Emergent hydrodynamics in integrable systems out of equilibrium

Benjamin Doyon

Department of Mathematics, King's College London

Cargèse, Corsica, 25-26 July 2017

Collaborators on the works discussed in these lectures:

Alvise Bastianello, SISSA

Olalla Castro Alvaredo, City, University of London

Jean-Sébastien Caux, University of Amsterdam

Jérôme Dubail, Université de Lorraine

Robert Konik, Brookhaven National Laboratory

Herbert Spohn, Technische Universität München

Gerard Watts, KCL

Takato Yoshimura, KCL

and on related works:

Denis Bernard, École Normale Supérieure de Paris

Joe Bhaseen, KCL

Andy Lucas, Stanford University

Koenraad Schalm, Leiden University

Main papers

O. A. Castro-Alvaredo, B. Doyon, T. Yoshimura, Emergent hydrodynamics in integrable quantum systems out of equilibrium, Phys. Rev. X 6, 041065 (2016)

B. Bertini, M. Collura, J. De Nardis, M. Fagotti, Transport in Out-of-Equilibrium XXZ Chains: Exact Profiles of Charges and Currents, Phys. Rev. Lett. 117, 207201 (2016)

Both selected for a Viewpoint in Physics written by J. Dubail (`http://physics.aps.org/articles/v9/153#c1`)

Plan

Lecture I: the theory of generalized hydrodynamics (GHD).

Applications of hydrodynamic ideas, generalized thermalization, derivation of the main GHD equations. Pictures and videos.

Lecture II: solutions and applications to transport and correlations.

Some geometry at the core of GHD, an exact solution to the initial value problem in the form of integral equations, relation to gases of classical solitons, relation with conventional hydrodynamics, how GHD leads to exact results in transport problems including Drude weights and non-equilibrium currents, connection with the theory of hydrodynamic projections, and exact results for large-scale space-time correlations.

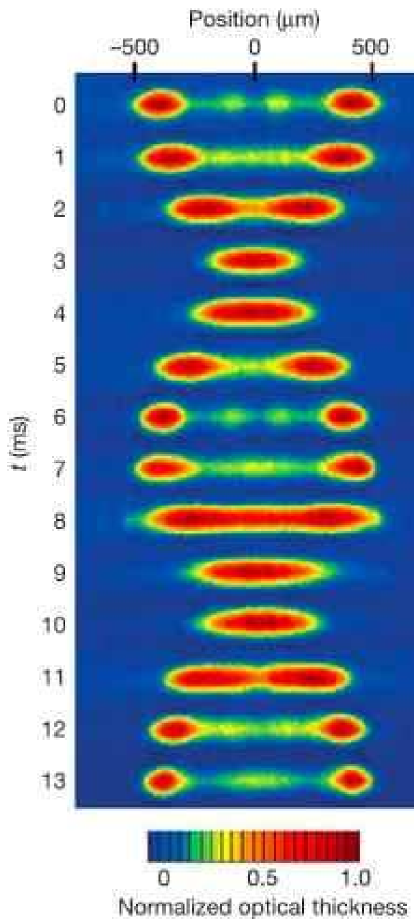
Lecture I: the theory of generalized hydrodynamics.

1. The problem: inhomogeneous dynamics of many-body integrable systems

1. The problem: inhomogeneous dynamics of many-body integrable systems

Effects of integrability in the famous “Quantum Newton Cradle” experiment:

[Kinoshita, Wenger, Weiss 2006]



“Our results are probably explainable by the well-known fact that a homogeneous 1D Bose gas with point-like collisional interactions is integrable”

“Until now, however, the time evolution of out-of-equilibrium 1D Bose gases has been a theoretically unsettled issue, as practical factors such as harmonic trapping and imperfectly point-like interactions may compromise integrability”

1. The problem: inhomogeneous dynamics of many-body integrable systems

As an example consider the Lieb-Liniger model, which describes point-like interactions of Galilean invariant Bose gases. Its Hamiltonian is

$$H = \int \mathrm{d}x\, \mathfrak{h}(x) = \int \mathrm{d}x \left(\frac{1}{2m} \partial_x \Psi^\dagger \partial_x \Psi + \frac{c}{2} \Psi^\dagger \Psi^\dagger \Psi \Psi \right).$$

It admits local conserved quantities Q_i:

$$Q_i = \int \mathrm{d}x\, \mathfrak{q}_i(x).$$

For instance, the number of particle N, the momentum P and the energy H:

- the particle density is $\mathfrak{q}_0(x) = \mathfrak{n}(x) = \Psi^\dagger(x)\Psi(x)$;
- the momentum density is $\mathfrak{q}_1(x) = \mathfrak{p}(x) = i\Psi^\dagger(x)\partial_x\Psi(x) + h.c.$;
- the energy density is $\mathfrak{q}_2(x) = \mathfrak{h}(x)$.

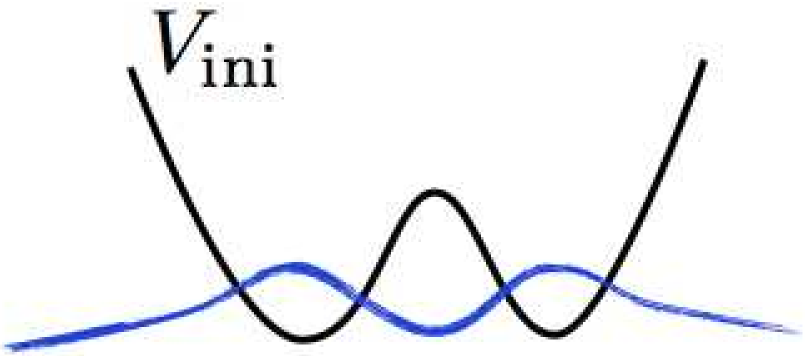
1. The problem: inhomogeneous dynamics of many-body integrable systems

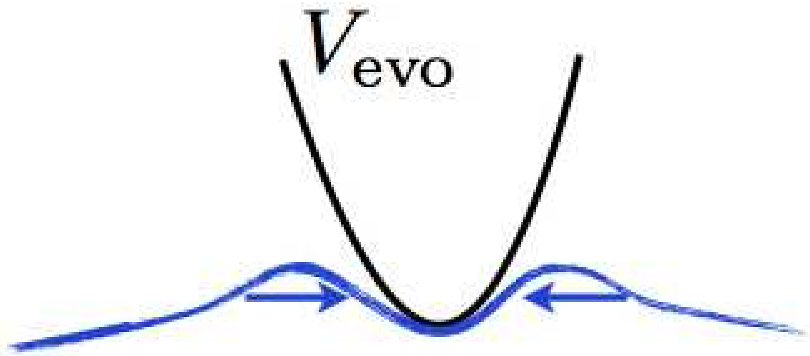
We can describe theoretically (a simplification of) the problem as follows. The initial state is an equilibrium state with a inhomogeneous potential

$$\langle A\rangle = \frac{\mathrm{Tr}\,(\rho_{\mathrm{ini}}A)}{\mathrm{Tr}\rho_{\mathrm{ini}}},\quad \rho_{\mathrm{ini}} = \exp\left[-\beta\left(H + \int \mathrm{d}x\, V_{\mathrm{ini}}(x)\mathfrak{n}(x)\right)\right].$$

Then the evolution occurs with the Hamiltonian in a different inhomogeneous potential

$$\langle A(t)\rangle = \langle e^{\mathrm{i}H_{\mathrm{evo}}t}Ae^{-\mathrm{i}H_{\mathrm{evo}}t}\rangle,\quad H_{\mathrm{evo}} = H + \int \mathrm{d}x\, V_{\mathrm{evo}}(x)\mathfrak{n}(x).$$





1. The problem: inhomogeneous dynamics of many-body integrable systems

How can we compute the evolution of such a gas?

Can we reproduce the effects seen in the quantum Newton cradle experiment?

What general theory would allow us to do so in a simple enough fashion, without using advanced computational techniques for one-dimensional quantum systems?

What are the general principles?

2. From Gibbs ensembles to conventional hydrodynamics

For a nice description of standard concepts in hydrodynamics, see e.g. A. Bressan, Hyperbolic conservation laws: An illustrated tutorial. In Modelling and Optimisation of Flows on Networks, Cetraro, Italy 2009, Lecture Notes in Mathematics 2062, Springer, 2013

2. From Gibbs ensembles to conventional hydrodynamics

Hydrodynamics is the natural framework to describe inhomogeneous phenomena in many-body systems, for instance waves in water. The main idea behind hydrodynamics is what is usually referred to as **“local thermodyanamic equilibrium”**.

