$su(2)$ cosets and Liouville theory

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Exact methods in low dimensional statistical physics

Z. Jaskolski, PS [arXiv: 1510.01773[hep-th]]

Coset construction of Virasoro Minimal Models

$$\mathrm{MM}_k = \frac{\widehat{su}(2)_k \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+1}}, \qquad (k = 1, 2, \dots)$$

- the algebra $\widehat{su}(2)_k \times \widehat{su}(2)_1$ can be decomposed into two mutually commuting algebras: $\widehat{su}(2)_{k+1}$ and Virasoro
- the energy-momentum tensor

$$T_{\frac{\widehat{su}(2)_k \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+1}}} = T_{\widehat{su}(2)_k} + T_{\widehat{su}(2)_1} - T_{\widehat{su}(2)_{k+1}}$$

It provides

- construction of symmetry generators
- branching rules (decomposition of representation)

To get information about **correlation functions** of a model given by a coset construction we need some relations to other CFT models

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old story: relations between minimal models

- minimal models

$$\mathrm{MM}_k = \frac{\widehat{su}(2)_k \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+1}},$$

- supersymmetric minimal models

$$\mathrm{SMM}_k = \frac{\widehat{su}(2)_k \times \widehat{su}(2)_2}{\widehat{su}(2)_{k+2}},$$

the relation between minimal models

$$\mathrm{SMM}_k \times \mathrm{MM}_1 \sim \mathrm{MM}_k \times \mathrm{MM}_{k+1}$$

$$\frac{\widehat{su}(2)_k \times \widehat{su}(2)_2}{\widehat{su}(2)_{k+2}} \times \frac{\widehat{su}(2)_1 \times \widehat{su}(2)_1}{\widehat{su}(2)_2} \sim \frac{\widehat{su}(2)_k \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+1}} \times \frac{\widehat{su}(2)_{k+1} \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+2}}$$

- Crnkovic, Sotkov, Stanishkov, Phys. Lett. B **226** (1989) 297; with Paunov Nucl. Phys. B **336** (1990) 637
- Lashkievich [hep-th/9301093], [hep-th/9304116]

continuous extension of

$$\frac{\widehat{su}(2)_k \times \widehat{su}(2)_2}{\widehat{su}(2)_{k+2}} \times \frac{\widehat{su}(2)_1 \times \widehat{su}(2)_1}{\widehat{su}(2)_2} \sim \frac{\widehat{su}(2)_k \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+1}} \times \frac{\widehat{su}(2)_{k+1} \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+2}}$$

to coset theories with a **free real parameter** κ

$$\frac{\widehat{su}(N)_\kappa \times \widehat{su}(N)_p}{\widehat{su}(N)_{\kappa+p}}$$

- $N = 2, p = 1$ Liouville theory
- $N = 2, p = 2$ superLiouville

superLiouville $\times$ fermion $\leftrightarrow$ Liouville $(c_1 > 1)$ $\times$ Liouville $(c_2 < 1)$

- $N = 2,\ p > 2$ – parafermionic Liouville theories
- general N, p – para-Toda theories

- N. Wyllard, arXiv:1109.4264 [hep-th]
- Belavin, Bershtein, Feigin, Litvinov, Tarnopolsky, Commun. Math. Phys. 319 (2013) 269 [arXiv:1111.2803 [hep-th]]

- these were relations between models that can be represented by different cosets
- our aim: based on the coset construction of minimal models

$$\mathrm{MM}_k = \frac{\widehat{su}(2)_k \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+1}}, \qquad (k = 1, 2, \dots)$$

find a relation between correlation functions in CFT models with chiral symmetries present in the coset construction:

relations between CFT models

- $\widehat{su}(2)_k \times \widehat{su}(2)_1$ models $\sim$ $\widehat{su}(2)_{k+1}$ model $\times$ MM_k

Extension to the real parameter κ

- $\widehat{su}(2)_\kappa \times \widehat{su}(2)_1$ models $\sim$ $\widehat{su}(2)_{\kappa+1}$ model $\times$ Liouville

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other relations between the models involved

n-**point correlators** in $su(2)_k$ WZW model	$\leftrightarrow$	$n+1$-**point correlators** in Virasoro minimal models

$k \in \mathbb{N}, j = \frac{1}{2}, \ldots, \frac{k}{2}$ [Zamolodchikov, Fateev]

finite spectrum of degenerate fields

- based on the relation between

KZ equations and **differential equations for Virasoro degenerate field**

"generalized" $su(2)_k$ model $\leftrightarrow$ generalized minimal models

$k \in \mathbb{C}, j_{m,n}$ degenerate [Andreev]

infinite spectrum of degenerate fields [Dotsenko,Fateev]

with continuous spectrum [Teschner; with Ribault; Hikida, Schomerus]

H_3^+ model $\leftrightarrow$ Liouville theory

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$su(2)_k$ model with integer k

[Zamolodchikov, Fateev]

- The model is invariant under two copies (holomorphic and antiholomorphic) of $\widehat{su}(2)_k$ current algebra
- The left moving symmetry is generated by the currents J^a with $a = 3, \pm$

$$\begin{aligned}
\left[J^3_m, J^3_n\right] &= \frac{k}{2} m \delta_{m+n,0} \\
\left[J^3_m, J^\pm_n\right] &= \pm J^\pm_{m+n} \\
\left[J^+_m, J^-_n\right] &= 2J^3_{m+n} + km\delta_{m+n,0}, \qquad m, n \in \mathbb{Z}.
\end{aligned} \tag{1}$$

and similarly for the right moving currents.

- The Sugawara construction

$$L_m = \frac{1}{2(k+2)} \sum_n \left(2 : J^3_n J^3_{m-n} : + : J^+_n J^-_{m-n} : + : J^-_n J^+_{m-n} :\right)$$

yields the associate Virasoro algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\ [L_m, J^a_n] &= -nJ^a_{m+n}, \end{aligned}$$

with the central charge

$$c = \frac{2k}{k+2} \ .$$

- For a given level k there are $(k+1)$ invariant tensor fields ϕ_j with $j = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$,

$$J_n^a \phi_j(z,\bar z) = \bar J_n^a \phi_j(z,\bar z) = 0, \quad n > 0,$$
$$L_0 \phi_j(z,\bar z) = \frac{j(j+1)}{k+2} \phi_j(z,\bar z)$$

The $(2j+1)^2$ components of the tensor field satisfy

$$J_0^3 \phi_j^{m,\bar m}(z,\bar z) = m \phi_j^{m,\bar m}(z,\bar z), \quad \bar J_0^3 \phi_j^{m,\bar m}(z,\bar z) = \bar m \phi_j^{m,\bar m}(z,\bar z),$$

with $m, \bar m = -j, -j+1, \ldots, j$

$$J_0^+ \phi_j^{m,\bar m}(z,\bar z) = (m-j)\, \phi_j^{m+1,\bar m}(z,\bar z),$$
$$J_0^- \phi_j^{m,\bar m}(z,\bar z) = (-m-j)\, \phi_j^{m-1,\bar m}(z,\bar z)$$

Isospin coordinates

It is convenient to introduce auxiliary coordinates $x, \bar{x}$ and define

- currents

$$\begin{aligned} J_n^+(x) &= J_n^+ - 2xJ_n^3 - x^2 J_n^-, \\ J_n^3(x) &= J_n^3 + xJ_n^-, \\ J_n^-(x) &= J_n^-, \end{aligned}$$

  For any x the currents satisfy the commutation relations of the $\widehat{su}(2)_k$ affine algebra (1).

- combination of the invariant fields $\phi_j^{m,\bar{m}}$

$$\Phi_j(x,\bar{x};z,\bar{z}) = \sum_{m,\bar{m}=-j}^{j} \sqrt{\binom{2j}{j+m}\binom{2j}{j+\bar{m}}}\; x^{j+m}\bar{x}^{j+\bar{m}}\, \phi_j^{m,\bar{m}}(z,\bar{z})$$

Isospin coordinates

In this representation

- zero modes of the currents act as **differential operators** on the fields Φ_j:

$$J_0^a(x)\Phi_j(y,z) = -\left((x-y)^{1+\epsilon(a)}\partial_y + (1+\epsilon(a))\, j\,(x-y)^{\epsilon(a)}\right)\Phi_j(y,z)$$

  where $\epsilon(\pm) = \pm 1, \epsilon(3) = 0$.

