Deformations of Q-systems, character formulas and the completeness problem

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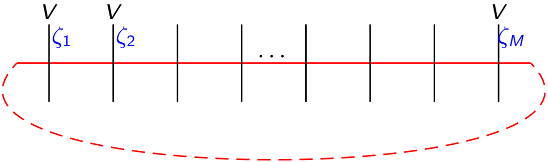
Outline

1. Counting graded dimensions of Heisenberg spin chains
2. The Q-system and its q-deformation
3. Discrete quantum integrability and exchange relations
4. Generalized Macdonald operators and quantum toroidal algebras

The Hilbert space of the XXX model

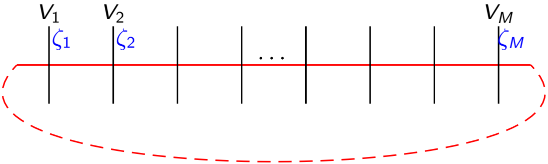


The algebra is $Y(\mathfrak{sl}_2)$, $V(\zeta_i) \simeq \mathbb{C}^2$ are representations of the Yangina and the Hilbert space is

$$\mathcal{H}_M \simeq V^{\otimes M} \simeq \bigoplus_{\lambda} \mathcal{H}_\lambda \otimes V(\lambda); \quad V(\lambda) = \text{irreducible } \mathfrak{sl}_2\text{-mod}$$

with dimension $\mathcal{H}_M = 2^M$.

Generalized inhomogeneous Heisenberg spin chain



The representations $V_i(\zeta_i)$ are arbitrary representations of $Y(\mathfrak{sl}_2)$, and the Hilbert space

$$\mathcal{H}_{\mathbf{n}} \simeq V_1 \otimes \cdots \otimes V_M = \bigotimes_{k \geq 1} V(k\omega_1)^{\otimes n_k} \simeq \bigoplus_{\lambda} \mathcal{H}_{\lambda,\mathbf{n}} \otimes V(\lambda)$$

where $V(k\omega_1) \simeq \mathbb{C}^{k+1}$ in the tensor product.

$$\dim \mathcal{H}_{\mathbf{n}} = \dim \prod_{i=1}^{k} V(i\omega_1)^{\otimes n_i} = \prod_{i=1}^{k} (i+1)^{n_i}.$$

Completeness problem

- The hamiltonian conserves spin: it acts on the multiplicity space

$$\mathcal{H}_{\lambda,\mathbf{n}} := \mathrm{Hom}_{\mathfrak{sl}_2}\left(\mathcal{H}_{\mathbf{n}}, V(\lambda)\right).$$

- Spectrum of the hamiltonian in the subspaces is parameterized by solutions to the Bethe ansatz equations.
- The "completeness conjecture" is that the dimension of $\mathcal{H}_{\lambda,\mathbf{n}}$ is bijection with the combinatorial data associated with the BAE.

Combinatorial content of BAE for $\mathfrak{sl}_2$

- Fix a partition μ of $S = \frac{1}{2}(\sum_i in_i - \ell)$, and the integers m_i are defined by

$$\mu = (1^{m_1}, 2^{m_2}, \cdots)$$

  We have

$$\mathcal{H}_{\lambda,\mathbf{n}} = \bigoplus_{\mu} \mathcal{H}_{\lambda,\mathbf{n}}(\mu)$$

- The basis of $\mathcal{H}_{\lambda,\mathbf{n}}(\mu)$ is parameterized by "riggings" of μ: m_i Distinct integers $I_j^{(i)} \in [1, p_i + m_i]$ for each row of length i. (Distinct partitions of length m_i and width at most $p_i + m_i$.)
- Grading: We weigh each rigging with a weight q^d where d is proportional to the sum of the integers.

Counting bosons vs. fermions

A bosonic Fock space has a basis parameterized by partitions:

$$a_{-\lambda_1} a_{-\lambda_2} \cdots a_{-\lambda_m} |0\rangle$$

with $\lambda_i \geq \lambda_{i+1}$.

Define the set of partitions $P(p|m)$ to be all sets of the form

$$\lambda = (p \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0)$$

Then

$$\sum_{\lambda \in P(p|m)} q^{|\lambda|} = \begin{bmatrix} p+m \\ m \end{bmatrix}_q,$$

where the Gaussian polynomial is

$$\begin{bmatrix} p+m \\ m \end{bmatrix}_q = \prod_{j=1}^{m} \frac{1-q^{p+j}}{1-q^j}.$$

Fermions

A fermionic Fock space is parameterized by distinct partitions:

$$\psi_{-\lambda_1} \cdots \psi_{-\lambda_m} |0\rangle$$

where $\lambda_i > \lambda_{i+1}$.

If $P_d(p+m|m)$ is the set of distinct partitions

$$\lambda = (p+m \geq \lambda_1 > \lambda_2 > \cdots > \lambda_m > 0)$$

then the partition $\widetilde{\lambda} = (\lambda_1 - m, \lambda_2 - m + 1, ..., \lambda_m - 1) \in P(p|m)$. The generating function

$$\sum_{\lambda \in P_d(p+m|m)} q^{|\lambda|} = \sum_{\widetilde{\lambda} \in P(p|m)} q^{|\widetilde{\lambda}| + \frac{1}{2}m(m+1)} = \begin{bmatrix} p+m \\ m \end{bmatrix}_q q^{\frac{1}{2}m(m+1)}.$$

Completeness of Bethe solutions

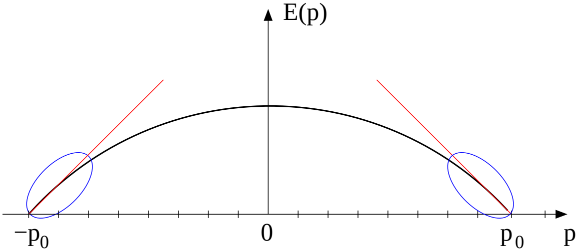
We are counting “fermions” with a difference:

- The “vacancy numbers” $\mathbf{p} = (p_1, p_2, ...)$ depend on $\mathbf{m}$, the number of “quasi-particles”:
$$\mathbf{p} = A\mathbf{n} - 2A\mathbf{m}, \quad [A]_{i,j} = \min(i,j).$$
- If Bethe integers parameterize spectrum of $\mathcal{H}_{\lambda,\mathbf{n}}$ then
$$\dim \mathcal{H}_{\lambda,\mathbf{n}} = \sum_{\substack{\mathbf{m} \\ |\mathbf{m}|=\frac{1}{2}|\mathbf{n}|-(\lambda_1-\lambda_2)}} \binom{p_i+m_i}{m_i}.$$
- We have the refined (graded) dimension formula:
$$Z_{\lambda,\mathbf{n}}(q) = \sum_{\mathbf{m}} q^{\mathbf{m}^t A\mathbf{m}} \begin{bmatrix} p_i+m_i \\ m_i \end{bmatrix}_q$$

  We call the refined counting $Z_{\lambda,\mathbf{n}}$ the **conformal partition function.**

Physical origin of grading

- In conformal limit, the partition function Z is dominated by order $1/M$ excitations: Massless quasi-particles, with linearized energy function, $E(P_i) \simeq v|(P_i - P_0)|$. ($P$=momentum and v=Fermi velocity).



- Periodic system: Momenta P_i are **quantized** in units of $\frac{2\pi}{N}$: $\Longrightarrow$ Dominant contribution to the chiral partition function is a series in $q = \exp(\frac{-2\pi v}{kNT})$. Conformal limit means $N \to \infty$, $T \to 0$, NT fixed.
- The momenta are proportional to (shifted) Bethe integers in this limit.

Infinite size limit (motivation for the term "conformal")

In the XXX model or its higher rank generalizations to $\mathfrak{sl}_N$,

$$\mathcal{H} \simeq V(\omega_1)^{\otimes M}.$$

Define the generating function

$$Z_M(\mathbf{x}; q) = \sum_{\lambda : |\lambda| = M} Z_{\lambda,M}(q) s_\lambda(\mathbf{x})$$

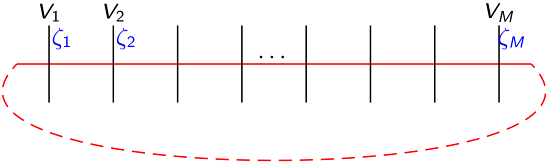
where the Schur functions $s_\lambda(\mathbf{x})$ are the characters of the irreducible representation $V(\lambda)$ of $\mathfrak{gl}_N$.

- **Theorem:** [K 2004]

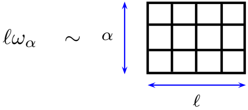
$$\lim_{M\to\infty} Z_{(1)^M}(\mathbf{x}, q^{-1}) \propto \mathrm{char} L(\Lambda_i), \quad i = M \mod N.$$

- The module $L(\Lambda_i)$ is the level-1 highest weight module of the affine algebra $\widehat{\mathfrak{sl}}_N$ with highest weight Λ_i.
- The character is a chiral partition function of the effective conformal field theory which describes the (critical) XXX model in the thermodynamic limit (WZW, $k = 1$).

Higher rank generalizations



The representations $V_i(\zeta_i)$ of $Y(\mathfrak{sl}_N)$ are special: **KR-modules** $V_i(\zeta_i) \simeq V(\ell_i \omega_{\alpha_i})$ as $\mathfrak{sl}_N$-modules:



$$\mathcal{H}_{\mathbf{n}} = \bigotimes_{a=1}^{N-1} \bigotimes_{k\geq 1} V(k\omega_a)^{\otimes n_{a,k}} \simeq \bigoplus_{\lambda} \mathcal{H}_{\lambda,\mathbf{n}} V(\lambda).$$

Bethe ansatz combinatorics for $\mathfrak{sl}_N$

Combinatorial data:

1. multi-partitions $\vec{\mu} = (\mu^{(1)}, ..., \mu^{(N-1)})$, where
$$\sum_\beta C_{\alpha,\beta}|\mu^{(\beta)}| = \sum_j n^{(\alpha)} - \lambda_\alpha, \quad \lambda = \sum_\alpha \lambda_\alpha \omega_\alpha.$$
2. Each $\mu^{(\alpha)}$ has a **rigging** as in the $\mathfrak{sl}_2$ case, with vacancy numbers $p_i^{(\alpha)}$ for the part of length i of partition $\mu^{(\alpha)}$.
3. We give each configuration a weight proportional to the sum of the riggings.

The result of counting such solutions is...

Explicit combinatorial formula for $Z_{\mathbf{n}}(\mathbf{x};q)$

$$Z_{\mathbf{n}}(\mathbf{x};q) = \sum_{\vec{\mu}} q^{\frac{1}{2}F(\vec{\mu})} \prod_{\alpha,i} \begin{bmatrix} p_i^{(\alpha)} + m_i^{(\alpha)} \\ m_i^{(\alpha)} \end{bmatrix}_q s_{\lambda(\mathbf{n})-C\mu}(\mathbf{x})$$

- The multi-partition $\lambda = (\lambda^{(1)}, ..., \lambda^{(N-1)})$ is determined by $\mathbf{n}$: $n_{\alpha,j}$ is the number of parts of $\lambda^{(\alpha)}$ of length j.
- The sum is over multi-partitions $\vec{\mu} = (\mu^{(1)}, ..., \mu^{(r)})$;
- $F(\vec{\mu}) = \sum \mu_i^{(\alpha)} C_{\alpha,\beta} \mu_i^{(\beta)}$, $C =$ Cartan matrix;
- $\mathbf{m} = \{m_i^{(\alpha)}\}$ with $m_i^{(\alpha)}$ the number of columns of $\mu^{(\alpha)}$ of length i.
- The integers $p_i^{(\alpha)}$: Sum over the first i columns of the composition $\lambda^{(\alpha)} - (C\vec{\mu})^{(\alpha)}$.