It says that **locally** and on **very short time scales** (in fluid cells), the many-body system **“equilibrates” or relaxes**. This means (naively) that locally we observe **Gibbs states**. Since things can be moving, then in general these will be **boosted** by the local fluid velocity. Thus, at every point x, t, the density matrix is

$$\rho_{\rm GE}(x,t) = e^{-\beta(x,t)(H-\mu(x,t)N-\nu(x,t)P)}$$

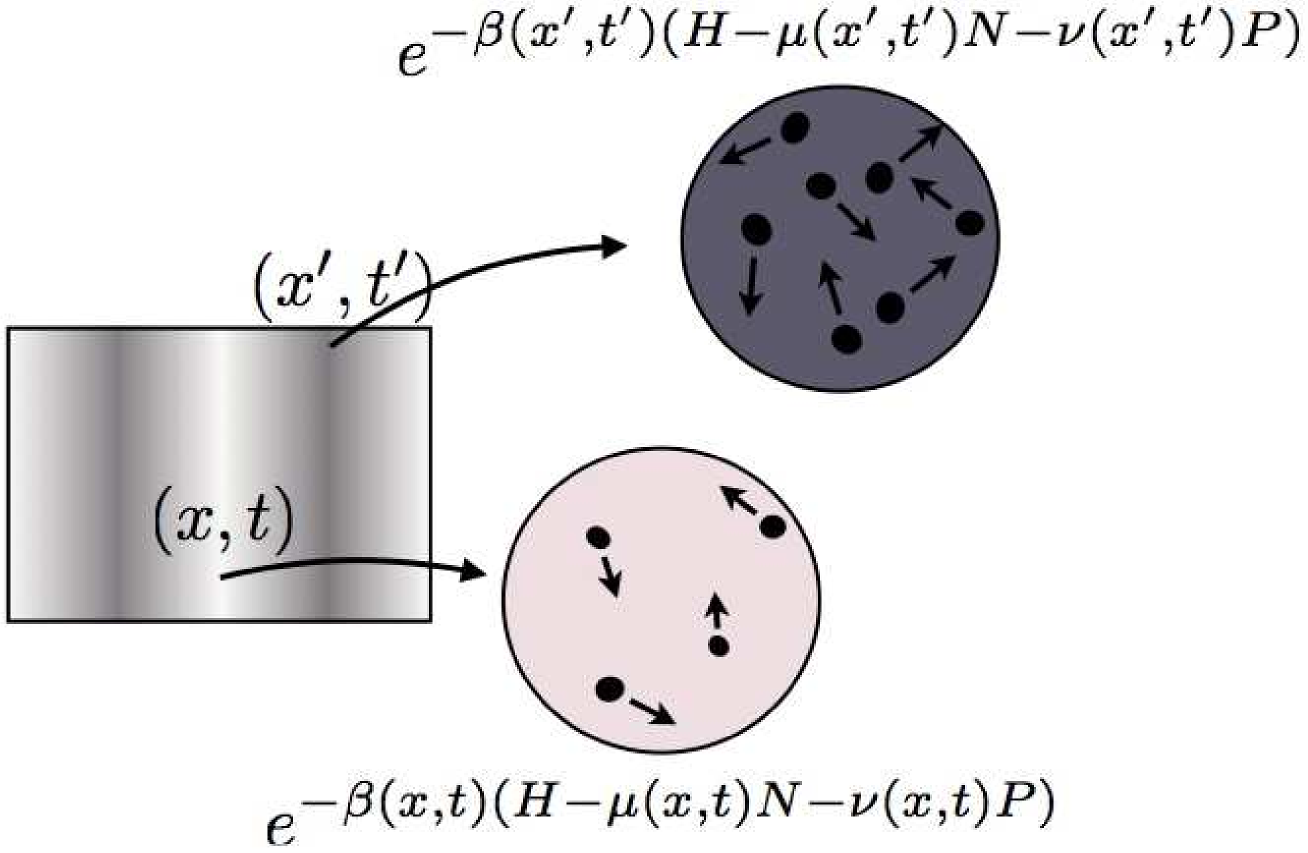
and the hydrodynamic approximation is

$$\langle \mathcal{O}(x,t)\rangle_{\rm ini} \approx \frac{{\rm Tr}\,(\rho_{\rm GE}(x,t)\mathcal{O}(0,0))}{{\rm Tr}\rho_{\rm GE}(x,t)}.$$

For instance:

$$\rho_{\rm ini} = e^{-\beta(H+\int {\rm d}x\, V_{\rm ini}(x)\mathfrak{n}(x))} \ \Rightarrow\ \rho_{\rm GE}(x,0) = e^{-\beta(H+V_{\rm ini}(x)N)}.$$

2. From Gibbs ensembles to conventional hydrodynamics



2. From Gibbs ensembles to conventional hydrodynamics

Let us show that hydrodynamics follows from local thermodynamic equilibrium combined with current conservation. Here for simplicity we suppose there are no force terms: homogeneous evolution Hamiltonian $H_{\rm evo}$.

There is an energy current $\mathfrak{j}_h$ such that

$$\partial_t \mathfrak{h}(x,t) + \partial_x \mathfrak{j}_{\mathfrak{h}}(x,t) = 0.$$

Denote ${\rm Tr}(\rho_{\rm GE}(x,t)\mathfrak{h})/{\rm Tr}\rho_{\rm GE}(x,t) = \mathtt{h}(x,t)$ and similarly for $\mathtt{j}_{\mathtt{h}}(x,t)$. By Stokes theorem and local thermodynamic equilibrium, on macroscopic paths,

$$0 = \oint {\rm d}\vec{x} \wedge \begin{pmatrix} \langle \mathfrak{h}(x,t)\rangle_{\rm ini} \\ \langle \mathfrak{j}_{\mathfrak{h}}(x,t))\rangle_{\rm ini} \end{pmatrix} = \oint {\rm d}\vec{x} \wedge \begin{pmatrix} \mathtt{h}(x,t) \\ \mathtt{j}_{\mathtt{h}}(x,t) \end{pmatrix}.$$

Therefore, $\mathtt{h}(x,t),\ \mathtt{j}_{\mathtt{h}}(x,t)$ satisfy a conservation equation at the macroscopic scale:

$$\partial_t \mathtt{h}(x,t) + \partial_x \mathtt{j}_{\mathtt{h}}(x,t) = 0.$$

2. From Gibbs ensembles to conventional hydrodynamics

The same holds with the momentum current $\mathtt{j}_{\mathfrak{p}}$ and the particle current $\mathtt{j}_{\mathfrak{n}} = \mathfrak{p}$ (equal to the momentum density in Galilean systems), giving the macroscopic conservation laws:

$$\partial_t \mathtt{h}(x,t) + \partial_x \mathtt{j}_{\mathtt{h}}(x,t) = 0$$
$$\partial_t \mathtt{p}(x,t) + \partial_x \mathtt{j}_{\mathtt{p}}(x,t) = 0$$
$$\partial_t \mathtt{n}(x,t) + \partial_x \mathtt{p}(x,t) = 0.$$

But also, since there are only three parameters μ, ν, β to determine a boosted Gibbs state, there must be two relations:

$$\mathtt{j}_{\mathtt{h}} = F(\mathtt{h}, \mathtt{p}, \mathtt{n}), \quad \mathtt{j}_{\mathtt{p}} = G(\mathtt{h}, \mathtt{p}, \mathtt{n}).$$

These are the **equations of state of the gas** (which are highly model-dependent), and combined with the above give the **hydrodynamic equations** for $\mathtt{h},\ \mathtt{p},\ \mathtt{n}$.

Note that $\mathtt{h},\ \mathtt{p},\ \mathtt{n}$ fix the potentials $\beta,\ \mu,\ \nu$. Thus hydrodynamics fixes the local space-time dependent state.

2. From Gibbs ensembles to conventional hydrodynamics

Remarks:

- This is valid at the **Euler scale**: all variations in space and time must be very smooth. Beyond this scale, there are higher derivative corrections, such as viscosity terms. But at large scales, such higher derivative terms are scaled out.