- fields Φ_j are primaries of the **highest weight** representations with respect to $J^a(x)$ currents:

$$J_0^+(x)\Phi_j(x,z) = 0, \qquad J_n^a(y)\Phi_j(x,z) = 0, \qquad n > 0$$

$$J_0^3(x)\Phi_j(x,z) = j\Phi_j(x,z), \quad J_0^-(x)\Phi_j(x,z) = -\partial_x\Phi_j(x,z)$$

Correlation functions

2-point and 3-point functions

$$\langle \Phi_{j_1}(x_1,\bar{x}_1;z_1,\bar{z}_1)\Phi_{j_2}(x_2,\bar{x}_2;z_2,\bar{z}_2)\rangle = \delta_{j_1,j_2}\frac{(x_{12}\bar{x}_{12})^{2j_1}}{(z_{12}\bar{z}_{12})^{2\Delta_1}},$$

$$\langle \prod_{p=1}^{3}\Phi_{j_p}(x_p,\bar{x}_p;z_p,\bar{z}_p)\rangle = C[j_1,j_2,j_3]\prod_{p<q}^{3}\frac{(x_{pq}\bar{x}_{pq})^{j_{pq}}}{(z_{pq}\bar{z}_{pq})^{\Delta_{pq}}},$$

with $x_{pq}=x_p-x_q$, $z_{pq}=z_p-z_q$, $j_{12}=j_1+j_2-j_3$, $\Delta_{12}=\Delta_1+\Delta_2-\Delta_3$.

Correlation functions satisfy two types of differential equations:

- Knizhnik-Zamolodchikov equations (from the construction of L_{-1} in terms of currents J^a)
- Zamolodchikov-Fateev equations (from the affine null-vector decoupling)

$$\left(J^{+}_{-1}(x)\right)^{k-2j+1}\Phi_j(x,\bar{x};z,\bar{z})=0, \qquad \text{for} \quad j\leq\frac{k}{2}.$$

structure constants

$$C^{(k)}[j_1,j_2,j_3] \sim P(j_{123}+1)\prod_{a=1}^{3}\frac{P(j_{123}-2j_a)}{\sqrt{P(2j_a)P(2j_a+1)}}$$

with

$$P(n)=\prod_{a=1}^{n}\gamma(a/(k+2)), \qquad \gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)}, \qquad P(0)=1\,.$$

- from the KZ and ZF equations for the 4-point function (related to 5-point function in MM)
- using the relation to the Υ function $P(n)=\frac{\Upsilon_b(nb+b)}{\Upsilon_b(b)}b^{n((n+1)b^2-1)}$

$$C^{(k)}[j_1,j_2,j_3] \sim \Upsilon_b(b(j_{123}+2))\prod_{a=1}^{3}\frac{\Upsilon_b(b(j_{123}-2j_a+1))}{\sqrt{\Upsilon_b(b(2j_a+1))\,\Upsilon_b(b(2j_a+2))}},$$

with $b=\frac{1}{\sqrt{k+2}}$, $j_{123}=j_1+j_2+j_3$

- it is well defined not only for half-integer spins j_i and $k\in\mathbb{N}$

Upsilon function

- can be defined in terms of Barnes' double Gamma function $\Gamma_2(x|\omega_1, \omega_2)$:

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma(b+\frac{1}{b}-x)}, \quad \Gamma_b(x) = \Gamma_2(x|b, b^{-1})$$

- shift properties

$$\Upsilon_b(x+b) = \gamma(bx)\, b^{1-2bx}\, \Upsilon_b(x),$$
$$\Upsilon_b(x+b^{-1}) = \gamma(b^{-1}x)\, b^{-1+2b^{-1}x}\, \Upsilon_b(x)$$

- $\Upsilon_b(x)$ as a function of b is analytic in the whole complex plane of b^2 except for the negative part of the real axis

structure constants

$$C^{(k)}[j_1, j_2, j_3] \sim \Upsilon_b(b(j_{123}+2)) \prod_{a=1}^{3} \frac{\Upsilon_b(b(j_{123}-2j_a+1))}{\sqrt{\Upsilon_b(b(2j_a+1))\,\Upsilon_b(b(2j_a+2))}},$$

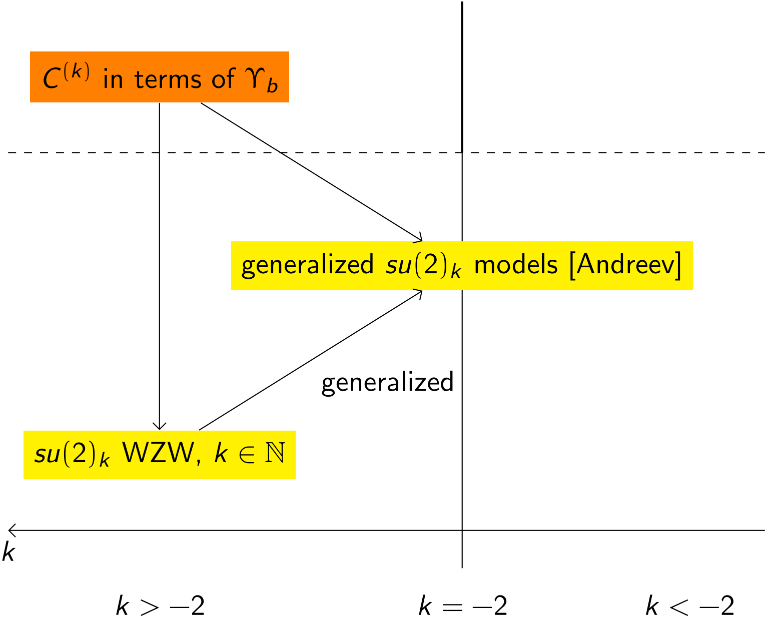
with $b = \frac{1}{\sqrt{k+2}}$, $j_{123} = j_1 + j_2 + j_3$

- for $k > -2$ and degenerate reps $j^{\pm}_{n,m}$ from Kac-Kazhdan formula

  $$j^{+}_{n,m} = -\tfrac{n-1}{2}(k+2) + \tfrac{m-1}{2}, \quad j^{-}_{n,m} = \tfrac{n}{2}(k+2) - \tfrac{m+1}{2}, \quad m, n \in \mathbb{N}$$

  it gives the structure constants from generalized $su(2)$ models by [Andreev]

- for $k \in \mathbb{N}$ and degenerate reps $j^{+}_{1,m}$ it gives the structure constants in $su(2)$ WZW model by [Zamolodchikov, Fateev]


$C^{(k)}$ in terms of Υ_b
generalized $su(2)_k$ models [Andreev]
generalized
$su(2)_k$ WZW, $k \in \mathbb{N}$
k
$k > -2$
$k = -2$
$k < -2$

$\widehat{su}(2)_1$ **model**

- representations: $j = 0, \frac{1}{2}$
- fusion rules $\Phi_{j_1}\, \Phi_{j_2} = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} [\Phi_j]$

- 3-point correlation functions

$$
\begin{aligned}
\langle \Phi_0^1(x_3, z_3)\, \Phi_0^1(x_2, z_2)\, \Phi_0^1(x_1, z_1)\rangle &= 1, \\
\langle \Phi_0^1(x_3, z_3) \Phi_{\frac{1}{2}}^1(x_2, z_2) \Phi_{\frac{1}{2}}^1(x_1, z_1)\rangle &= \frac{(x_1-x_2)(\bar{x}_1-\bar{x}_2)}{\sqrt{(z_2-z_1)(\bar{z}_2-\bar{z}_1)}}, \\
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Virasoro Minimal Models

- the set of unitary CFT models with central charge

$$c = 1 - \frac{6}{m(m+1)}, \qquad m \geq 3.$$

- for each c there are $\binom{m}{2}$ highest weight representations with

$$\Delta_{rs}(m) = \frac{((m+1)r - sm)^2 - 1}{4m(m+1)}$$

where $1 \leq r \leq m-1$ and $1 \leq s \leq m$.

- the corresponding primary fields

$$\phi_{rs} \sim |\Delta_{rs}\rangle \otimes |\bar{\Delta}_{rs}\rangle$$

Structure constants

- calculated by [Dotsenko, Fateev],

generalized for a **continuous family of models** parametrized by a (complex) parameter β, with central charge

$$c = 1 - 6(\beta^{-1} - \beta)^2 = 1 - 6\hat{Q}^2, \qquad \hat{Q} = \beta^{-1} - \beta$$

and with a larger set of degenerate fields.