Special case: "Level 1"

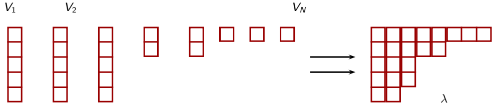
Choose all representations to be fundamental representations with highest weight ω_α for various α.

$$\mathcal{H} \simeq \bigotimes_{\alpha=1}^{N-1} V(\omega_\alpha)^{\otimes n_{\alpha,1}}.$$

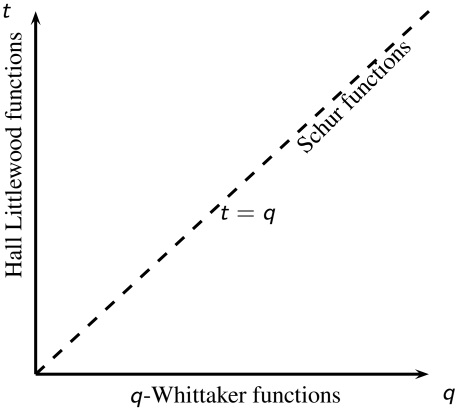
- The functions $Z_{\mathbf{n}}(\mathbf{x}; q)$ are polynomial versions of q-Whittaker functions (eigenfunctions of q-Toda).
- In terms of the modified Macdonald polynomials,

$$Z_{\mathbf{n}}(\mathbf{x}; q) = H_\lambda(\mathbf{x}; q, 0) = P_\lambda(\mathbf{x}; q, 0).$$

where λ is the partition with $n_{\alpha,1}$ columns of length α.



Macdonald symmetric functions



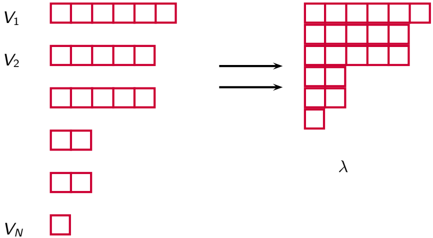
Special case: Symmetric power representations

Take all representations V_i to be symmetric power representations

$$\mathcal{H} \simeq \bigotimes_{\ell=1}^{k} V(\ell\omega_1)^{\otimes n_{1,\ell}}.$$

- The functions $Z_{\mathbf{n}}(\mathbf{x}; q)$ are modified Hall-Littlewood symmetric functions.
- A specialization of the modified Macdonald polynomial

$$Z_{\mathbf{n}}(\mathbf{x}; q) = \widetilde{H}_{\lambda}(\mathbf{x}; q, 0).$$



Status of proofs of completeness and $Z_{\mathbf{n}}(\mathbf{x}; q)$

1. The refined counting function $Z_{\mathbf{n}}(\mathbf{x}; q)$ has a representation-theoretical definition in terms of the crystal bases for $U_q(\widehat{\mathfrak{g}})$ (for some $\mathfrak{g}$) or the representation theory of $\mathfrak{g}[t]$ (all simple $\mathfrak{g}$). [Feigin Loktev 99].
2. Dimension formula proved for all $\mathfrak{g}$ [DFK08] and formula for $Z_{\mathbf{n}}(\mathbf{x}, q)$ proved for simply-laced $\mathfrak{g}$ [DFK11] and all $\mathfrak{g}$ [K. Lin 17].

The formulas can be rewritten as a constant term formula in the variables$\{Q_{a,k} : a \in [1, \mathrm{rank}\ \mathfrak{g}], k \in \mathbb{Z}\}$ and are equivalent to the fact that they satisfy the Q-system:

$$Q_{a,k+1} Q_{a,k-1} = Q_{a,k}^2 - \prod_{b \sim a} Q_{b,k}, \quad \mathfrak{g} \text{ simply-laced.}$$

A discrete, integrable evolution with a canonical quantization.

From combinatorics to algebra

Switch point of view: Look for the algebra of "raising operators".

Define $Q_{a,k} = \mathrm{ch}\, V(k\omega_a)$:

$$Z_{\mathbf{n}}(\mathbf{x};1) = \prod Q_{a,j}^{n_{a,j}} = \sum_{\lambda} Z_{\lambda,\mathbf{n}}(1) s_\lambda(\mathbf{x}).$$

So that adding one more site to the spin chain means multiplying by $Q_{a,k}$:

$$Z_{\mathbf{n}+\epsilon_{\alpha,k}}(\mathbf{x};1) = Q_{a,k} Z_{\mathbf{n}}(\mathbf{x};1).$$

Is there q-deformed version of this multiplication which produces the polynomials $Z_{\mathbf{n}}(\mathbf{x};q)$?

Outline of the answer:

Theorem: The characters of KR-modules $\{Q_{a,k}\}$ satisfy recursion equations called Q-systems. These are:

1. Cluster algebra mutations.
2. Discrete integrable equations.

Since we have a cluster algebra, we have a canonical quantization.

Theorem: [Di Francesco, K.]

1. The quantum Q-system is the correct q-deformation to generate $Z_{\mathbf{n}}(\mathbf{x}; q)$.
2. Integrability survives quantization.
3. The integrals of motion give q-difference equations for the functions $Z_{\mathbf{n}}(\mathbf{x}; q)$. Special cases: Toda q-difference equations.

The Q-system: The classical case

Theorem: The characters of KR-modules (in the case of $\mathfrak{sl}_N$, $Q_{a,k} = s_{(k^a)}(\mathbf{x})$) satisfy the the Q-system

$$Q_{a,k+1}Q_{a,k-1} = Q_{a,k}^2 - Q_{a+1,k}Q_{a-1,k}, \quad Q_{0,k} = Q_{N,k} = 1$$

together with the **initial data**

$$Q_{i,0} = 1 \quad (i = 1, ..., N-1).$$

Definition: The algebra R_N is the commutative, associative algebra generated by $\{Q_{a,k}^{\pm 1}\}$ with relations given by the Q-system.