- These equations can be re-written in standard hydrodynamic form. Defining a velocity v via $\mathtt{p} = \mathtt{n}\, v$, the $\mathtt{n}$ and $\mathtt{p}$ conservation laws imply

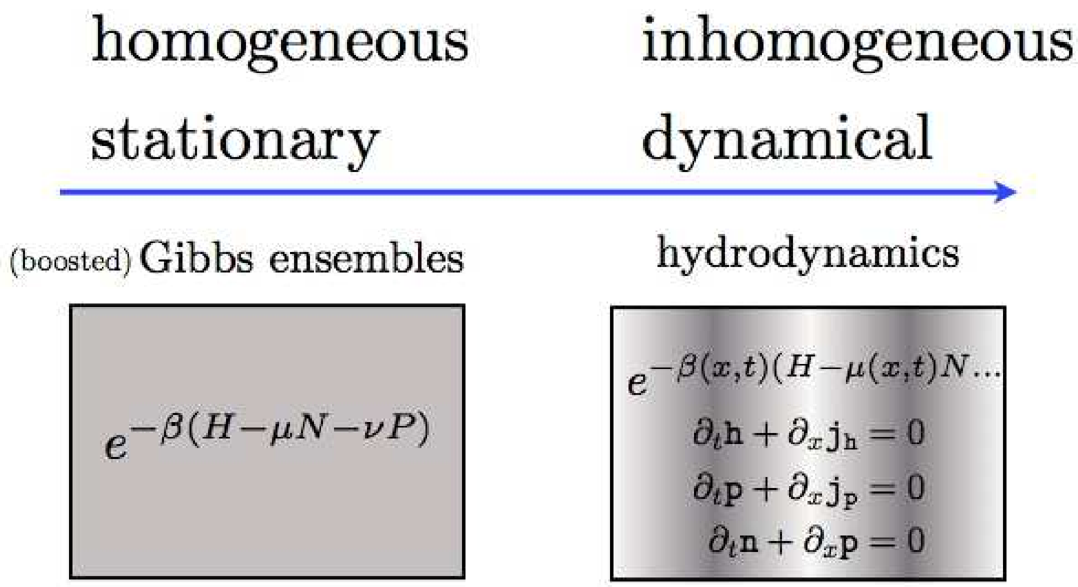
$$\partial_t v + v\partial_x v = -\frac{1}{\mathtt{n}}\partial_x \mathcal{P}$$

  where the pressure is $\mathcal{P} = \mathtt{j}_{\mathtt{p}} - \mathtt{n}v^2$. This is the usual Euler equation. Combined with $\partial_t \mathtt{n} + \partial_x(v\mathtt{n}) = 0$ these are the usual hydrodynamic equations (without viscosity).

- The pressure $\mathcal{P}$ in a boosted Gibbs state can be evaluated in the Lieb-Liniger model using Bethe ansatz. This Lieb-Liniger conventional hydrodynamics has been used, with partial success.

2. From Gibbs ensembles to conventional hydrodynamics

Thus there is an unambiguous procedure to go from homogeneous, stationary states to inhomogeneous, dynamical states, describing the large (Euler) scale space-time variations.



3. From Gibbs ensembles to Generalized Gibbs ensembles (GGEs)

Main paper: M. Rigol, V. Dunjko, V. Yurovsky, M. Olshanii, Phys. Rev. Lett. 97, 050405 (2007).

Reviews:

- A. Polkovnikov, K. Sengupta, A. Silva, M. Vengalattore, Rev. Mod. Phys. 83, 863 (2011)
- J. Eisert, M. Friesdorf, C. Gogolin, Nat. Phys. 11, 124 (2015);
- C. Gogolin, J. Eisert, Rep. Prog. Phys. 79, 056001 (2016);
- F. Essler, M. Fagotti, J. Stat. Mech. 2016, 064002 (2016);
- L. Vidmar, M. Rigol, J. Stat. Mech. 2016, 064007 (2016);
- E. Ilievski, M. Medenjak, T. Prosen, L. Zadnik, J. Stat. Mech. 2016, 064008 (2016).

3. From Gibbs ensembles to Generalized Gibbs ensembles (GGEs)

But **the Lieb-Liniger model is integrable**. Integrable models possess an **infinite number of local conserved quantities**

$$Q_i = \int \mathrm{d}x\, \mathfrak{q}_i(x), \qquad \partial_t \mathfrak{q}_i + \partial_x \mathfrak{j}_i = 0, \qquad i = 0, 1, 2, 3, \ldots \text{ unboundedly.}$$

Because of the presence of all these conserved quantities, integrable models **do not generically relax to Gibbs ensembles**.

3. From Gibbs ensembles to Generalized Gibbs ensembles (GGEs)

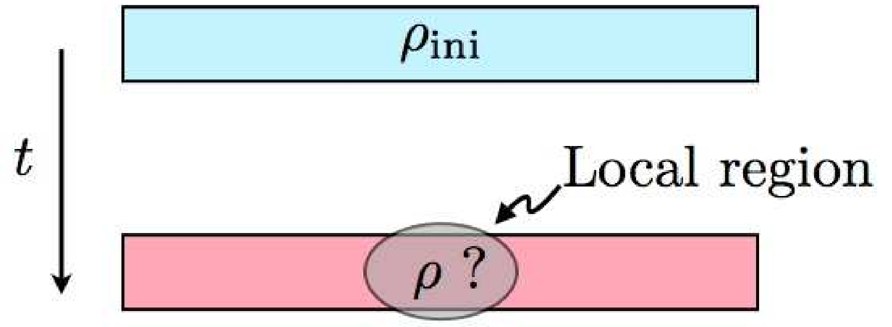
Let us explain what replaces Gibbs ensembles using the following "quench protocol".

A system is in some homogeneous initial state $\rho_{\rm ini}$. Then this state is evolved with a homogeneous Hamiltonian $H_{\rm evo}$. Consider a local observable $\mathcal{O}(x)$ in the evolved state:

$$\langle \mathcal{O}(x,t)\rangle_{\rm ini} = \langle e^{{\rm i}H_{\rm evo}t}\mathcal{O}(x)e^{-{\rm i}H_{\rm evo}t}\rangle_{\rm ini}.$$

What is the limit $\lim_{t\to\infty}\langle \mathcal{O}(x,t)\rangle_{\rm ini}$?

Although the state of a closed quantum system itself cannot relax as a whole, **it does from the viewpoint of local observables** in infinite volume.



3. From Gibbs ensembles to Generalized Gibbs ensembles (GGEs)

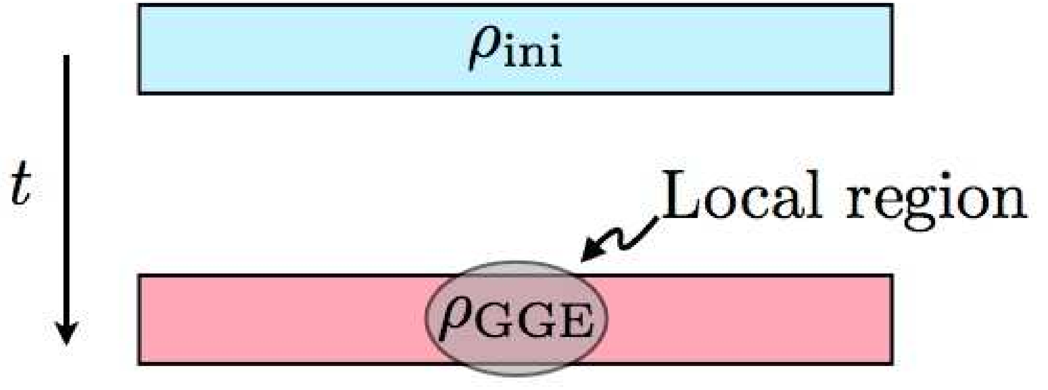
Every conserved density is preserved. Thus the stationary state depends on all $\langle \mathfrak{q}_i(0,0)\rangle_{\rm ini}$.

$$\partial_t \langle \mathfrak{q}_i(x,t)\rangle = -\partial_x \langle \mathfrak{j}_i(x,t)\rangle = 0 \qquad \text{(by homogeneity).}$$

Ergodicity suggests stationary states should **maximize entropy**. Here **under the condition of conservation of all local charges**. The maximal entropy state thus looks like

$$\rho_{\rm GGE} = \exp\left[-\sum_{i=0}^{\infty} \beta_i Q_i\right].$$

The resulting stationary state is called a generalized Gibbs ensemble (GGE).