- unitary set of minimal models recovered for $\beta = \sqrt{\frac{m+1}{m}}$

generalization to the **continuous spectrum of fields** parametrized by highest weights $\Delta_j = \hat{Q}^2 j(1+j)$ [Zamolodchikov; Kostov,Petkova]

$$C_\beta[j_3, j_2, j_1] \sim \Upsilon_\beta(\beta - \hat{Q}(j_{123}+1)) \frac{\prod_{a=1}^{3} \Upsilon_\beta(\beta - \hat{Q}(j_{123} - 2j_a))}{\sqrt{\Upsilon_\beta(\beta - 2\hat{Q}j_a)\Upsilon_\beta(\beta - \hat{Q}(2j_a+1))}}$$

- DF minimal models structure constants are recovered for $j_{rs} = \frac{(s-1)\beta^{-1}}{2\hat{Q}} - \frac{(r-1)\beta}{2\hat{Q}}$

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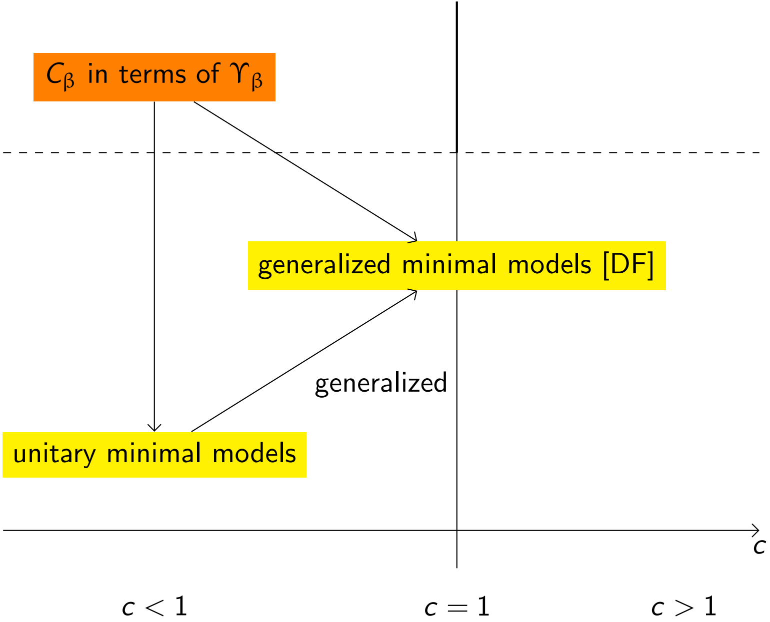
C_β in terms of Υ_β
generalized minimal models [DF]
generalized
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c
$c < 1$
$c = 1$
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Coset construction of minimal models

Goddard, Kent, Olive construction

Virasoro minimal models with $c = 1 - \frac{6}{m(m+1)}$ can be described by the coset

$$\frac{\widehat{su}(2)_k \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+1}}, \qquad m = k + 2$$

- the algebra $\widehat{su}(2)_{k+1}$ is generated by the sum of currents for $\widehat{su}(2)_k$ and $\widehat{su}(2)_1$: $J^a + K^a$
- from each set of currents there are corresponding Virasoro generators L_n^k, L_n^1, L_n^{k+1} given by the Sugawara construction
- the Virasoro generators of the coset:

$$L_n^V = L_n^k + L_n^1 - L_n^{k+1}$$

with the central charge

$$c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{(k+3)} = 1 - \frac{6}{(k+2)(k+3)}$$

Coset construction of minimal models

$$MM(m) \sim \frac{\widehat{su}(2)_k \times \widehat{su}(2)_1}{\widehat{su}(2)_{k+1}}, \qquad m = k+2$$

- for a fixed value of k, all highest weight representations of the minimal model $(\Delta_{r,s}(m))$ appear in the decomposition of the product of the highest weight representations of $\widehat{su}(2)_k \times \widehat{su}(2)_1$

branching rules

$$\left(\frac{r-1}{2}\right)_k \otimes (\epsilon)_1 \quad = \bigoplus_{\substack{0 \le (s-1) \le k+1, \\ r-s+2\epsilon = 0 \ mod\ 2}} \left(\frac{s-1}{2}\right)_{k+1} \otimes (\Delta_{r,s}(m))$$

- with $\epsilon = 0, \frac{1}{2}$ and $1 \le r \le k+1$.
- $(j)_k$ denotes spin j representation of $\widehat{su}(2)_k$ with conformal dimension of the highest weight state $\Delta_j^{(k)} = \frac{j(j+1)}{(k+2)}$.

relation between states on both sides of the decomposition

$$\left(\frac{r-1}{2}\right)_k \otimes (\epsilon)_1 \quad = \bigoplus_{\substack{0 \le (s-1) \le k+1, \\ r-s+2\epsilon = 0 \ mod\ 2}} \left(\frac{s-1}{2}\right)_{k+1} \otimes (\Delta_{r,s}(m))$$

- two highest weight states on the lhs are the highest weight states with $s = r$ and $s = r+1$, respectively:

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(check the action of $J_n^a + K_n^a$ and the condition $L_0^k + L_0^1 = L_0^{k+1} + L_0^V$)

- the other Virasoro highest weight states with will correspond to some descendant states on the lhs

$$|j, n\rangle^\star = |j+n\rangle_{k+1} \otimes |\Delta_{r,r+2n}\rangle \ , \qquad j = \tfrac{r-1}{2}$$

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- the other Virasoro highest weight states with will correspond to some descendant states on the lhs

$$|j, n\rangle^\star = |j+n\rangle_{k+1} \otimes |\Delta_{r,r+2n}\rangle \ , \qquad j = \tfrac{r-1}{2}$$

- **descendant states** $|j,n\rangle^\star$ in the product theory $\widehat{su}(2)_k \times \widehat{su}(2)_1$ that are **highest weight** with respect to $\widehat{su}(2)_{k+1}$ currents and Virasoro algebra:

$$\begin{aligned}
\left(J_0^+ + K_0^+\right)|j,n\rangle^\star &= \left(J_m^a + K_m^a\right)|j,n\rangle^\star = L_m^{\mathrm{Vir}}|j,n\rangle^\star = 0, \qquad n>0 \\
\left(J_0^3 + K_0^3\right)|j,n\rangle^\star &= (j+n)\,|j,n\rangle^\star\,, \qquad L_0^{\mathrm{Vir}}|j+n\rangle^\star = \Delta_{r,r+2n}|j,n\rangle^\star
\end{aligned}$$

with $j=\frac{r-1}{2}$ and $0\le j+n\le \frac{k+1}{2}$

- in general: $|j,n\rangle^\star = \mathcal{O}_{j,n}(J^a,K^a)\,|j\rangle_k \otimes |\epsilon\rangle_1$
- the first examples

$$\begin{aligned}
|j,0\rangle^\star &= |j\rangle_k \otimes |0\rangle_1\,, \qquad \left|j,\tfrac12\right\rangle^\star = |j\rangle_k \otimes \left|\tfrac12\right\rangle_1\,, \\
\left|j,-\tfrac12\right\rangle^\star &= \left(J_0^- - 2jK_0^-\right)\left|j,\tfrac12\right\rangle^\star\,, \\
|j,1\rangle^\star &= \left(J_{-1}^+ - (k-2j)K_{-1}^+\right)|j,0\rangle^\star\,, \\
|j,-1\rangle^\star &= \Big(-J_{-1}^+(J_0^-)^2 - 2(2j-1)J_{-1}^3J_0^- + 2j(2j-1)J_{-1}^- \\
&\quad +(k+2j+2)\left(K_{-1}^+(J_0^-)^2 + 2(2j-1)K_{-1}^3J_0^- - 2j(2j-1)K_{-1}^-\right)\Big)|j,0\rangle^\star
\end{aligned}$$

- **descendant states** $|j,n\rangle^\star$ in the product theory $\widehat{su}(2)_k \times \widehat{su}(2)_1$ that are **highest weight** with respect to $\widehat{su}(2)_{k+1}$ currents and Virasoro algebra:

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\left(J_0^3 + K_0^3\right)|j,n\rangle^\star &= (j+n)\,|j,n\rangle^\star\,, \qquad L_0^{\mathrm{Vir}}|j+n\rangle^\star = \Delta_{r,r+2n}|j,n\rangle^\star
\end{aligned}$$

with $j = \frac{r-1}{2}$ and $0 \le j+n \le \frac{k+1}{2}$

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- the first examples

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|j,0\rangle^\star &= |j\rangle_k \otimes |0\rangle_1\,, \qquad \left|j,\tfrac{1}{2}\right\rangle^\star = |j\rangle_k \otimes \left|\tfrac{1}{2}\right\rangle_1\,,\\
\left|j,-\tfrac{1}{2}\right\rangle^\star &= \left(J_0^- - 2jK_0^-\right)\left|j,\tfrac{1}{2}\right\rangle^\star\,,\\
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&+(k+2j+2)\left(K_{-1}^+(J_0^-)^2 + 2(2j-1)K_{-1}^3 J_0^- - 2j(2j-1)K_{-1}^-\right)\Big)|j,0\rangle^\star
\end{aligned}$$

relation between correlation functions

- for the fields corresponding to the excited states

$$\Phi^{\star}_{j,n}(x,\bar{x};z,\bar{z}) \;\leftrightarrow\; N_{j,n}\;\, |j,n\rangle^{\star} \otimes \overline{|j,n\rangle^{\star}}$$

we expect the correspondence:

$$\Phi^{\star}_{j,n}(x,\bar{x};z,\bar{z}) = \Phi^{(k+1)}_{j+n}(x,\bar{x};z,\bar{z}) \otimes \phi_{r,r+2n}(z,\bar{z}), \qquad r = 2j+1$$

- explicit checks of the equality between 3-point functions containing the fields with $n = 0, \pm\frac{1}{2}, \pm 1$
- $n = 0$ means the equality between particular structure constants
- checks for $n \neq 0$ involve using the Ward identities in the $\widehat{su}(2)_k$ and $\widehat{su}(2)_1$ models

the case $n = 0$

$$\Phi^{\star}_{j,0}(x,\bar{x};z,\bar{z}) = \Phi^{(k)}_{j}(x,\bar{x};z,\bar{z}) \otimes \Phi^{1}_{0}(x,\bar{x};z,\bar{z})$$
$$= \Phi^{(k+1)}_{j}(x,\bar{x};z,\bar{z}) \otimes \phi_{r,r}(z,\bar{z}), \qquad r = 2j+1$$

- in the j parametrisation of the degenerate Virasoro weight

$$\Delta_{r,s} = \hat{Q}^2 j_{rs}(1+j_{rs}), \quad j_{rs} = \frac{(s-1)\beta^{-1}}{2\hat{Q}} - \frac{(r-1)\beta}{2\hat{Q}}, \quad \hat{Q} = \beta^{-1} - \beta$$

we have $j_{rr} = \frac{r-1}{2} = j$.

$$\langle \prod_{p=1}^{3} \Phi^{(k)}_{j_p}(x_p,\bar{x}_p;z_p,\bar{z}_p)\rangle = C^{(k)}[j_1,j_2,j_3] \prod_{p<q}^{3} \frac{(x_{pq}\bar{x}_{pq})^{j_{pq}}}{(z_{pq}\bar{z}_{pq})^{\Delta_{pq}}},$$
$$\langle \prod_{p=1}^{3} \phi_{r_p,r_p}(z_p,\bar{z}_p)\rangle = C_\beta[j_1,j_2,j_3] \prod_{p<q}^{3} (z_{pq}\bar{z}_{pq})^{-\Delta_{pq}}, \qquad j_p = \frac{r_p-1}{2}$$

relation between structure constants

$$C^{(k)}[j_1,j_2,j_3] = C^{(k+1)}[j_1,j_2,j_3]\, C_\beta[j_1,j_2,j_3], \qquad \beta = \sqrt{\frac{m+1}{m}} = \sqrt{\frac{k+3}{k+2}}$$

the case $n = 0$

$$\begin{aligned}\Phi^{\star}_{j,0}(x,\bar{x};z,\bar{z}) &= \Phi^{(k)}_{j}(x,\bar{x};z,\bar{z}) \otimes \Phi^{1}_{0}(x,\bar{x};z,\bar{z}) \\ &= \Phi^{(k+1)}_{j}(x,\bar{x};z,\bar{z}) \otimes \phi_{r,r}(z,\bar{z}), \qquad r = 2j+1\end{aligned}$$

- in the j parametrisation of the degenerate Virasoro weight

$$\Delta_{r,s} = \hat{Q}^2 j_{rs}(1+j_{rs}), \quad j_{rs} = \frac{(s-1)\beta^{-1}}{2\hat{Q}} - \frac{(r-1)\beta}{2\hat{Q}}, \quad \hat{Q} = \beta^{-1} - \beta$$

we have $j_{rr} = \frac{r-1}{2} = j$.

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$$\langle \prod_{p=1}^{3} \phi_{r_p,r_p}(z_p,\bar{z}_p)\rangle = C_{\beta}[j_1,j_2,j_3] \prod_{p<q}^{3} (z_{pq}\bar{z}_{pq})^{-\Delta_{pq}}, \qquad j_p = \frac{r_p - 1}{2}$$

relation between structure constants

$$C^{(k)}[j_1,j_2,j_3] = C^{(k+1)}[j_1,j_2,j_3]\ C_{\beta}[j_1,j_2,j_3], \qquad \beta = \sqrt{\frac{m+1}{m}} = \sqrt{\frac{k+3}{k+2}}$$

Structure constants of $\widehat{su}(2)_k$ model and (generalized) minimal models written in terms of the Υ functions:

$$C^{(k)}[j_1, j_2, j_3] \sim \Upsilon_b(b(j_{123}+2)) \prod_{a=1}^{3} \frac{\Upsilon_b(b(j_{123}-2j_a+1))}{\sqrt{\Upsilon_b(b(2j_a+1))\,\Upsilon_b(b(2j_a+2))}},$$

$$C_\beta[j_3, j_2, j_1] \sim \Upsilon_\beta(\beta - \hat{Q}(j_{123}+1)) \prod_{a=1}^{3} \frac{\Upsilon_\beta(\beta - \hat{Q}(j_{123}-2j_a))}{\sqrt{\Upsilon_\beta(\beta - 2\hat{Q}j_a)\Upsilon_\beta(\beta - \hat{Q}(2j_a+1))}}$$

- relation for the structure constants

$$C^{(k)}[j_1, j_2, j_3] = C^{(k+1)}[j_1, j_2, j_3]\ C_\beta[j_1, j_2, j_3]$$

true due to:

$$\frac{\Upsilon_{b_1}(b_1(1+j))}{\Upsilon_{b_2}(b_2(1+j))} \sim \Upsilon_\beta(\beta - \hat{Q}j)$$

with parameters exactly as in our case:

$$b_1 = \frac{1}{\sqrt{k+2}}, \quad b_2 = \frac{1}{\sqrt{k+3}}, \quad \beta = \frac{b_1}{b_2} = \sqrt{\frac{k+3}{k+2}}, \quad \hat{Q} = \beta^{-1} - \beta = b_1 b_2$$

$$\Phi^{\star}_{j,n}(x,\bar{x};z,\bar{z}) = \Phi^{(k+1)}_{j+n}(x,\bar{x};z,\bar{z}) \otimes \phi_{r,r+2n}(z,\bar{z}), \qquad r = 2j+1$$

- in the j parametrisation of the degenerate Virasoro weight $j_{rs} = \frac{(s-1)\beta^{-1}}{2\hat{Q}} - \frac{(r-1)\beta}{2\hat{Q}}$ we have $j_{r,r+2n} = \frac{r-1}{2} + \frac{n}{\beta\hat{Q}}$.
- the 3-point functions of the primary fields on the rhs:

$$\langle \prod_{p=1}^{3} \Phi^{(k+1)}_{j_p+n_p}(x_p,\bar{x}_p;z_p,\bar{z}_p)\rangle = C^{(k+1)}_{[j_1+n_1,j_2+n_2,j_3+n_3]} \prod_{p<q}^{3} \frac{(x_{pq}\bar{x}_{pq})^{j_{pq}+n_{pq}}}{(z_{pq}\bar{z}_{pq})^{\Delta_{pq}}},$$

$$\langle \prod_{p=1}^{3} \phi_{r_p,r_p+2n_p}(z_p,\bar{z}_p)\rangle = C_\beta\left[j_1+\tfrac{n_1}{\beta\hat{Q}},j_2+\tfrac{n_2}{\beta\hat{Q}},j_3+\tfrac{n_3}{\beta\hat{Q}}\right] \prod_{p<q}^{3} (z_{pq}\bar{z}_{pq})^{-\Delta_{pq}}$$

- correlator of the descendant fields on the lhs

$$\langle \prod_{p=1}^{3} \Phi^{\star}_{j_p,n_p}(x_p,\bar{x}_p;z_p,\bar{z}_p)\rangle = (P(j_i,n_i))^2 C^{(k)}_{[j_1,j_2,j_3]} \prod_{p<q}^{3} \frac{(x_{pq}\bar{x}_{pq})^{j_{pq}+n_{pq}}}{(z_{pq}\bar{z}_{pq})^{\Delta_{pq}}}$$

with polynomials $P(j_i,n_i)$ determined by chiral Ward identities

Idea for the checks

- using shift relations for Υ functions calculate the ratio of structure constants

$$\frac{C^{(k+1)}_{[j_1+n_1,j_2+n_2,j_3+n_3]}\ C_\beta[j_1+\frac{n_1}{\beta\hat{Q}},j_2+\frac{n_2}{\beta\hat{Q}},j_3+\frac{n_3}{\beta\hat{Q}}]}{C^{(k+1)}[j_1,j_2,j_3]\ C_\beta[j_1,j_2,j_3]}$$

