nd moment, } n=2) . \end{cases} \end{aligned} \quad (18)$$

# FV via Box Asymmetry, XPT Estimate for Various $L$ II

c.f. Aubin et.al., PRD (2016).



$$\Delta_{n=1,2}^i(L) = [\Pi_{n=1,2}^{\text{xpt},i}(L) - \Pi_{n=1,2}^{\text{xpt}}(\infty)] , \quad i = \text{ss}, \text{ts}, \text{st} ,$$

$$FV.(L) \pm dFV.(L) = [\max\{\Delta_i\} + \min\{\Delta_i\}] / 2 \pm [\max\{\Delta_i\} - \min\{\Delta_i\}] / 2$$

$$\xrightarrow[L \rightarrow 6 \text{ fm}]{} \begin{cases} 0.0006(22) & (\text{for the 1st moment, } n = 1) , \\ -0.015(19) & (\text{for the 2nd moment, } n = 2) . \end{cases} \quad (19)$$

# Summary Table of Moments with FV I (Preliminary)

	$\Pi_1[\text{GeV}^{-2}]$	$\Pi_2[\text{GeV}^{-4}]$
light	0.1657(16)(18)	-0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$I = 0$	0.0166(2)(2)	-0.017(1)(1)
$I = 1$	0.0828(8)(9)	-0.148(5)(2)
<b>total</b>	<b>0.0995(9)(10)</b>	<b>-0.166(6)(3)</b>
$I = 1$ FV corr.	0.0006(23)	-0.015(10)
$I = 1 +$ FV corr.	0.0834(8)(9)(23)	-0.164(5)(2)(10)
<b>total + FV corr.</b>	<b>0.1001(9)(10)(23)</b>	<b>-0.182(6)(3)(10)</b>

**Table:** Preliminary results on the first two moments of the HVP function.

c.f. HPQCD(arXiv:1601.03071 and PRD2014):

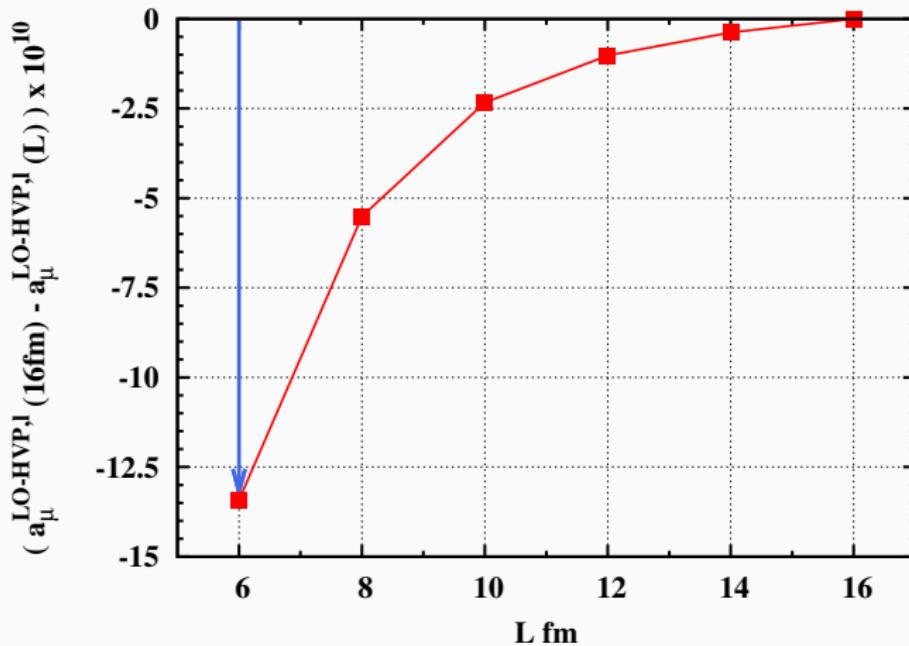
$$\Pi_1^I = 0.1606(25) \text{ GeV}^{-2}, \quad \Pi_2^I = -0.368(16) \text{ GeV}^{-4}, \\ \Pi_1^S = 0.06625(74) \text{ GeV}^{-2}, \quad \Pi_2^S = -0.0526(11) \text{ GeV}^{-4}.$$

## Summary Table of Moments with FV II (Preliminary)

	$\Pi_1[\text{GeV}^{-2}]$	$\Pi_2[\text{GeV}^{-4}]$
light	0.1657(16)(18)	-0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
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**Table:** Preliminary results on the first two moments of the HVP function.

c.f. Phenomenology(Benayoun et.al.1605.04474):  
 $\Pi_1 = 0.990(7) \text{ GeV}^{-2}, \quad \Pi_2 = -0.206(2) \text{ GeV}^{-4}.$