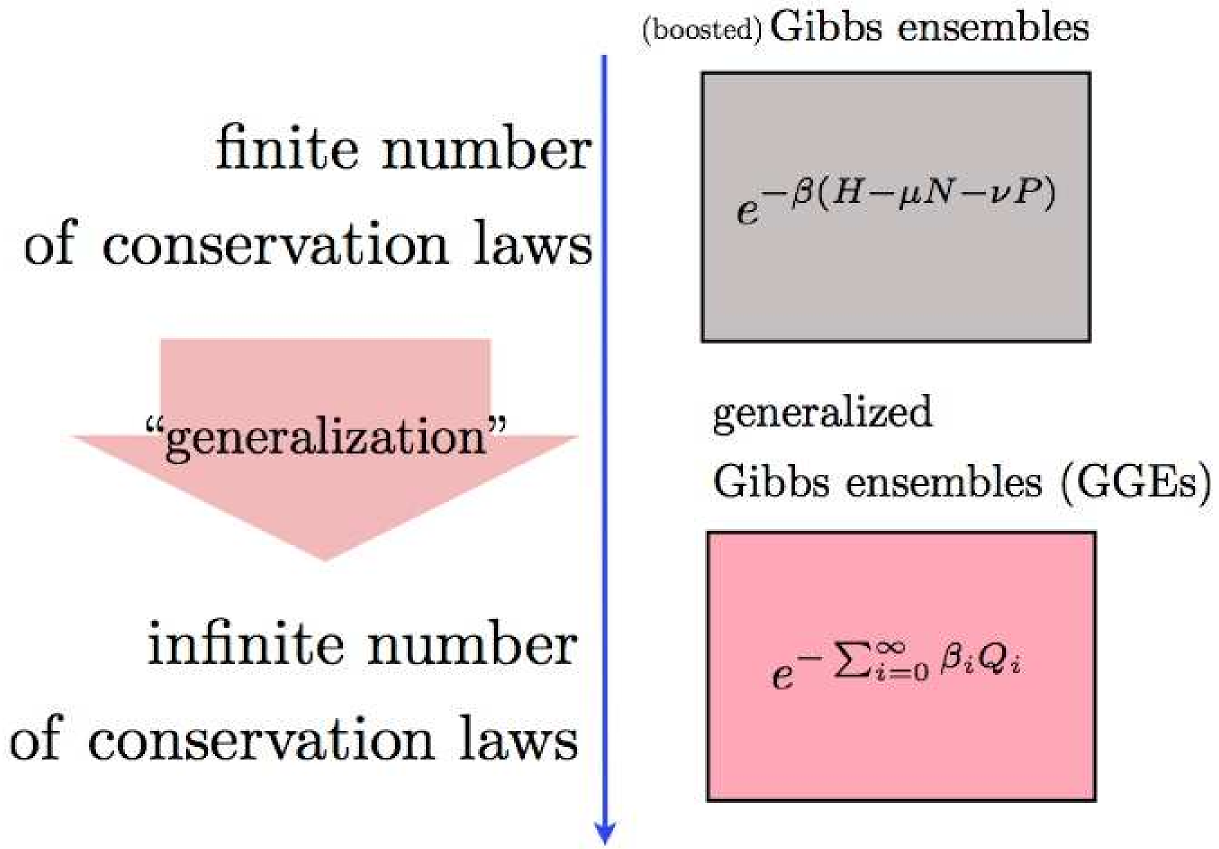
3. From Gibbs ensembles to Generalized Gibbs ensembles (GGEs)

Remarks

- The set of conserved charges $\{Q_i\}$ and the infinite series $\sum_i \beta_i Q_i$ must be defined carefully. The correct definition is that of **pseudolocal charges**, which form a Hilbert space in which $\sum_i \beta_i Q_i$ is interpreted as a basis decomposition [BD 2017].
- According to standard results of the operator algebra approach to quantum statistical mechanics, any extremal H_{evol}-stationary state is a Gibbs state (or Kubo-Martin-Schwinger (KMS) state) with respect to a conserved "Hamiltonian" H_{sta} (not necessarily local, but generating a one-parameter group of unitaries).
- There is the related **eigenstate thermalization hypothesis**: in the large-volume limit, for generic quantum lattices, $\langle E|\mathcal{O}|E\rangle = \mathrm{Tr}\left(e^{-\beta H}\mathcal{O}\right)/\mathrm{Tr}\, e^{-\beta H}$. This is generalized to integrable systems with the replacement $e^{-\beta H} \mapsto e^{-\sum_i \beta_i Q_i}$.
- The space of all GGEs with respect to a given integrable H is **infinite-dimensional**. It is probably an infinite-dimensional Riemannian manifold.

3. From Gibbs ensembles to Generalized Gibbs ensembles (GGEs)

Thus there is an unambiguous procedure to construct homogeneous, stationary states of integrable models: a “generalization” to infinitely-many conservation laws.



Combining: inhomogeneous dynamics of integrable systems:

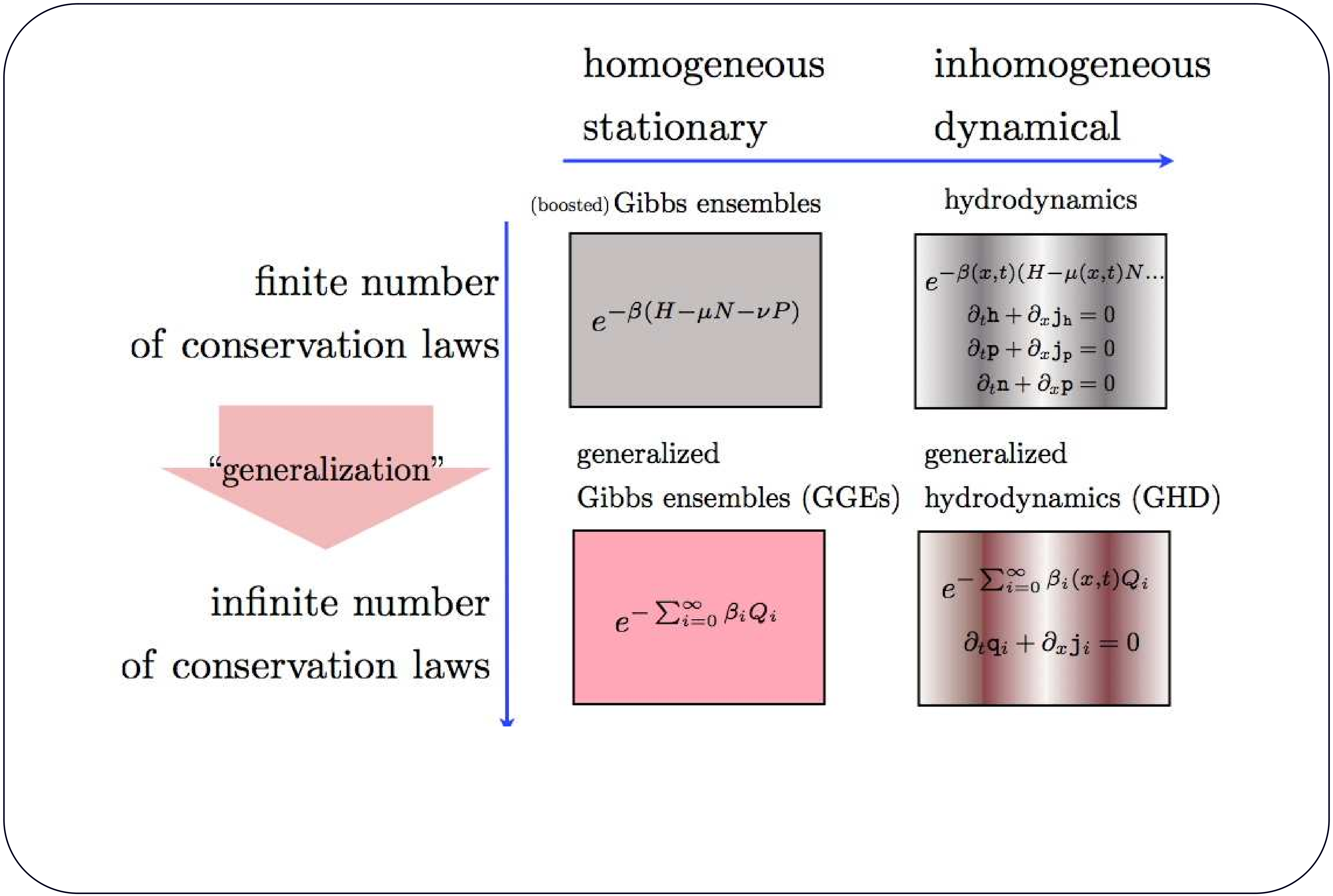
$$\partial_t \mathtt{q}_i + \partial_x \mathtt{j}_\mathtt{i} = 0, \quad \mathtt{q}_i, \mathtt{j}_i = \frac{\mathrm{Tr}\left[\rho_{\mathrm{GGE}}(x,t)\, \mathtt{q}_i, \mathtt{j}_i\right]}{\mathrm{Tr}\, \rho_{\mathrm{GGE}}(x,t)}, \quad i = 0, 1, 2, \ldots$$

This is **generalized hydrodynamics (GHD)**.