- calculate polynomials $P(j_i,n_i)$ from Ward identities
- compare the results

The simplest examples:

- $[n_1=0,n_2=-n_3=-\frac{1}{2}]$: $\quad P(j_i,n_i)=(j_2+j_1-j_3)$
- $[n_1=0,n_2=n_3=-\frac{1}{2}]$:
  $P(j_i,n_i)=(j_2-j_1+j_3)(j_1+j_2+j_3+1)$
- $[n_1=n_2=0,n_3=1]$: $\quad P(j_i,n_i)=(j_2+j_1-j_3)$
- $[n_1=n_2=0,n_3=1]$:
  $P(j_i,n_i)=(j_1-j_2+j_3)(j_2-j_1+j_3)(j_1+j_2+j_3+1)$

Idea for the checks

- using shift relations for Υ functions calculate the ratio of structure constants

$$\frac{C^{(k+1)}_{[j_1+n_1,j_2+n_2,j_3+n_3]}\ C_\beta[j_1+\frac{n_1}{\beta\hat{Q}},j_2+\frac{n_2}{\beta\hat{Q}},j_3+\frac{n_3}{\beta\hat{Q}}]}{C^{(k+1)}[j_1,j_2,j_3]\ C_\beta[j_1,j_2,j_3]}$$

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Summary

The coset construction of minimal models suggests the relation between CFT models:

$\widehat{su}(2)_k \times \widehat{su}(2)_1$ models $\sim$ $\widehat{su}(2)_{k+1}$ model $\times$ Vir minimal model

We have:

- first examples of descendant fields in the product theory on the lhs that correspond to primary fields on the rhs
- checks of equality of 3-point correlators containing these fields $(n = 0, \pm\frac{1}{2}, \pm 1)$

$$\langle \prod_{p=1}^{3} \Phi^{\star}_{j_p,n_p}(x_p,\bar{x}_p;z_p,\bar{z}_p)\rangle = \langle \prod_{p=1}^{3} \Phi^{(k+1)}_{j_p+n_p}(x_p,\bar{x}_p;z_p,\bar{z}_p)\rangle\langle \prod_{p=1}^{3} \phi_{r_p,r_p+2n_p}(z_p,\bar{z}_p)\rangle$$

Reconstruction of minimal models' structure constants for any $\phi_{r,s}$ possible only with:

- general construction of descendant fields $\Phi^{\star}_{j,n}$ and explicit formula for their 3-point functions

Extension to real parameter κ

Now we want to investigate the relation between models with continuous spectrum:

$$\widehat{su}(2)_\kappa \times \widehat{su}(2)_1 \text{ models} \quad \sim \quad \widehat{su}(2)_{\kappa+1} \text{ model } \times \text{Liouville}$$

1. Minimal Models
   - $\widehat{su}(2)_k$ model
   - Virasoro Minimal Models
   - coset construction of minimal models

2. Liouville theory
   - Liouville theory
   - $\widehat{su}(2)_\kappa$ model
   - coset construction of Liouville theory

Liouville theory

$$S_L[\varphi] \;=\; \frac{1}{4\pi}\int d^2z\left(\partial\varphi\bar{\partial}\varphi + 4\pi\mu_L e^{2b\varphi}\right)$$

- b, μ_L - two real parameters of the model
- two copies of Virasoro algebra with central charge

$$c = 1 + 6Q^2, \qquad Q = b + b^{-1}, \qquad (c \geq 25 \text{ for } b \in \mathbb{R})$$

- Vir highest weight states: $L_n\,|j\rangle = 0,\; L_0\,|j\rangle = \Delta_j\,|j\rangle\,, \quad n > 0$
- **primary fields**: $V_j =: e^{2\alpha\varphi}:, \quad \alpha = -Qj, \quad j = -\frac{1}{2} + \mathrm{i}\mathbb{R}$
  conformal dimensions: $\Delta_j = \alpha(Q-\alpha) = -Q^2 j(1+j),$

solution of Liouville theory

2-point functions canonically normalized

$$\langle V_{j_1}(z_1)V_{j_2}(z_2)\rangle = (z_{12}\bar{z}_{12})^{-2\Delta_1}\left(2\pi\delta(j_1+j_2+1) + D_L(j_1)\delta(j_2-j_1)\right)$$

3-point functions: [Dorn, Otto; Zamolodchikov^2] structure constants

$$\langle \textstyle\prod_{p=1}^3 V_{j_p}(z_p,\bar{z}_p)\rangle = C_{\mathrm{L}}[j_1,j_2,j_3]\textstyle\prod_{p<q}^3 (z_{pq}\bar{z}_{pq})^{-\Delta_{pq}}$$

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DOZZ structure constants

- rederived by Teschner by means of conformal bootstrap technique, as a solution of shift relations

$$C_{\mathrm{L}}(j_3, j_2, j_1) \sim \frac{1}{\Upsilon_b\left(-Q(j_{123}+1)\right)} \prod_{a=1}^{3} \frac{\sqrt{\Upsilon_b\left(-2Qj_a\right)\Upsilon_b\left(-Q(2j_a+1)\right)}}{\Upsilon_b\left(-Q(j_{123}-2j_a)\right)}$$

The explicit expressions for 3-point functions admit analytic continuation to **complex values of** b^2 excluding the negative real axis

- in that case $c = 1 + 6(b + b^{-1})^2 > 1$

For $b \to i\beta$ the central charge $c \to 1 - 6(\beta^{-1} - \beta)^2 < 1$

- the shift relations can be analytically continued
- the solution is given by $C_\beta[j_3, j_2, j_1]$ (the same function as in the generalized minimal models)

$$C_\beta[j_3, j_2, j_1] \sim \Upsilon_\beta\left(\beta - \hat{Q}(j_{123}+1)\right) \prod_{a=1}^{3} \frac{\Upsilon_\beta\left(\beta - \hat{Q}(j_{123}-2j_a)\right)}{\sqrt{\Upsilon_\beta\left(\beta - 2\hat{Q}j_a\right)\Upsilon_\beta\left(\beta - \hat{Q}(2j_a+1)\right)}}$$

DOZZ structure constants

- rederived by Teschner by means of conformal bootstrap technique, as a solution of shift relations

$$C_{\mathrm{L}}(j_3, j_2, j_1) \sim \frac{1}{\Upsilon_b(-Q(j_{123}+1))} \prod_{a=1}^{3} \frac{\sqrt{\Upsilon_b(-2Qj_a)\,\Upsilon_b(-Q(2j_a+1))}}{\Upsilon_b(-Q(j_{123}-2j_a))}$$