Example of Q-system for $\mathfrak{sl}_2$

For $\mathfrak{g} = \mathfrak{sl}_2$, there is only one simple root, $Q_k := Q_{1,k}$:

$$Q_{k+1}Q_{k-1} = Q_k^2 - 1.$$

Given initial data (Q_0, Q_1),

$$\begin{aligned}
Q_2 &= \frac{Q_1^2 - 1}{Q_0} \xrightarrow[Q_0=1]{} Q_1^2 - 1, \\
Q_3 &= \frac{(Q_1^2-1)^2 - Q_0}{Q_0^2 Q_1} \xrightarrow[Q_0=1]{} Q_1^3 - 2Q_1, \\
Q_4 &= \frac{(Q_1^3 - Q_1^2 - Q_1 - Q_0^2 + 1)(Q_1^3 + Q_1^2 - Q_1 + Q_0^2 - 1)}{Q_0^3 Q_1^2} \xrightarrow[Q_0=1]{} Q_1^4 - 3Q_1^2 + 1.
\end{aligned}$$

1. All Q_k are Laurent polynomials in (Q_0, Q_1).
2. When $Q_0 = 1$, all Q_k are polynomials in Q_1. (Chebyshev polynomials of second kind in x if $Q_1 = 2x$).

Denanot-Jacobi and discrete integrability I

The Q-system for $\mathfrak{sl}_N$ is a discrete integrable system in the time variable k:

$$Q_{i+1,k}Q_{i-1,k} = Q_{i,k}^2 - Q_{i,k+1}Q_{i,k-1}, \quad Q_{0,k} = 1, Q_{N,k} = 1$$

is satisfied by the minors of the discrete Wronskian matrix ($Q_k := Q_{1,k}$):

$$W_{i+1,k} = \begin{pmatrix} Q_k & Q_{k+1} & & \cdots & Q_{k+i} \\ Q_{k-1} & Q_k & & \cdots & Q_{k+i-1} \\ \vdots & \vdots & \ddots & \vdots & \\ Q_{k-i} & Q_{k-i+1} & & \cdots & Q_k \end{pmatrix}$$

The Q-system is the Denanot-Jacobi relation for the determinant of the $n \times n$ matrix M:

$$|M||M_{1,n}^{1,n}| = |M_1^1||M_n^n| - |M_1^n||M_n^1|$$

under the identification $Q_{i,k} = |W_{i,k}|$.

Conserved quantities

- The relation $|W_{N,k}| = 1$ for all k implies $|W_{N+1,k}| = 0$, a linear recursion relation:

$$\sum_{j=0}^{N} (-1)^j C_{k,j} Q_{k+N-j}.$$

- The relation $|W_{N,k+1}| - |W_{N,k}| = 1 - 1 = 0$ implies the coefficients in the linear recursion are independent of k.

Example: $N = 2$

$$0 = 1 - 1 = |W_{2,k+1}| - |W_{2,k}| = \begin{vmatrix} Q_{k+1} & Q_{k+2} \\ Q_k & Q_{k+1} \end{vmatrix} - \begin{vmatrix} Q_k & Q_{k+1} \\ Q_{k-1} & Q_k \end{vmatrix}$$

$$= \begin{vmatrix} Q_{k+1} & Q_k + Q_{k+2} \\ Q_k & Q_{k-1} + Q_{k+1} \end{vmatrix}$$

so the first column is proportional to the second:

$$c_k Q_{k+1} = Q_k + Q_{k+2} \text{ and } c_k Q_k = Q_{k-1} + Q_{k+1}$$

therefore c_k is independent of k:

$$c = \frac{Q_{k-1} + Q_{k+1}}{Q_k} \text{ is a conserved quantity.}$$

The cluster algebra structure

Theorem: [K 2009] Each of the relations in R_N

$$Q_{i,k+1}Q_{i,k-1} = Q_{i,k}^2 - Q_{i+1,k}Q_{i-1,k}, \quad k \in \mathbb{Z}, i \in 1, ..., N-1,$$

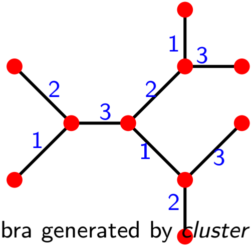
with $Q_{0,k} = Q_{N,k} = 1$, is a mutation relation in the cluster algebra with the exchange matrix

$$B = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}, \qquad C = \text{the } \mathfrak{sl}_N \text{ Cartan matrix.}$$

Aside on cluster algebras

Aside on cluster algebras

Fix an integer n (the rank) and let T be the complete n-tree with each vertex $t \in T$ having incident edges labeled $1, ..., n$.



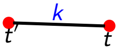
Definition: A cluster algebra is an associative commutative algebra generated by *cluster variables* $\{x_i(t), i \in [1, n], t \in T\}$, with relations between them, defined as follows:

To each vertex t of the tree T we associate the cluster $(\mathbf{x}(t), \Gamma(t))$ where $\Gamma(t)$ is a quiver with no 1- or 2-cycles.

Clusters in vertices connected by an edge are related by an involution called a *mutation*.