- The hydrodynamic principle is the emergence of local entropy maximization with respect to all available conserved charges, valid when variation lengths are large enough.
- Local averages are fixed by space-time dependent (generalized) Gibbs ensembles,

$$\langle \mathcal{O}(x,t) \rangle_{\mathrm{ini}} \approx \frac{\mathrm{Tr}\left[\rho_{\mathrm{GGE}}(x,t)\, \mathcal{O}\right]}{\mathrm{Tr}\, \rho_{\mathrm{GGE}}(x,t)}$$

- There are equations of states: $\mathtt{j}_i = F_i(\{\mathtt{q}_j\})$, and a bijective relation $\mathtt{q}_i \leftrightarrow \beta_i$.
- Equations of conservations give dynamical equations determining the space-time dependent (generalized) Gibbs ensembles.


homogeneous
stationary
inhomogeneous
dynamical
finite number
of conservation laws
"generalization"
infinite number
of conservation laws
(boosted) Gibbs ensembles
$e^{-\beta(H-\mu N-\nu P)}$
hydrodynamics
$e^{-\beta(x,t)(H-\mu(x,t)N\ldots}$
$\partial_t \mathtt{h} + \partial_x \mathtt{j_h} = 0$
$\partial_t \mathtt{p} + \partial_x \mathtt{j_p} = 0$
$\partial_t \mathtt{n} + \partial_x \mathtt{p} = 0$
generalized
Gibbs ensembles (GGEs)
$e^{-\sum_{i=0}^{\infty} \beta_i Q_i}$
generalized
hydrodynamics (GHD)
$e^{-\sum_{i=0}^{\infty} \beta_i(x,t) Q_i}$
$\partial_t \mathtt{q}_i + \partial_x \mathtt{j}_i = 0$

4. GGEs via quasi-particles

(generalized thermodynamic Bethe ansatz)

A. Zamolodchikov, Nucl. Phys. B 342, 695 (1990);

J. Mossel, J.-S. Caux, J. Phys. A 45, 255001 (2012).

For an understanding the relation GGE = TBA in the XXZ chain, see

E. Ilievski, E. Quinn, J.-S. Caux, Phys. Rev. B 95, 115128 (2017).

4. GGEs via quasi-particles

Gibbs ensembles and generalized Gibbs ensembles can be described, in Bethe ansatz integrable models, by using the **(generalized) thermodynamic Bethe ansatz**.

This is based on the fact that there emerge **quasi-particles**. The set of their momenta and other quantum numbers is preserved under scattering, thus giving good quantum numbers used to describe GGEs.

These quantum numbers are gathered into a **spectral parameter** θ characterizing the quasi-particle, and we imagine states to be of the form

$$|\theta_1, \theta_2, \ldots\rangle.$$

A model is fully defined by giving the space of spectral parameters, the momentum $p(\theta)$ and energy $E(\theta)$ functions, and the **differential scattering phase** $\varphi(\theta, \theta')$.

4. GGEs via quasi-particles

For instance, in many models with a single-particle spectrum we may take $\theta \in \mathbb{R}$ and

- with Galilean invariance $p(\theta) = m\theta,\ E(\theta) = m\theta^2/2$
- with relativistic invariance $p(\theta) = m\sinh(\theta),\ E(\theta) = m\cosh\theta$

with in both case $\varphi(\theta - \theta') = -\mathrm{i}\mathrm{d}\log S(\theta-\theta')/\mathrm{d}\theta$ where $S(\theta-\theta')$ is the two-particle scattering amplitude.

4. GGEs via quasi-particles

Each quasi-particle θ carry a quantity $h_i(\theta)$ of the conserved charge Q_i. That is, conserved charges act as

$$Q_i|\theta_1,\theta_2,\ldots\rangle = \left(\sum_k h_i(\theta_k)\right)|\theta_1,\theta_2,\ldots\rangle.$$

A GGE can be seen as a single state with infinitely-many quasi-particles of a given density. It is fully characterized by the number $L\rho_{\rm p}(\theta){\rm d}\theta$ of quasiparticles in the element $[\theta,\theta+{\rm d}\theta]$ (where L is the infinite volume).

In a GGE (where integral symbol includes sum over quasi-particle species),

$$\mathsf{q}_i = \int {\rm d}\theta\, h_i(\theta)\rho_{\rm p}(\theta).$$

The set of functions $h_i(\theta)$ is assumed to be "complete" in some sense.

Thus **the set $\{\mathsf{q}_i\}$ and the function $\rho_{\rm p}(\theta)$ are both complete characterization of a GGE**.

4. GGEs via quasi-particles

For instance:

- In the **repulsive Lieb-Liniger model** ($c > 0$), there is a single particle specie. We take $\theta \in \mathbb{R}$. We have

$$h_0(\theta) = m, \quad h_1(\theta) = p(\theta) = m\theta, \quad h_2(\theta) = E(\theta) = m\theta^2/2$$

and

$$\varphi(\theta, \alpha) = \frac{2c}{(\theta - \alpha)^2 + c^2}.$$

- In the **attractive Lieb-Liniger model** ($c < 0$), there are infinitely-many particle species (Bethe-ansatz "strings"). We take $\theta \in \mathbb{R} \times \mathbb{N}$ with $\theta = (v, j)$. We have (here with $m = 1/2$ for simplicity)

$$h_0(\theta) = \frac{j}{2}, \quad p(\theta) = \frac{jv}{2}, \quad E(\theta) = \frac{jv^2}{2} - \frac{c^2}{48} j(j^2 - 1)$$

and $\varphi(\theta, \alpha)$ takes a known, complicated form.

4. GGEs via quasi-particles

The relation to the Lagrange parameters β_i is obtained as follows. Here we use **fermionic statistics**, for instance as used for the quasi-particles of the Lieb-Liniger model.

Average of local conserved densities are evaluated using a **free energy**:

$$\mathsf{q}_i = -\frac{\partial}{\partial \beta_i} F$$

where

$$F = \int \mathrm{d}p(\theta) \log(1 + e^{-\epsilon(\theta)})$$

with **pseudo-energy**

$$\epsilon(\theta) = \sum_i \beta_i h_i(\theta) - \int \frac{\mathrm{d}\alpha}{2\pi} \varphi(\theta, \alpha) \log(1 + e^{-\epsilon(\alpha)}).$$

5. GGE equations of states

- O. A. Castro-Alvaredo, BD, T. Yoshimura, Phys. Rev. X 6, 041065 (2016);
- B. Bertini, M. Collura, J. De Nardis, M. Fagotti, Phys. Rev. Lett. 117, 207201 (2016);

5. GGE equations of states

For GHD, we need the **GGE equations of state**: the expressions of the currents.

Since $\rho_{\rm p}$ fully determines the GGE, one can always write, for some $v^{\rm eff}(\theta) = v^{\rm eff}_{[\rho_{\rm p}]}(\theta)$,

$$\mathtt{j}_i = \int {\rm d}\theta\, h_i(\theta) v^{\rm eff}(\theta)\rho_{\rm p}(\theta).$$

Using a variety of arguments including form factor expansions of GGE averages in relativistic QFT, and with numerical verifications in the XXZ chain, one finds

$$v^{\rm eff}(\theta) = \frac{E'(\theta)}{p'(\theta)} + \int {\rm d}\alpha\, \frac{\varphi(\theta,\alpha)\,\rho_{\rm p}(\alpha)}{p'(\theta)}\,(v^{\rm eff}(\alpha) - v^{\rm eff}(\theta))$$

This can be seen as the GGE equations of state.

5. GGE equations of states

Define the **occupation function**:

$$n(\theta) = \frac{\rho_{\rm p}(\theta)}{\rho_{\rm s}(\theta)}, \quad 2\pi\rho_{\rm s}(\theta) = p'(\theta) + \int {\rm d}\alpha\, \varphi(\theta,\alpha)\rho_{\rm p}(\alpha).$$

Here $\rho_{\rm s}$ as the interpretation as a **density of states**: the "availabilities" for quasi-particles.

Define the all-important **"dressing" operation**:

$$h^{\rm dr}(\theta) = h(\theta) + \int \frac{{\rm d}\alpha}{2\pi} \varphi(\theta,\alpha) n(\alpha) h^{\rm dr}(\alpha).$$

Then $2\pi\rho_{\rm s} = (p')^{\rm dr}(\theta)$ where $p'(\theta) = {\rm d}p(\theta)/{\rm d}\theta$.