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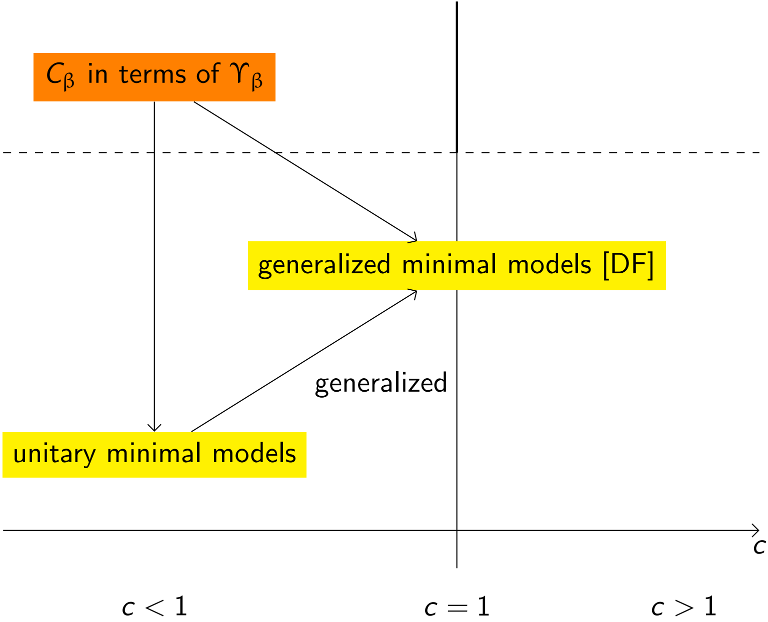
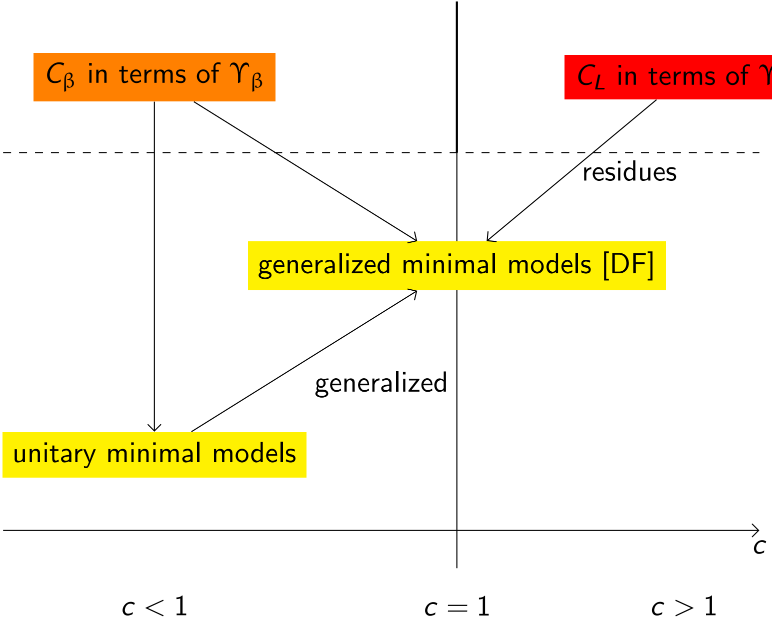

C_β in terms of Υ_β
generalized minimal models [DF]
generalized
unitary minimal models
c
$c < 1$
$c = 1$
$c > 1$


C_β in terms of Υ_β
C_L in terms of Υ_b
residues
generalized minimal models [DF]
generalized
unitary minimal models
c
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1. Minimal Models
   - $\widehat{su}(2)_k$ model
   - Virasoro Minimal Models
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2. Liouville theory
   - Liouville theory
   - $\widehat{su}(2)_\kappa$ model
   - coset construction of Liouville theory

$\widehat{su(2)}_\kappa$ model with $\kappa < -2$

Two choices of continuous spectrum based on different representations:

- principal unitary series of $sl(2,\mathbb{C})$ reps $\mathcal{P}_j$ with $j = -\frac{1}{2} + i\mathbb{R}$
- one can construct the $su(2)_\kappa \oplus su(2)_\kappa$ module over $\mathcal{P}_j$ (no division for chiral parts)
- the class of reps used in the quantization of the H_3^+ model

- principal unitary series of $sl(2,\mathbb{R})$ reps $\mathcal{D}_{j,\epsilon}$ with $j = -\frac{1}{2} + i\mathbb{R},\ \epsilon = 0, \frac{1}{2}$
- this is also a series of $su(2)$ reps but non-unitary
- provides a representation of $su(2)_\kappa \oplus su(2)_\kappa$ that factorizes as a tensor product of two modules

$$\widehat{\mathcal{D}}^\kappa_{j,\epsilon} \otimes \widehat{\mathcal{D}}^\kappa_{j,\epsilon}, \quad j \in -\frac{1}{2} + i\mathbb{R},\ \epsilon = 0, \frac{1}{2},$$

of the left and right chiral symmetries

$\widehat{su(2)}_\kappa$ model

spectrum of the model

- $\phi_{j,\epsilon}$ with $j = -\frac{1}{2} + i\mathbb{R}$, $\epsilon = 0, \frac{1}{2}$ has infinitely many "components" with $m, \bar{m} \in \mathbb{Z}$

$$J_0^3 \phi_{j,\epsilon}^{m,\bar{m}}(z,\bar{z}) = (m+\epsilon)\phi_{j,\epsilon}^{m,\bar{m}}(z,\bar{z}), \quad \bar{J}_0^3 \phi_{j,\epsilon}^{m,\bar{m}}(z,\bar{z}) = (\bar{m}+\epsilon)\phi_{j,\epsilon}^{m,\bar{m}}(z,\bar{z})$$
$$J_0^+ \phi_{j,\epsilon}^{m,\bar{m}}(z,\bar{z}) = (m+\epsilon-j)\,\phi_{j,\epsilon}^{m+1,\bar{m}}(z,\bar{z}),$$
$$J_0^- \phi_{j,\epsilon}^{m,\bar{m}}(z,\bar{z}) = (-m-\epsilon-j)\,\phi_{j,\epsilon}^{m-1,\bar{m}}(z,\bar{z})$$

- since j is not a (half)integer: $m+\epsilon \neq \pm j$
- introducing **isospin coordinates** $x, \bar{x}$, one defines currents $J^a(x)$ and fields:

$$\Phi_{j,\epsilon}(x,\bar{x};z,\bar{z}) = \sum_{m,\bar{m}\in\mathbb{Z}} x^{j-m-\epsilon}\bar{x}^{j-\bar{m}-\epsilon}\,\phi_{j,\epsilon}^{m,\bar{m}}(z,\bar{z})$$

  which are primaries of the highest weight reps with respect to $J^a(x)$
- the fields with j and $-j-1$ are identified (related by a reflection operator)

correlation functions

2-point functions

- for integer k and (half)integer j_i

$$\langle \Phi_{j_1}(x_1,\bar{x}_1;z_1,\bar{z}_1)\Phi_{j_2}(x_2,\bar{x}_2;z_2,\bar{z}_2)\rangle = \delta_{j_1,j_2}\frac{(x_{12}\bar{x}_{12})^{2j_1}}{(z_{12}\bar{z}_{12})^{2\Delta_1}},$$

- for real κ and $j_i = -\frac{1}{2} + i\mathbb{R}$

$$\langle \Phi_{j_1,\epsilon_1}(x_1,\bar{x}_1;z_1,\bar{z}_1)\Phi_{j_2,\epsilon_2}(x_2,\bar{x}_2;z_2,\bar{z}_2)\rangle = \frac{\delta_{\epsilon_1,\epsilon_2}}{(z_{12}\bar{z}_{12})^{2\Delta_1}}$$

$$\times\left(\delta(j_1-j_2)S_{j_1,\epsilon_1}(x_1,x_2)\,S_{j_1,\epsilon_1}(\bar{x}_1,\bar{x}_2)+\delta(j_2-j_1-1)\,\delta(x_1-x_2)\right)$$

- special care needed due to **complex values** of exponents
- $S_{j_1,\epsilon_1}(x_1,x_2)$ is a properly defined bilinear invariant satisfying equations given by global Ward identities

3-point functions

- for integer k and (half)integer j_i

$$\left\langle \prod_{p=1}^{3} \Phi_{j_p}(x_p,\bar{x}_p;z_p,\bar{z}_p)\right\rangle = C^{(k)}[j_1,j_2,j_3] \prod_{p<q}^{3} \frac{(x_{pq}\bar{x}_{pq})^{j_{pq}}}{(z_{pq}\bar{z}_{pq})^{\Delta_{pq}}} ,$$

- for real κ and $j_i = -\frac{1}{2} + i\mathbb{R}$

$$\left\langle \prod_{p=1}^{3} \Phi_{j_p,\epsilon_p}(x_p,\bar{x}_p;z_p,\bar{z}_p)\right\rangle = C^{(\kappa)}[j_1,j_2,j_3] \prod_{p<q}^{3} (z_{pq}\bar{z}_{pq})^{-\Delta_{pq}}$$
$$\times \left(\sum_{\epsilon=0,\frac{1}{2}} S_\epsilon \begin{bmatrix} j_3 & j_2 & j_1 \\ \epsilon_3 & \epsilon_2 & \epsilon_1 \\ x_3 & x_2 & x_1 \end{bmatrix} S_\epsilon \begin{bmatrix} j_3 & j_2 & j_1 \\ \epsilon_3 & \epsilon_2 & \epsilon_1 \\ \bar{x}_3 & \bar{x}_2 & \bar{x}_1 \end{bmatrix} \right)$$

- $S_\epsilon \begin{bmatrix} j_3 & j_2 & j_1 \\ \epsilon_3 & \epsilon_2 & \epsilon_1 \\ x_3 & x_2 & x_1 \end{bmatrix}$ is a properly defined three-linear invariant
- the structure constants $C[j_1,j_2,j_3]$ for $\kappa < -2$ are expected to be the same as in the H_3^+ model