Mutations

If two vertices of the tree are connected by an edge k:



then $(\mathbf{x},\Gamma)(t') = \mu_k((\mathbf{x},\Gamma)(t))$, where μ_k is defined as follows:

- Quiver mutation:
  1. For each subquiver $k \longrightarrow i \longrightarrow j$ in Γ, add an arrow $k \longrightarrow j$.
  2. Reverse all arrows incident to vertex i.
  3. Erase all 2-cycles.
- Cluster mutations: $x(t')$ is obtained from $x(t)$ as follows:

$$x_j(t') = \begin{cases} x_j(t), & j \neq i; \\[2ex] \dfrac{\prod_{j\to i} x_j + \prod_{i \to j} x_j}{x_i(t)}, & i = j. \end{cases}$$

Insight from cluster algebras

The cluster algebra structure gives us some information:

1. (Laurent phenomenon) All cluster variables in a cluster algebra are **Laurent polynomials** in any cluster variable $\mathbf{x}(t)$. [Fomin-Zelevinsky].
2. **Theorem:** [DFK] The evaluation $Q_{i,0} = 1$ (i.e. $Q_{i,-1} = 0$) for all i reduces each of the Laurent polynomials $Q_{i,k}$ to a **polynomial** in $\{Q_{i,1} : i \in [1, N-1]\}$. (Laurent property applied to this particular algebra).
3. If the exchange matrix of the cluster algebra is non-degenerate, there is a canonical quantization of the cluster algebra.

Quantization of the Q-system

Classical:

$$Q_{i,k+1}Q_{i,k-1} = Q_{i,k}^2 - Q_{i+1,k}Q_{i-1,k}, \quad Q_{0,k} = Q_{N,k} = 1.$$

Definition: The quantized algebra $\mathcal{R}_N$ is the algebra generated by the (invertible) elements $\{\mathcal{Q}_{i,k}, 1 \leq i \leq N, k \in \mathbb{Z}\}$ modulo the relations (1), (2), (3) below:

- The quantized variables commute as

$$\mathcal{Q}_{i,k}\mathcal{Q}_{j,k+m} = q^{\min(i,j)m}\mathcal{Q}_{j,k+m}\mathcal{Q}_{i,k}, \quad |m| \leq |i-j|+1. \quad (1)$$

- The quantum mutation relation **(Quantum Q-system)**:

$$q^i\mathcal{Q}_{i,k+1}\mathcal{Q}_{i,k-1} = \mathcal{Q}_{i,k}^2 - \mathcal{Q}_{i+1,k}\mathcal{Q}_{i-1,k}, \quad (2)$$

- Boundary conditions:

$$\mathcal{Q}_{0,k} = 1, \quad \mathcal{Q}_{N+1,k} = 0. \quad (3)$$

Conformal partition function from Q-system

Classical case:

$$Z_{\mathbf{n}}(\mathbf{x};1) = \prod_{i,k>0} (Q_{i,k})^{n_{i,k}}\Big|_{Q_{i,1}=e_i(\mathbf{x}),Q_{i,0}=1}$$

Theorem [Di Francesco, K 2014] There is a linear functional

$$\Pi: \ U(\{\mathcal{Q}_{i,k}, k \geq 0\}) \to \mathbb{Z}[q,q^{-1}][x_1,\ldots,x_N]^{\mathcal{S}_N}$$

which maps the product of solutions to the quantum Q-system to the conformal partition function:

$$\Pi: \prod_{j=k}^{1} \prod_{i} \mathcal{Q}_{i,j}^{n_{i,j}} \mapsto Z_{\mathbf{n}}(\mathbf{x};q^{-1}).$$

Note: The functional uses (Laurent, polynomiality theorems) structure of quantum cluster algebra to impose the analogue of $Q_{i,0}=1$ and to extract the coefficients of $s_\lambda(\mathbf{x})$.

The quantum determinant

Theorem:[Di Francesco, RK] The quantum Q-system

$$q^i \mathcal{Q}_{i,k+1}\mathcal{Q}_{i,k-1} = \mathcal{Q}_{i,k}^2 - \mathcal{Q}_{i+1,k}\mathcal{Q}_{i-1,k}$$

is a quantum Desnanot-Jacobi relation: The elements $\mathcal{Q}_{i,k}$ are **quantum determinants** of $\{\mathcal{Q}_{1,j}\}$s:

In terms of generating currents

$$\mathcal{Q}(z) := \sum_{n\in\mathbb{Z}} z^n \mathcal{Q}_{1,n},$$

$$\mathcal{Q}_{a,n} = CT_{z_1,\ldots,z_a} \prod_{i=1}^{a} z_i^{-n} \prod_{1\le i<j\le a} (1 - q\frac{z_j}{z_i})\mathcal{Q}(z_1)\cdots\mathcal{Q}(z_a).$$

Corollary: $\mathcal{R}_N$ is generated as a polynomial algebra by the elements $\{\mathcal{Q}_{1,k}\}_{k\in\mathbb{Z}}$.

Discrete integrable system

The quantum Q-system is a discrete integrable system:

$$q^i \mathcal{Q}_{i,k+1} = (\mathcal{Q}_{i,k}^2 - \mathcal{Q}_{i+1,k}\mathcal{Q}_{i-1,k})\mathcal{Q}_{i,k-1}^{-1}, \quad i \in \{1, ..., N\}$$

with

$$\mathcal{Q}_{0,k} = 1, \mathcal{Q}_{N+1,k} = 0$$

is an evolution equation for the variables in the discrete time variable $k \in \mathbb{Z}$. Time translation is

$$D: \ (\mathcal{Q}_{1,k}, ..., \mathcal{Q}_{N,k}) \mapsto (\mathcal{Q}_{1,k+1}, ..., \mathcal{Q}_{N,k+1})$$

Theorem: The discrete quantum evolution above has N integrals of motion in involution.

Miura operator

For fixed k, define the N commuting elements in $\mathcal{R}_N$:

$$x_{i,k} = \mathcal{Q}_{i,k+1}\mathcal{Q}_{i,k}^{-1}\mathcal{Q}_{i-1,k}\mathcal{Q}_{i-1,k+1}^{-1}, \qquad i \in \{1, ..., N\}.$$

Theorem: [Di Francesco, K 2016] The operator acting on $\mathcal{R}_N$

$$\mu_k = (D - x_{N,k})(D - x_{N-1,k}) \cdots (D - x_{1,k})$$

is independent of k.