The mapping $n(\theta) \leftrightarrow \rho_{\rm p}(\theta)$ is a change of coordinate in the space of GGEs.

5. GGE equations of states

Then some not-too-hard calculations give, in these new coordinates,

$$\mathsf{q}_i = \int \mathrm{d}p(\theta)\, n(\theta) h_i^{\mathrm{dr}}(\theta), \quad \mathsf{j}_i = \int \mathrm{d}E(\theta)\, n(\theta) h_i^{\mathrm{dr}}(\theta), \quad v^{\mathrm{eff}}(\theta) = \frac{(E')^{\mathrm{dr}}(\theta)}{(p')^{\mathrm{dr}}(\theta)}.$$

Densities and currents can also be obtained from "**free energies**":

$$\mathsf{q}_i = -\frac{\partial}{\partial \beta_i} F, \quad \mathsf{j}_i = -\frac{\partial}{\partial \beta_i} G,$$

$$F = \int \mathrm{d}p(\theta) \log(1 + e^{-\epsilon(\theta)}), \qquad G = \int \mathrm{d}E(\theta) \log(1 + e^{-\epsilon(\theta)})$$

with **pseudo-energy** as before, solving the nonlinear integral equation

$$\epsilon(\theta) = \sum_i \beta_i h_i(\theta) - \int \frac{\mathrm{d}\alpha}{2\pi} \varphi(\theta, \alpha) \log(1 + e^{-\epsilon(\alpha)}).$$

Also, it turns out that

$$n(\theta) = \frac{1}{1 + e^{\epsilon(\theta)}}.$$

6. GHD in the quasi-particle language

- O. A. Castro-Alvaredo, BD, T. Yoshimura, Phys. Rev. X 6, 041065 (2016);
- B. Bertini, M. Collura, J. De Nardis, M. Fagotti, Phys. Rev. Lett. 117, 207201 (2016);

and corrections due to external force fields, temperature fields etc. were found in

- BD, T. Yoshimura, SciPost Phys. 2, 014 (2017).

6. GHD in the quasi-particle language

We now make GGEs space-time dependent. This means we promote

$$\rho_{\rm p}(\theta) \mapsto \rho_{\rm p}(x,t;\theta) \qquad \text{or equivalently} \qquad n(\theta) \mapsto n(x,t;\theta).$$

The quantity $\rho_{\rm p}(x,t;\theta){\rm d}x{\rm d}\theta$ is the number of quasi-particles in the “phase-space” element $[\theta,\theta+{\rm d}\theta]\times[x,x+{\rm d}x]$.

Using

$$\mathtt{q}_i(x,t) = \int {\rm d}\theta\, h_i(\theta)\rho_{\rm p}(x,t;\theta), \qquad \mathtt{j}_i(x,t) = \int {\rm d}\theta\, h_i(\theta)v^{\rm eff}(\theta)\rho_{\rm p}(x,t;\theta)$$

and **completeness of** $\{h_i(\theta)\}$, the fundamental GHD equations $\partial_t \mathtt{q}_i + \partial_x \mathtt{j}_i = 0$ become

$$\partial_t \rho_{\rm p}(x,t;\theta) + \partial_x\left[v^{\rm eff}(x,t;\theta)\rho_{\rm p}(x,t;\theta)\right] = 0.$$

These are the **GHD hydrodynamic equations in the quasi-particle language**.

6. GHD in the quasi-particle language

Parenthesis: normal modes or Riemann invariants of hydrodynamics.

Recall that

$$\partial_t \underline{\mathtt{q}} + \partial_x \underline{\mathtt{j}} = 0, \qquad \underline{\mathtt{j}} = \underline{\mathcal{F}}(\underline{\mathtt{q}}).$$

The hydrodynamic equations then involve the Jaciobian of the transformation $\underline{\mathtt{q}} \to \underline{\mathtt{j}}$:

$$\partial_t \underline{\mathtt{q}}(x,t) + J(\underline{\mathtt{q}}(x,t))\,\partial_x \underline{\mathtt{q}}(x,t) = 0, \quad J(\underline{\mathtt{q}})_{ij} = \partial_j \mathcal{F}_i(\underline{\mathtt{q}})$$

The spectrum of J is the set $\{v_i^{\rm eff}\}$ of **effective propagation velocities**. It is independent of the choice of fluid coordinates. If we are lucky, we may find the right fluid coordinates to diagonalize it: the **normal modes**. Say $\underline{\mathtt{q}} = \mathcal{F}^q(\underline{\mathtt{n}})$, $\underline{\mathtt{j}} = \mathcal{F}^j(\underline{\mathtt{n}})$:

$$\partial_t \mathtt{n}_i + v_i^{\rm eff}(\underline{\mathtt{n}})\partial_x \mathtt{n}_i = 0 \qquad \forall\ i.$$

This is useful for solving the fluid equations in various situations.

6. GHD in the quasi-particle language

Surprisingly, it is possible to find the **normal modes or Riemann invariants** of GHD in the quasi-particle langiage.

Indeed a not-too-hard calculation gives

$$\partial_t n(x,t;\theta) + v^{\rm eff}(x,t;\theta)\partial_x n(x,t;\theta) = 0.$$

That is, the occupation function is the fluid coordinate that diagonalizes GHD. It is convectively transported by the fluid, with propagation velocities $v^{\rm eff}(x,t;\theta)$.

6. GHD in the quasi-particle language

All of this generalizes to the presence of **force fields, temperature fields, etc.**

It is the energy function that controls the time evolution. Assume that it is explicitly space dependent $E(\theta) = E(x;\theta)$. For instance in the repulsive Lieb-Liniger model,

$$H_{\rm evo} = H + \int {\rm d}x\, V_{\rm evo}(x)\mathfrak{n}(x) \quad \Rightarrow \quad E(x;\theta) = m\theta^2/2 + V_{\rm evo}(x).$$

Then the two following equivalent equations hold (here suppressing $x,t;\theta$ dependence):

$$\partial_t \rho_{\rm p} + \partial_x \left[v^{\rm eff}\rho_{\rm p}\right] + \partial_\theta \left[a^{\rm eff}\rho_{\rm p}\right] = 0$$

$$\partial_t n + v^{\rm eff}\partial_x n + a^{\rm eff}\partial_\theta n = 0$$

where the **effective acceleration** is

$$a^{\rm eff} = \frac{F^{\rm dr}}{(p')^{\rm dr}}, \qquad F = -\partial_x E.$$

6. GHD in the quasi-particle language

Remarks:

- This is the full Euler-scale hydrodynamics with force or external fields. It is valid assuming both that the fluid variables and the external fields vary only on large distances. Beyond the Euler scale, there are higher-derivative terms (such as viscosity).
- The equations look a little bit like Boltzmann equations if we interpret $v^{\rm eff}$ as giving rise to collision terms. However, they are not of Boltxmann type. The GHD equations are rather Euler-type hydrodynamic equations: they are time-reversal invariant, and their validity necessitates the assumption of local entropy maximization, which Boltzmann equations are not / do not.
- The state density $\rho_{\rm s}$ satisfy the same equation $\partial_t \rho_{\rm s} + \partial_x \left[v^{\rm eff} \rho_{\rm s}\right] + \partial_\theta \left[a^{\rm eff} \rho_{\rm s}\right] = 0$.
- The Yang-Yang entropy of thermodynamic Bethe ansatz is also conserved.