Structure constants

- for $\kappa < -2$

$$C_H^{(\kappa)}[j_1, j_2, j_3] \sim \frac{1}{\Upsilon_b(-b(j_{123}+1))} \prod_{a=1}^{3} \frac{\sqrt{\Upsilon_b(-2j_a b)\,\Upsilon_b(-b(2j_a+1))}}{\Upsilon_b(-b(j_{123}-2j_a))},$$

with $b = \frac{1}{\sqrt{-(\kappa+2)}}$

- for $\kappa > -2$ (the same formula as for "generalized" $\widehat{su}(2)_k$ model)

$$C^{(\kappa)}[j_1, j_2, j_3] \sim \Upsilon_b(b(j_{123}+2)) \prod_{a=1}^{3} \frac{\Upsilon_b(b(j_{123}-2j_a+1))}{\sqrt{\Upsilon_b(b(2j_a+1))\,\Upsilon_b(b(2j_a+2))}},$$

with $b = \frac{1}{\sqrt{\kappa+2}}$

- the second formula is not an analytic continuation of the first one (but the equations that determine both formulas can be analytically continued to each other)

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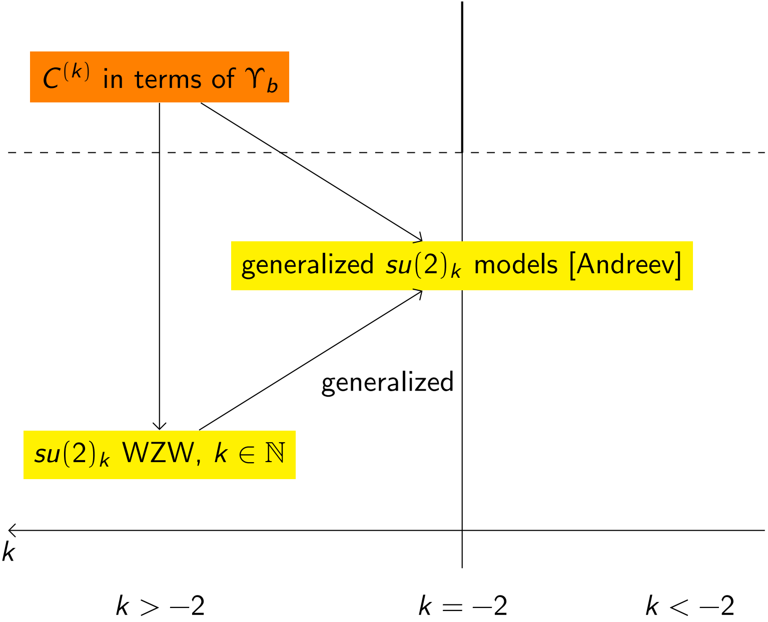
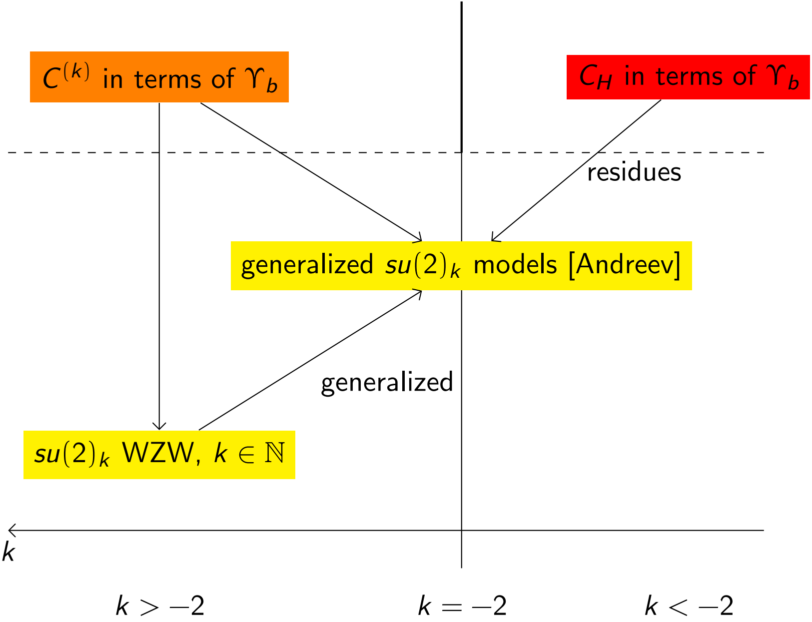

$C^{(k)}$ in terms of Υ_b
generalized $su(2)_k$ models [Andreev]
generalized
$su(2)_k$ WZW, $k \in \mathbb{N}$
k
$k > -2$
$k = -2$
$k < -2$


$C^{(k)}$ in terms of Υ_b
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1. Minimal Models
   - $\widehat{su}(2)_k$ model
   - Virasoro Minimal Models
   - coset construction of minimal models

2. Liouville theory
   - Liouville theory
   - $\widehat{su}(2)_\kappa$ model
   - coset construction of Liouville theory

continuous extension of the minimal models coset construction

$$\frac{\widehat{su}(2)_\kappa \times \widehat{su}(2)_1}{\widehat{su}(2)_{\kappa+1}}$$

- the Virasoro generators of the coset can be constructed in the same way as is the case of integer k
- the central charge

$$c = 1 - \frac{6}{(\kappa+2)(\kappa+3)} = 1 + 6Q^2$$

for $b^2 = -\frac{\kappa+3}{\kappa+2}$. Assuming real b, we get a condition for the levels:

$$\kappa < -2 < \kappa + 1$$

branching rules

$$(j,\epsilon)_\kappa \otimes (\delta)_1 \;=\; \bigoplus_{n\in\mathbb{Z}+\delta} (j+n, |\epsilon-\delta|)_{\kappa+1} \otimes (\Delta_{j,n})$$

- with reps of $\widehat{su}(2)_1$ denoted by $\delta = 0, \frac{1}{2}$,
- $(j,\epsilon)_\kappa$ denotes $\widehat{\mathcal{D}}^{\kappa}_{j,\epsilon}$ representation of $\widehat{su}(2)_\kappa$ with $j = -\frac{1}{2} + i\mathbb{R}$,
- $(j+n,\epsilon)_{\kappa+1}$ denotes $\widehat{\mathcal{D}}^{\kappa+1}_{j,\epsilon}$ representation of $\widehat{su}(2)_{\kappa+1}$ with $j = -\frac{1}{2} + n + i\mathbb{R}$, $n \in \frac{1}{2}\mathbb{Z}$
- $(\Delta_{j,n})$ denotes the Vir highest weight representation with $\Delta_{j+\frac{n}{bQ}} = -Q^2(j+\frac{n}{bQ})(1+j+\frac{n}{bQ})$
- the product theory on the rhs contains representations out of the standard spectrum of Liouville theory (for $n \neq 0$) - there are copies of "shifted" spectrum with respect to discrete variable n

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relation between states on both sides of the decomposition

$$(j,\epsilon)_\kappa \otimes (\delta)_1 \quad = \quad \bigoplus_{n\in\mathbb{Z}+\delta} (j+n, |\epsilon-\delta|)_{\kappa+1} \otimes (\Delta_{j,n})$$

- the highest weight states on the lhs are the highest weight states with $n=0$ and $n=\frac{1}{2}$, respectively:

$$|j,\epsilon\rangle_\kappa \otimes |0\rangle_1 = |j,\epsilon\rangle_{\kappa+1} \otimes |j\rangle$$

$$|j,\epsilon\rangle_\kappa \otimes \left|\tfrac{1}{2}\right\rangle_1 = \left|j+\tfrac{1}{2}, |\epsilon-\tfrac{1}{2}|\right\rangle_{\kappa+1} \otimes \left|j+\frac{1/2}{bQ}\right\rangle$$

- the other Virasoro highest weight states will correspond to some **descendant states** on the lhs

$$|j,n,\epsilon\rangle^\star = \mathcal{O}_{j,n}(J^a,K^a)\,|j,\epsilon\rangle_\kappa \otimes |\delta\rangle_1 = |j+n, |\epsilon-\delta|\rangle_{\kappa+1} \otimes \left|j+\frac{n}{bQ}\right\rangle$$

- operators $\mathcal{O}_{j,n}(J^a,K^a)$ the same as for integer k, known explicit formulas for $n=\pm\frac{1}{2},\pm 1$

relation between correlation functions

- for fields corresponding to the **descendant states**

$$\Phi^{\star}_{j,n,\epsilon}(x,\bar{x};z,\bar{z}) \;\leftrightarrow\; N_{j,n,\epsilon}\;\; |j,n,\epsilon\rangle^{\star} \otimes \overline{|j,n,\epsilon\rangle}^{\star}$$

we expect the correspondence:

$$\Phi^{\star}_{j,n,\epsilon}(x,\bar{x};z,\bar{z}) = \Phi^{(\kappa+1)}_{j+n,\epsilon}(x,\bar{x};z,\bar{z}) \otimes V_{j+\frac{n}{bQ}}(z,\bar{z}),$$