Sketch of proof: Define $\xi_{i,k} = \mathcal{Q}_{i,k}\mathcal{Q}_{i,k+1}^{-1}$ so that $x_{i,k} = \xi_{i,k}^{-1}\xi_{i-1,k}$. The relation

$$(D - x_{i+1,k})(D - \xi_{i,k}^{-1}\xi_{i-1,k-1}) = (D - \xi_{i+1,k}^{-1}\xi_{i,k-1})(D - x_{i,k-1})$$

is a consequence of two applications of the quantum Q-system. Together with the boundary terms

$$\xi_{1,k}^{-1}\xi_{0,k-1}^{-1} = \xi_{1,k}^{-1}\xi_{0,k}^{-1} = x_{1,k}$$

and

$$\xi_{N+1,k}^{-1}\xi_{N,k-1} = \xi_{N+1,k-1}^{-1}\xi_{N,k-1} = x_{N,k-1}$$

this gives a "zipper proof" that $\mu_k = \mu_{k-1}$.

Conserved quantities

Corollary: The coefficients $C_j := C_{j,n}$ in the "Miura operator"

$$\mu = \mu_k = (D - x_{N,k})(D - x_{N-1,k}) \cdots (D - x_{1,k}) = \sum_{j=0}^{N} (-1)^j C_{j,k} D^{N-j}$$

are independent of k.

Example: For $\mathfrak{sl}_2$ ($N = 2$),

$$C_0 = 1, \quad C_2 = \mathcal{Q}_{2,k+1}\mathcal{Q}_{2,k}^{-1}, \quad C_1 = (C_2\mathcal{Q}_{1,k} + \mathcal{Q}_{1,k+2})\mathcal{Q}_{1,k+1}^{-1}.$$

Lemma: The elements

$$x_a := \lim_{n\to\infty} x_{a,n}$$

are **well-defined** and commute with each other.

So $C_i = e_i(x_1, \ldots, x_N)$ are the elementary symmetric functions in these variables.

Time evolution

Theorem: The elements $\{\mathcal{Q}_k := \mathcal{Q}_{1,k}\}$ which generate $\mathcal{R}_N$ are in the kernel of the Miura operator μ:

$$\mu \mathcal{Q}_k = 0, \qquad n \in \mathbb{Z}.$$

Proof: Since $\mu = \mu_k$ for all k,

$$\mu_k \mathcal{Q}_k = \cdots (D - x_{1,k}) \mathcal{Q}_k = \cdots (D - \mathcal{Q}_{k+1} \mathcal{Q}_k^{-1}) \mathcal{Q}_k = \cdots (D \mathcal{Q}_k - \mathcal{Q}_{k+1}) = 0.$$

Theorem: the coefficients C_j act as Hamiltonians:

$$[C_1, \mathcal{Q}_k] = (1-q) \mathcal{Q}_{k+1},$$

and in general

$$(1-q)^{-1} [C_j, \mathcal{Q}_k] = \sum_{a=1}^{j} (-1)^a C_{j-a} \mathcal{Q}_{k+a}.$$

Generating functions

Define

$$\mathcal{Q}(z) := \sum_{k\in\mathbb{Z}} \mathcal{Q}_k z^k, \quad C(z) = \sum_{j=0}^{N} (-z)^j C_j = \prod_{i=1}^{N} (1 - zx_i).$$

then

$$C(z) = \exp\left(-\sum_{j\geq 1} \frac{z^j}{j} P_j \right),$$

where P_j are the power-sum symmetric functions, and

$$[P_j, \mathcal{Q}_k] = (1 - q^k)\mathcal{Q}_{k+j}, \quad j \in \mathbb{N}.$$

Application of integrability: Exchange relations

Theorem [DFK16]: The generating functions $\mathcal{Q}(z)$ satisfy the quadratic relations

$$(z - qw)\mathcal{Q}(z)\mathcal{Q}(w) + (w - qz)\mathcal{Q}(w)\mathcal{Q}(z) = 0.$$

Proof: The exchange relation is equivalent to $\phi_{k,\ell} = [\mathcal{Q}_k, \mathcal{Q}_{k+\ell}]_q + [\mathcal{Q}_{k+\ell-1}, \mathcal{Q}_{k+1}]_q = 0, \quad \ell > 0.$

$$\ell = 1: \qquad \phi_{k,1} = 2(\mathcal{Q}_k\mathcal{Q}_{k+1} - q\mathcal{Q}_{k+1}\mathcal{Q}_k) = 0.$$

$$\ell = 2: \qquad \phi_{k,2} = \mathcal{Q}_k^2 - q\mathcal{Q}_{k+1}\mathcal{Q}_{k-1} - (q\mathcal{Q}_k^2 - \mathcal{Q}_{k-1}\mathcal{Q}_{k+1}) = \mathcal{Q}_{2,k} - \mathcal{Q}_{2,k} = 0,$$

which follows from the quantum Q-system and its counterpart

$$\mathcal{Q}_{k-1}\mathcal{Q}_{k+1} = q\mathcal{Q}_k^2 - \mathcal{Q}_{2,k}.$$

By induction,

$$[H_1, [\mathcal{Q}_k, \mathcal{Q}_{k+\ell}]_q] = ([\mathcal{Q}_{k+1}, \mathcal{Q}_{k+\ell}]_q + [\mathcal{Q}_k, \mathcal{Q}_{k+\ell+1}]_q)$$

to $\phi_{k,\ell}$:

$$\phi_{k,\ell+1} = [H_1, \phi_{k,\ell}] - \phi_{k+1,\ell-1}$$

with $\phi_{k,1} = \phi_{k,2} = 0$, this vanishes by induction for all $\ell > 0$.

Quantum determinant relation

The relation
$$(z-qw)\mathcal{Q}(z)\mathcal{Q}(w)+(w-qz)\mathcal{Q}(w)\mathcal{Q}(z)=0.$$
is the defining relation in $U_{\sqrt{q}}(\mathfrak{n}[u,u^{-1}])\subset U_{\sqrt{q}}(\widehat{sl}_2)$.