FV for  $a_\mu^{\text{LO-HVP}, I}$  by XPT

# Why $\hat{\Pi}^f$ ?

$$a_\ell^{\text{LO-HVP},f} = \left(\frac{\alpha}{\pi}\right)^2 \left( \int_0^{Q_{max}} + \int_{Q_{max}}^{\infty} \right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2), \quad (20)$$

$$= a_\ell^{\text{LO-HVP},f}(Q \leq Q_{max}) + a_\ell^{\text{LO-HVP},f}(Q > Q_{max}), \quad (21)$$

$a_\ell^{\text{LO-HVP},f}(Q \leq Q_{max})$ : computed by lattice simulations ,

$a_\ell^{\text{LO-HVP},f}(Q > Q_{max})$ : computed by lattice  $\hat{\Pi}^f(Q_{max})$  and perturbations .

# Why $\hat{\Pi}^f$ ?

$$a_\ell^{\text{LO-HVP},f} = \left(\frac{\alpha}{\pi}\right)^2 \left( \int_0^{Q_{max}} + \int_{Q_{max}}^{\infty} \right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2), \quad (20)$$

$$= a_\ell^{\text{LO-HVP},f}(Q \leq Q_{max}) + a_\ell^{\text{LO-HVP},f}(Q > Q_{max}), \quad (21)$$

$a_\ell^{\text{LO-HVP},f}(Q \leq Q_{max})$ : computed by lattice simulations ,

$a_\ell^{\text{LO-HVP},f}(Q > Q_{max})$ : computed by lattice  $\hat{\Pi}^f(Q_{max})$  and perturbations .

# Why $\hat{\Pi}^f$ ?

$$a_\ell^{\text{LO-HVP}, f}(Q > Q_{\max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2), \quad (22)$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left[ (\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) + (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \right],$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 (\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) \quad (\rightarrow \Delta a_\ell^{\text{LO-HVP}, f})$$

$$+ \left[ \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \right] (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \quad (\rightarrow \gamma_\ell(Q_{\max}^2) \hat{\Pi}^f(Q_{\max}^2)). \quad (23)$$

# Why $\hat{\Pi}^f$ ?

$$a_\ell^{\text{LO-HVP}, f}(Q > Q_{\max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2), \quad (22)$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left[ (\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) + (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \right],$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 (\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) \quad (\rightarrow \Delta a_\ell^{\text{LO-HVP}, f})$$

$$+ \left[ \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \right] (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \quad (\rightarrow \gamma_\ell(Q_{\max}^2) \hat{\Pi}^f(Q_{\max}^2)). \quad (23)$$

# Why $\hat{\Pi}^f$ ?

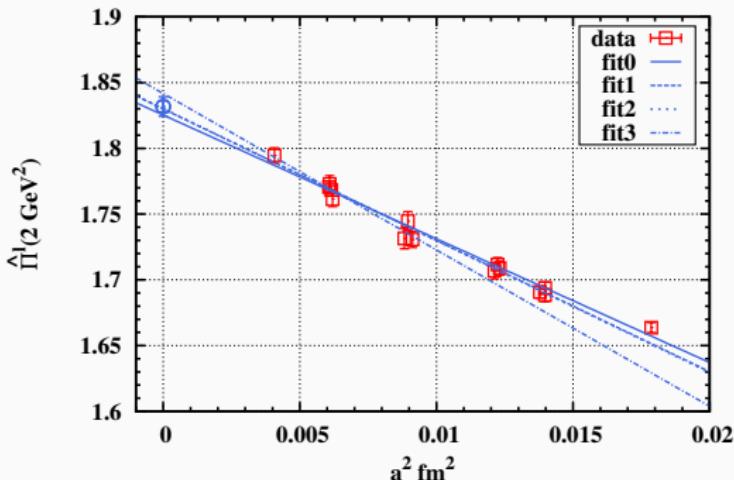
$$a_\ell^{\text{LO-HVP}, f}(Q > Q_{\max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2), \quad (22)$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left[ (\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) + (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \right],$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 (\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) \quad (\rightarrow \Delta a_\ell^{\text{LO-HVP}, f})$$

$$+ \left[ \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^\infty \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \right] (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \quad (\rightarrow \gamma_\ell(Q_{\max}^2) \hat{\Pi}^f(Q_{\max}^2)). \quad (23)$$