- explicit checks of the equality between 3-point functions containing the fields with $n = 0, \pm\frac{1}{2}, \pm 1$
- $n = 0$ means the equality between the structure constants
- checks for $n \neq 0$ involve using the Ward identities in the $\widehat{su}(2)_\kappa$ and $\widehat{su}(2)_1$ models

relation for primary fields

$$\begin{aligned}\Phi^{\star}_{j,0,\epsilon}(x,\bar{x};z,\bar{z}) &= \Phi^{(\kappa)}_{j,\epsilon}(x,\bar{x};z,\bar{z}) \otimes \Phi^{1}_{0}(x,\bar{x};z,\bar{z}) \\ &= \Phi^{(\kappa+1)}_{j,\epsilon}(x,\bar{x};z,\bar{z}) \otimes V_j(z,\bar{z}),\end{aligned}$$

with the condition for the levels:

$$\kappa < -2 < \kappa + 1$$

In 3-point correlators it implies

relation between structure constants

$$C^{(\kappa)}_{\mathrm{H}}[j_1,j_2,j_3] = C^{(\kappa+1)}[j_1,j_2,j_3]\; C_{\mathrm{L}}[j_1,j_2,j_3]$$

- the three-linear invariants are κ-independent

Structure constants of $\widehat{su}(2)_\kappa$ model and Liouville theory:

- for $\kappa < -2$, with $b = \frac{1}{\sqrt{-(\kappa+2)}}$

$$C_{\mathrm{H}}^{(\kappa)}[j_1, j_2, j_3] \sim \frac{1}{\Upsilon_b(-b(j_{123}+1))} \prod_{a=1}^{3} \frac{\sqrt{\Upsilon_b(-2j_a b)\,\Upsilon_b(-b(2j_a+1))}}{\Upsilon_b(-b(j_{123}-2j_a))},$$

- for $\kappa > -2$ (the same as for $\widehat{su}(2)_k$ model), with $b = \frac{1}{\sqrt{\kappa+2}}$

$$C^{(\kappa)}[j_1, j_2, j_3] \sim \Upsilon_b(b(j_{123}+2)) \prod_{a=1}^{3} \frac{\Upsilon_b(b(j_{123}-2j_a+1))}{\sqrt{\Upsilon_b(b(2j_a+1))\,\Upsilon_b(b(2j_a+2))}},$$

- Liouville theory with $c > 1$

$$C_{\mathrm{L}}(j_3, j_2, j_1) \sim \frac{1}{\Upsilon_b\left(-Q(j_{123}+1)\right)} \prod_{a=1}^{3} \frac{\sqrt{\Upsilon_b\left(-2Qj_a\right)\Upsilon_b\left(-Q(2j_a+1)\right)}}{\Upsilon_b\left(-Q(j_{123}-2j_a)\right)}$$

relation $\quad C_{\mathrm{H}}^{(\kappa)}[j_1, j_2, j_3] = C^{(\kappa+1)}[j_1, j_2, j_3]\; C_{\mathrm{L}}[j_1, j_2, j_3]$

true due to:

$$\Upsilon_{b_1}(-b_1 j)\Upsilon_{b_2}\left(b_2 j + b_2\right) \sim \Upsilon_b(-Qj)$$

$$b_1 = \frac{1}{\sqrt{-(\kappa+2)}}, \quad b_2 = \frac{1}{\sqrt{\kappa+3}}, \quad b = \frac{b_1}{b_2} = \sqrt{-\frac{\kappa+3}{\kappa+2}}, \quad Q = b + b^{-1} = b_1 b_2$$

relation between structure constants

$$C_{\mathrm{H}}^{(\kappa)}[j_1, j_2, j_3] = C^{(\kappa+1)}[j_1, j_2, j_3]\ C_{\mathrm{L}}[j_1, j_2, j_3], \quad \kappa < -2 < \kappa + 1$$

- this relation provides formulation of the Liouville structure constants in terms of $su(2)_\kappa$ structure constants
- since we are considering representations with $j = -\frac{1}{2} + i\mathbb{R}$ it is valid for the standard spectrum of Liouville theory ($\alpha = -jQ = \frac{Q}{2} + i\mathbb{R}$)

Correspondence between 3-point functions with $n \neq 0$

$$\langle \prod_{p=1}^{3} \Phi^{\star}_{j_p, n_p, \epsilon_p}(x_p, \bar{x}_p; z_p, \bar{z}_p) \rangle = \langle \prod_{p=1}^{3} \Phi^{(\kappa+1)}_{j_p+n_p, \epsilon_p}(x_p, \bar{x}_p; z_p, \bar{z}_p) \rangle \otimes \langle \prod_{p=1}^{3} V_{j_p + \frac{n_p}{bQ}}(z_p, \bar{z}_p) \rangle$$

Idea of the check:

- check the (x, z)-dependent terms (the three-linear invariants)
- calculate the rhs (from shift relations for Υ functions)

$$C^{(\kappa+1)}[j_1+n_1, j_2+n_2, j_3+n_3]\ C_{\mathrm{L}}[j_1+\tfrac{n_1}{bQ}, j_2+\tfrac{n_2}{bQ}, j_3+\tfrac{n_3}{bQ}]$$

$$\sim I(j_{123}+1, n_{123})^2 \prod_{a=1}^{3} I(j_{123}-2j_a, n_{123}-2n_a)^2\ C^{(\kappa+1)}[j_1, j_2, j_3]\ C_{\mathrm{L}}[j_1, j_2, j_3]$$

where

$$I(x,n) = \begin{cases} \prod_{p=2}^{n} \prod_{q=1}^{p-1} (x - p(\kappa+2) + q(\kappa+3)) & , \quad n > 1, \\ 1 & , \quad n = 0, 1 \\ \prod_{p=0}^{|n|-1} \prod_{q=0}^{p} (x + p(\kappa+2) - q(\kappa+3)) & , \quad n < 0, \end{cases}$$

- calculate the lhs from chiral Ward identities

$$\langle \prod_{p=1}^{3} \Phi^{\star}_{j_p, n_p, \epsilon_p}(x_p, \bar{x}_p; z_p, \bar{z}_p) \rangle \sim (P(j_i, n_i))^2\ C^{(\kappa)}[j_1, j_2, j_3]$$

Summary

We were investigating the relations between CFT models

$\widehat{su}(2)_k \times \widehat{su}(2)_1$ models $\sim$ $\widehat{su}(2)_{k+1}$ model $\times$ Vir minimal model

$\widehat{su}(2)_\kappa \times \widehat{su}(2)_1$ models $\sim$ $\widehat{su}(2)_{\kappa+1}$ model $\times$ Liouville

In the second case we have to consider an extension of Liouville theory - apart from the standard spectrum there are fields from spectrum "shifted" by a discrete variable (as in the relation SL $\sim$ L $\times$ L).

In both cases we have:

- branching rules (decomposition of representations)
- first examples of descendant fields in the product theory on the lhs that correspond to primary fields on the rhs
- checks of the relation for 3-point correlators containing these fields ($n = 0, \pm\frac{1}{2}, \pm 1$)
- equality between structure constants (special cases from spectrum of minimal models, but general formula for spectrum of Liouville)

Summary

- we were talking about diagonal fields and their correlation functions
- since we are considering models with representations that factorize as tensor products of two chiral modules (both for $su(2)$ and Vir models), it is possible to define chiral correlators and focus only on the chiral part

Open questions

- general formula for the descendant states $|j, n, \epsilon\rangle^\star$ in the product theory $\widehat{su}(2)_\kappa \times \widehat{su}(2)_1$
- higher n-point functions of primary fields
- similar relations for other theories:

$$\widehat{\mathrm{su}}(2)_\kappa \; \times \; \widehat{\mathrm{su}}(2)_2 \sim N = 1 \text{ super-Liouville } \times \; \widehat{\mathrm{su}}(2)_{\kappa+2}\,,$$
$$\widehat{\mathrm{su}}(2)_\kappa \; \times \; \widehat{\mathrm{su}}(2)_p \sim \text{para-Liouville } \times \; \widehat{\mathrm{su}}(2)_{\kappa+p}\,, \qquad p > 2$$

- generalizations to symmetry $\widehat{su}(N)$ for $N > 2$

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