Here, we have one more relation in $\mathcal{R}_N$: A degree $N+1$ polynomial relation coming from the $(N+1)$st quantum determinant:
$$\mathcal{Q}_{N+1,k}=0, \quad k\in\mathbb{Z}.$$

Theorem: The algebra $\mathcal{R}_N$ is isomorphic to a quotient of $U_{\sqrt{q}}(\mathfrak{n}[t,t^{-1}])$ by the rank-dependent quantum determinant relations.

Action on characters

Recall: There is a linear functional from the quantum Grothendieck ring to the ring of symmetric polynomials, such that

$$\Pi : \prod_{j=k}^{1} \prod_{i=1}^{N-1} \mathcal{Q}_{i,j}^{n_{i,j}} \mapsto Z_{\mathbf{n}}(\mathbf{x}; q^{-1}).$$

The elements $\mathcal{Q}_{i,k}$ act as maps between *graded characters of*:

$$\mathcal{Q}_{i,k} : Z_{\mathbf{n}}(\mathbf{x}; q^{-1}) \to Z_{\mathbf{n}'}(\mathbf{x}; q^{-1})$$

where $\mathbf{n}'$ differs from $\mathbf{n}$ only by $n'_{i,k} = n_{i,k} + 1$. (Corresponding to **adding one representation** $V(k\omega_i)$.)

Difference operator Solutions of quantum Q-system

How does $\mathcal{Q}_{i,k}$ act on symmetric functions?

For any $k \in \mathbb{Z}$ and $i \in [1, N]$, Define the q-difference operators on the space of functions in $x_1, ..., x_N$:

$$D_{a,k} = \sum_{\substack{I \subset [1,N] \\ |I|=a}} \left(\prod_{i \in I} x_i^k \prod_{j \notin I} \frac{x_i}{x_i - x_j} \right) \prod_{i \in I} \Gamma_i, \qquad \text{where } \Gamma_i x_j = q^{\delta_{ij}} x_j \Gamma_i.$$

For example $D_{0,k} = 1$, $D_{N,k} = (x_1 \cdots x_N)^k \Gamma_1 \cdots \Gamma_N$, $D_{N+1,k} = 0$ and $D_{1,k}$ is a linear combination of x_i^k with non-commuting right coefficients.

Theorem: The elements $D_{i,k}$ form a representation π of $\mathcal{R}_N$ where

$$\pi(\mathcal{Q}_{a,k}) = D_{a,k}.$$

To prove this one must show that $D_{a,k}$ satisfy the quantum Q-system (hard) and the quantum determinant conditions (automatic).

Graded characters from difference operators

Theorem: [Di Francesco, K 2015] The graded characters of $\mathcal{H}_{\mathbf{n}}$ is be generated by the action of difference operators on the trivial polynomial, as follows:

$$Z_{\mathbf{n}}(\mathbf{x}; q^{-1}) = q^{-\frac{1}{2}p(\mathbf{n})} \prod_{i=1}^{N-1} (D_{i,k})^{n_{i,k}} \cdots \prod_{i=1}^{N-1} (D_{i,1})^{n_{i,1}} \ 1.$$

where

$$p(\mathbf{n}) = \sum_{i,j,a,b} n_{a,i} \min(i,j) \min(a,b) n_{b,j} - \sum_{i,a} i a n_{a,b},$$

with $n_{a,i}$ being the number of modules in the tensor product with highest weight $i\omega_a$.

So we have the raising operators for the family of symmetric functions $Z_{\mathbf{n}}(\mathbf{x}; q)$ as q-difference operators.

Difference (Toda-like) equations for $Z_{\mathbf{n}}(\mathbf{x};q)$

The existence of Hamiltonians C_i implies q-difference equations:

Example: Let $\mathfrak{g}=\mathfrak{sl}_2$.

- The quantum Q-system is an equation for $\mathcal{Q}_k := \mathcal{Q}_{1,k}$ and $\mathcal{Q}_{2,k}=A^k\Delta$ where $\Delta\mathcal{Q}_{a,k}=q^{ak}\mathcal{Q}_{a,k}\Delta$.

$$\mathcal{Q}_k\mathcal{Q}_{k+1}=q\mathcal{Q}_{k+1}\mathcal{Q}_k,\quad q\mathcal{Q}_{k+1}\mathcal{Q}_{k-1}=\mathcal{Q}_k^2-\mathcal{Q}_{2,k}.$$

- The hamiltonian

$$C_1=(\mathcal{Q}_{k-1}+\mathcal{Q}_{k+1})\mathcal{Q}_k^{-1}=\mathcal{Q}_k\mathcal{Q}_{k+1}^{-1}+\mathcal{Q}_{k+1}\mathcal{Q}_k^{-1}-\Delta\mathcal{Q}_{k+1}^{-1}\mathcal{Q}_k^{-1}$$

  acts with eigenvalue $e_1(x_1,x_2)$ on the eigenfunction $Z_{\mathbf{n}}(\mathbf{x};q)$. If

$$\mathcal{H}=V((k-1)\omega_1)^{\otimes m}\otimes V(k\omega_1)^{\otimes n},\text{ and }\ Z_{m,n}(\mathbf{x};q^{-1})=q^{-\frac{1}{2}p(m,n)}\mathcal{Q}_k^n\mathcal{Q}_{k-1}^m\cdot 1,$$

$$C_1Z_{m,n}=Z_{m-1,n+1}+Z_{m+1,n-1}-q^{k(1-n-m)-m+1}Z_{m-1,n-1}=e_1(\mathbf{x})Z_{m,n}.$$

- When $k=1$, $Z_{m,n}=Z_n(\mathbf{x};q)$ satisfies

$$e_1Z_n=Z_{n+1}+(1-q^{-n})Z_{n-1},\quad \text{“}q\text{-difference quantum Toda.”}$$

t-deformation: DIM, sDAHA, elliptic Hall algebra...