6. GHD in the quasi-particle language

- Since external space-dependent fields generically break integrability, in their presence, beyond the Euler scale, there are also integrability-breaking terms. These will eventually cause the system to relax towards the Gibbs ensemble of the evolution Hamiltonian. Writing $E(x;\theta) = \sum_i h_i(\theta) V_i(x)$, at very large times, after corrections to Euler hydrodynamics accumulate, the system relaxes to the Gibbs state of the corresponding Hamiltonian, $\exp\left[-\beta \sum_i \int \mathrm{d}x\, V_i(x)\mathfrak{q}_i(x)\right]$. In the hydrodynamic approximation, this is

$$\exp\left[-\beta \sum_i \int \mathrm{d}x\, V_i(x)\mathfrak{q}_i(x)\right] \quad \Rightarrow \quad \exp\left[-\beta \sum_i V_i(x) Q_i\right].$$

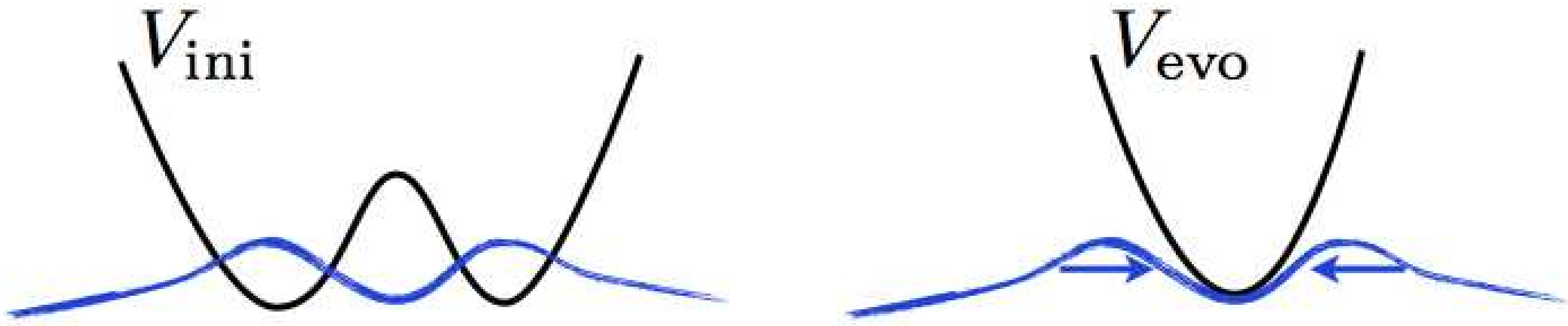
One can show that this is a stationary solution of the GHD equation in the corresponding external inhomogeneous fields.

7. An example: quantum Newton cradle-like setups

J.-S. Caux, BD, J. Dubail, R. Konik, T. Yoshimura, in preparation

7. An example: quantum Newton cradle-like setups

We are now ready to write the full dynamics for the original problem.



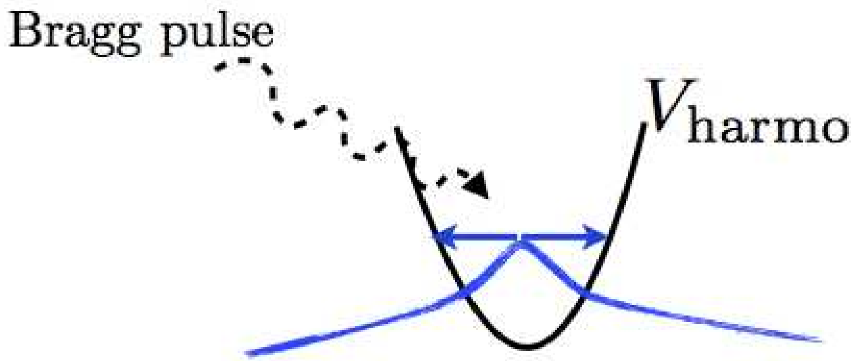
In order to fix the initial state

$$\rho_{\rm ini} = e^{-\beta(H+\int {\rm d}x\, V_{\rm ini}(x)\mathfrak{n}(x))} \quad \overset{\rm hydro}{\Rightarrow} \quad \rho_{\rm GE}(x,0) = e^{-\beta(H+V_{\rm ini}(x)N)},$$

we set $n(x,0;\theta) = \frac{1}{1+e^{\epsilon(x;\theta)}}$ where

$$\epsilon(x;\theta) = m\theta^2/2 + V_{\rm ini}(x) - \int \frac{{\rm d}\alpha}{2\pi}\,\varphi(\theta,\alpha)\log(1+e^{-\epsilon(x;\alpha)}).$$

7. An example: quantum Newton cradle-like setups



Another way of initializing would be by representing the Bragg pulse used in experiment. Here we calculate $\rho_{\rm p}(\theta)$ associated to $e^{-\beta(H+V_{\rm harmo}N)}$, and then set, as the effect of the Bragg pulse followed by fast local entropy maximization,

$$\rho_{\rm p}(x,0;\theta) = \frac{1}{2}\big[\rho_{\rm p}(\theta+\theta_{\rm Bragg}) + \rho_{\rm p}(\theta-\theta_{\rm Bragg})\big].$$

7. An example: quantum Newton cradle-like setups

The evolution is then done through

$$H_{\rm evo} = H + \int {\rm d}x\, V_{\rm evo}(x)\mathfrak{n}(x) \quad \overset{\rm hydro}{\Rightarrow} \quad E(x;\theta) = m\theta^2/2 + V_{\rm evo}(x).$$

Using $v^{\rm eff} = (\partial_\theta E)^{\rm dr}/(\partial_\theta p)^{\rm dr}$ and $a^{\rm eff} = -(\partial_x E)^{\rm dr}/(\partial_\theta p)^{\rm dr}$, we then solve

$$\partial_t n + v^{\rm eff}\partial_x n + a^{\rm eff}\partial_\theta n = 0$$

with $v^{\rm eff}$ and $a^{\rm eff}$ evaluated using $n(x,t;\theta)$.

Recall the dressing operation:

$$h^{\rm dr}(\theta) = h(\theta) + \int \frac{{\rm d}\alpha}{2\pi}\varphi(\theta,\alpha)n(\alpha)h^{\rm dr}(\alpha).$$

Lecture II: solutions and application to transport and correlations.

8. Exact solution to the initial value problem of GHD

BD, H. Spohn, T. Yoshinura, arXiv:1704.04409

8. Exact solution to the initial value problem of GHD

It turns out that one can solve the initial value problem of GHD (without acceleration term) in terms of integral equations.

Recall that a given GHD model, in the quasi-particle language, is determined by a spectral parameter θ that lies in the appropriate space, a differential scattering phase $\varphi(\theta,\alpha)$, a momentum function $p(\theta)$ (which tells us how quasi-particles fit wihtin physical space), and an energy function $E(\theta)$ (which tells us how to evolve in time).

Recall the dressing operation, which we will denote $h^{\rm dr}_{[n]}(\theta)$ to emphasize its dependence on the function $\theta \mapsto n(\theta)$,

$$h^{\rm dr}_{[n]}(\theta) = h(\theta) + \int \frac{{\rm d}\alpha}{2\pi}\varphi(\theta,\alpha)n(\alpha)h^{\rm dr}_{[n]}(\alpha).$$

Recall the effective velocity

$$v^{\rm eff}(x,t;\theta) = \frac{(E')^{\rm dr}_{[n]}(\theta)}{(p')^{\rm dr}_{[n]}(\theta)}.$$

8. Exact solution to the initial value problem of GHD

The **exact solution** to the initial-value problem

$$\partial_t n(x,t;\theta) + v^{\rm eff}(x,t;\theta)\partial_x n(x,t;\theta) = 0, \qquad n(x,0;\theta) = n^0(x;\theta)$$

is the solution (unique?) to the set of nonlinear integral equations

$$n(x,t;\theta) = n^0(u(x,t;\theta);\theta)$$

$$\int_{x_0}^{u(x,t;\theta)} {\rm d}y\, (p')^{\rm dr}_{[n^0(y)]}(\theta) + (E')^{\rm dr}_{[n^0(x_0)]}(\theta)\, t = \int_{x_0}^{x} {\rm d}y\, (p')^{\rm dr}_{[n(y,t)]}(\theta).$$

Here the point x_0 is an **asymptotic stationary point**, which must be chosen **far enough on the left** such that $\rho_{\rm p}(x_0,t;\theta)$ is **independent of** t. We solve this by iteration as follows:

1. Set $n(x,t;\theta) = n^0(x;\theta)$.
2. Solve the second equation for $u(x,t;\theta)$.
3. Set $n(x,t;\theta) = n^0(u(x,t;\theta);\theta)$, and repeat from the second step until convergence.

8. Exact solution to the initial value problem of GHD

This is essentially a version of the solution by characteristics.

Special cases:

- Free models: $\varphi = 0$, giving $u(x,t;\theta) = x - v(\theta)t$ where $v(\theta) = \mathrm{d}E(\theta)/\mathrm{d}p(\theta)$ is the group velocity.
- Homogeneous initial state: $u(x,t;\theta) = x - v^{\mathrm{eff}}(\theta)t$ (but without acceleration field, the state just stays the same for all times!)

8. Exact solution to the initial value problem of GHD

This has a beautiful geometric interpretaion.