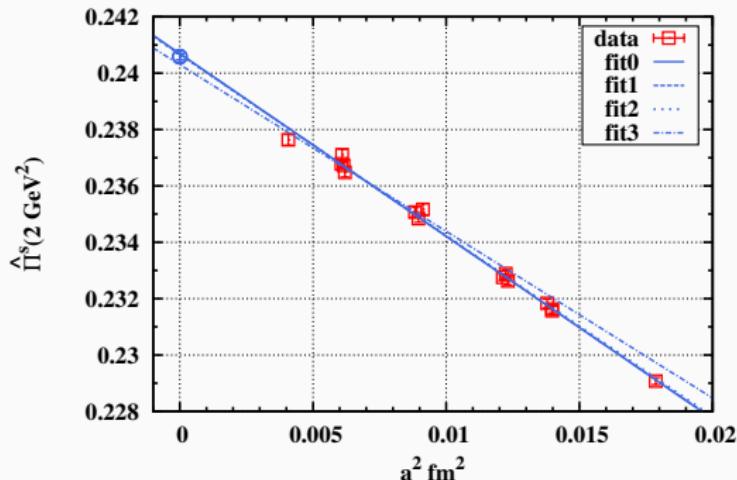
# Continuum Extrap. of Light Component: $\hat{\Pi}'$



$$F(\hat{\Pi}', A, B, C_{M_\pi}, C_{M_K}) = \frac{\hat{\Pi}'(1 + Aa^2 + \dots)}{1 + Ba^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\hat{\Pi}' = 1.8318(42)(60) , \quad \chi^2/\text{d.o.f.} = 8.2/12 \text{ (fit1 case).}$$

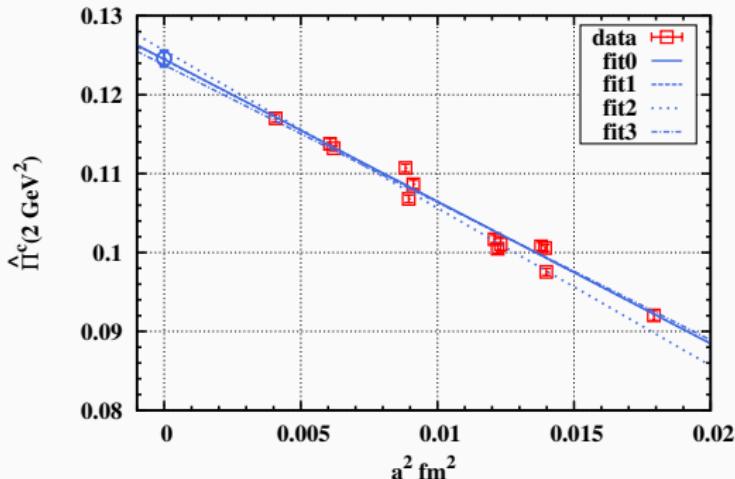
# Continuum Extrap. of Strange Component: $\hat{\Pi}^s$



$$F(\hat{\Pi}^s, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^s \frac{1 + A a^2 + \dots}{1 + B a^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\hat{\Pi}^s = 0.2406(1)(2) , \quad \chi^2/\text{d.o.f.} = 13.6/11 \text{ (fit1 case).}$$

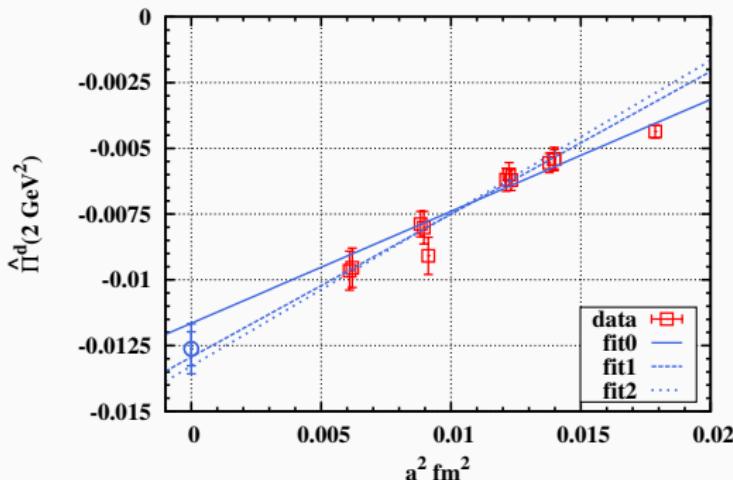
# Continuum Extrap. of Charm Component: $\hat{\Pi}^c$



$$F(\hat{\Pi}^c, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^c \frac{1 + A a^2 + \dots}{1 + B a^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\hat{\Pi}^c = 0.1246(9)(7) , \quad \chi^2/\text{d.o.f.} = 26.1/9 \text{ (fit1 case).}$$

# Continuum Extrap. of Disc. Component: $\hat{\Pi}^d$



$$F(\hat{\Pi}^d, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^d \frac{1 + A a^2 + \dots}{1 + B a^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\hat{\Pi}^d = -0.0126(6)(7) , \quad \chi^2/\text{d.o.f.} = 3.5/9 \text{ (fit1 case).}$$