The operators $D_{a,n}$ are the $t \to \infty$ limit of "generalized Macdonald operators:" [c.f. Miki in the context of q,t W-algebras]

$$\mathcal{D}_{a,k} = \prod_{\substack{I \subset [1,N] \\ |I|=a}} \prod_{i \in I} x_i^k \prod_{j \notin I} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} \Gamma_i.$$

- The Macdonald operators are $\{\mathcal{D}_{a,0}\}_a$ and they form a commuting family.
- For fixed k, the difference operators $\mathcal{D}_{1,k}, \ldots, \mathcal{D}_{N,k}$ also form a commuting family.
- The operators $\mathcal{D}_{a,k}$ are "raising operators" (in the elliptic Hall algebra).

The algebra of generalized Macdonald operators

The algebra $\mathfrak{R}_N$ whose generators are represented by the generalized Macdonald operators is a quotient of the DIM algebra at trivial central charge.

Theorem: [Di Francesco, K 2017] Let $x^+(z) = \sum_{k\in\mathbb{Z}} (q^{\frac{1}{2}}z)^k \mathcal{D}_{1,k}$. Then

$$g(z,w)x^+(z)x^+(w) + g(w,z)x^+(w)x^+(z) = 0,$$

where

$$g(z,w) = (z-qw)(z-t^{-1}w)(z-q^{-1}tw).$$

- The currents $x^+(z)$ also satisfy a cubic relation (Serre-type relation).
- If we stop here, this is a subalgebra of the quantum toroidal algebra or DIM when $N\to\infty$.
- For N finite, there is a set of relations $\mathcal{D}_{N+1,k}=0$ for all k, c.f. spherical DAHA at finite N.
- Add generators $x^-(z) = \mathcal{D}(z)|_{(q,t)\mapsto(1/q,1/t)}$ to get full algebra at trivial central charge.
- The algebra $\mathcal{R}_N$ is recovered in the limit $t\to\infty$ of $\mathfrak{R}_N$.

The DIM algebra

The quantum toroidal algebra of $\mathfrak{gl}_1$ [see Miki; Awata, Feigin, Shiraishi,...] is the algebra generated by currents $x^{\pm}(z), \psi^{\pm}(z)$ with non-trivial relations (Drinfeld-type)

$$\begin{aligned}
G(w/\gamma z)\psi^+(z)\psi^-(w) &= G(\gamma w/z)\psi^-(w)\psi^+(z) \\
\psi^\epsilon(z)x^{\pm}(w) &= G(\gamma^{\mp\epsilon}w/z)^{\mp 1}x^{\pm}(w)\psi^\epsilon(z) \\
g(z/w)^{\pm 1}x^{\pm}(z)x^{\pm}(w) &= g(w/z)^{\pm 1}x^{\pm}(w)x^{\pm}(z) \\
\frac{1-q/t}{(1-q)(1-t^{-1})}[x^+(z),x^-(w)] &= \big(\delta(\gamma w/z)\psi^+(z/\sqrt{\gamma}) - \delta(\gamma z/w)\psi^-(\sqrt{\gamma}z)\big) \\
\psi_0^{\pm} &= \delta^{\pm 1}
\end{aligned}$$

and Serre-type relations

$$Sym_{z_1,z_2,z_3}\left(z_2/z_3[x^{\pm}(z_1),[x^{\pm}(z_2),x^{\pm}(z_3)]]\right) = 0.$$

The generators γ, δ are central elements and

$$G(x) = \frac{g(1,x)}{g(x,1)}, \quad g(z,w) = (z-qw)(z-w/q)(z-tw/q).$$

From commuting hamiltonians to bosonization

We can compute explicitly from the difference operators:

$$[P_k, \mathcal{D}_{1,n}] = (1 - q^k)\mathcal{D}_{1,n+k}.$$

which leads to the equation for

$$x^+(z)p_k[X] = \left(p_k[X] + \frac{q^{k/2} - q^{-k/2}}{z^k}\right) x^+(z).$$

- If $F[X]$ is a symmetric function, this action of $x^+(z)$ is written in plethystic notation as
$$x^+(z)F[X] = F[X + \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{z^k}]x^+(z).$$
- In the limit $N \to \infty$, an infinite number of variables, P_k are algebraically independent, and
$$x^+(z) \propto \exp(\sum_{k \neq 0} \frac{q^{k/2} - q^{-k/2}}{z^k} \frac{d}{dP_k})$$

  when acting on the space of symmetric functions.

Limit $N \to \infty$: Non-trivial central charge

If we let $x^+(z)$ act on the space of symmetric polynomials, and consider its action on 1, we can write

$$x^+(z) = \frac{q^{\frac{1}{2}}}{(1-q)(1-1/t)} \exp\left(\sum_{k>0} a_k \frac{z^k}{k}\right) \exp\left(\sum_{k>0} a_k^* z^{-k}\right)$$

with $a_k = q^{k/2}(1-t^{-k})P_k[X]$ and $a_k^* = (q^{k/2} - q^{-k/2})\frac{d}{dP_k}$.

- See [Feigin, Jimbo, Miwa+,09].
- The currents $x^-(z)$ are obtained from $x^+(z)$ with $(q,t) \mapsto (q^{-1}, t^{-1})$.
- These currents generate the DIM algebra with **non-trivial central charge** $\gamma = \sqrt{t/q}$ ("horizontal representation") which is a fock space.

Summary

- The linearized-spectrum partition functions of the generalized Heisenberg spin chains can be generated by the action of q-difference operators acting on 1.
- These q-difference operators satisfy the quantum cluster algebra called the quantum Q-system.
- This algebra is isomorphic to a rank-dependent quotient of a quantum affine algebra.
- The t-deformation of these operators satisfies relations in the quantum toroidal algebra of $\mathfrak{gl}_1$.
- In the infinite-rank limit, they are the level $(1, 0)$ representation of this algebra.

Thank you!