Let us introduce a **family of metrics, which depend on the local state,** on physical space, parametrized by θ, with length square $\mathrm{d}q^2$ fixed by

$$\mathrm{d}q(x;\theta) = \frac{(p')^{\mathrm{dr}}_{[n(x)]}(\theta)}{p'(\theta)}\mathrm{d}x = \mathcal{K}_{[n(x)]}(\theta)\mathrm{d}x.$$

This can be seen as emerging from a volume element **proportional to the density of states**

$$\mathrm{d}V = \mathrm{d}q\,\mathrm{d}p = \rho_{\mathrm{s}}(x;\theta)\mathrm{d}x\,2\pi\mathrm{d}\theta.$$

Thanks to conservation of the state density, this is conserved by the dynamics.

Let $n(x;\theta)$ be the **density per unit invariant volume**. This is determined by ρ_{p} via

$$n(x;\theta)\mathrm{d}V = \rho_{\mathrm{p}}(x;\theta)\mathrm{d}x\,2\pi\mathrm{d}\theta.$$

Thus $n(x;\theta) = \rho_{\mathrm{p}}(x;\theta)/\rho_{\mathrm{s}}(x;\theta)$, equal to the **occupation function**!

8. Exact solution to the initial value problem of GHD

Thus the occupation function should satisfy a simple conservation equation in the q coordinate. One can show that **the following is equivalent to GHD**:

$$\partial_t n|_{q,\theta} + v^-(\theta)\partial_q n|_{t,\theta} = 0$$

where

$$v^-(\theta) = \mathcal{K}_{[n(x_0)]}(\theta)\, v^{\rm eff}(x_0;\theta)$$

and x_0 is an asymptotic stationary point.

Note that $v^-(\theta)$ is independent of the state $n(x,t;\theta)$. This is an equation for a density of **freely moving particles at velocities $v^-(\theta)$ within the space with coordinate q**.

The solution is straightforward:

$$n|_{q,t;\theta} = n^0|_{q-v^-(\theta)t;\theta}.$$

The said integral-equation solution is obtained from this with

$$q(u(x,t;\theta),0;\theta) = q(x,t;\theta) - v^-(\theta)t.$$

8. Exact solution to the initial value problem of GHD

Remarks:

- The integral equations can be solved on a laptop. It takes at most few minutes for every value of t to get $n(x,t;\theta)$ for all x,θ at high precision. It doesn't seem to matter how big t is.
- We do not know if the integral equations have a unique solution or not, but in examples we did the iterative procedure converges well.
- In principle one can analyze the large-t limit and approach to steady state.
- It seems to point to potential integrability of the GHD equations themselves. Connection with simultaneously discovered integrability structure?

  [Bulchandani, Vasseur, Karrasch, Moore 2017; Bulchandani 2017]

9. Molecular dynamics

BD, T. Yoshimura, J.-S. Caux, arXiv:1704.05482 (2017).

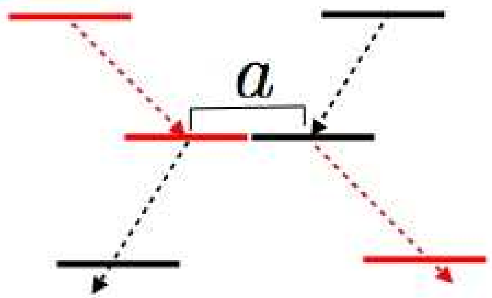
9. Molecular dynamics

There is a way of simulating the full GHD, with acceleration term, using a class of classical dynamics of particles in one dimension.

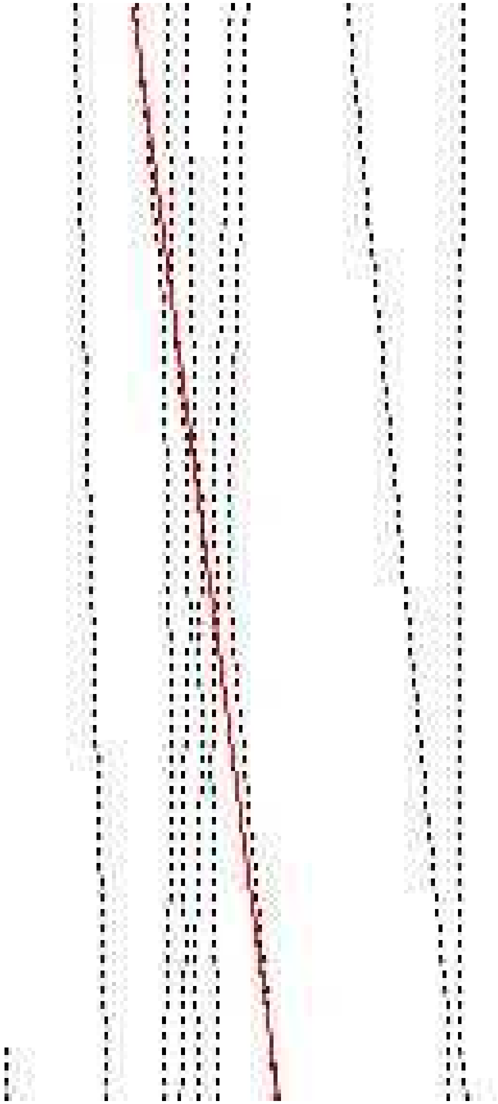
The story starts with **the hard rod gas.** This is a gas of segments, say of length a, lying on the line, and propagating freely except for elastic collisions.

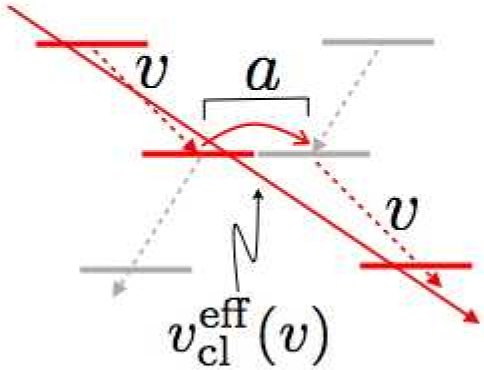
Upon collision, velocities are exchanged. We can think of a “quasi-particle” as the **tracer of a given velocity**. By elastic collision, the number of tracers in a given velocity interval $[v, v + \mathrm{d}v]$ is preserved by the dynamics, giving **infinitely many conservation laws**.

A tracer **jumps by a distance a at every collision**.



9. Molecular dynamics





It was shown rigorously that

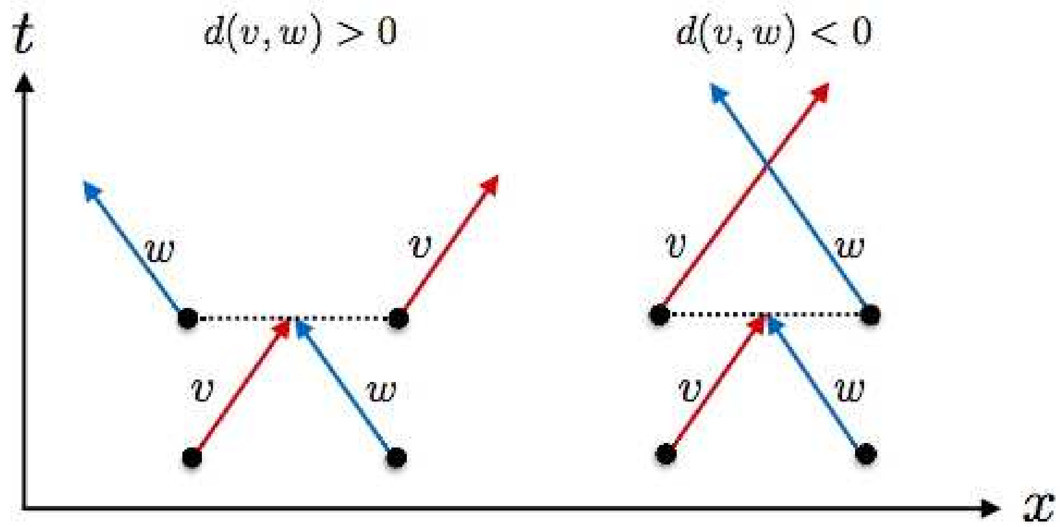
[Boldrighini, Dobrushin, Sukhov 1982]

$$v_{\mathrm{cl}}^{\mathrm{eff}}(v) = v - a \int \mathrm{d}w\, \rho_{\mathrm{p}}(w)(v_{\mathrm{cl}}^{\mathrm{eff}}(w) - v_{\mathrm{cl}}^{\mathrm{eff}}(v))$$

This is GHD if we identify $v = \theta$ and $\varphi(\theta, \alpha) = -a$!!!!

9. Molecular dynamics

We can generalize this to **velocity-dependent jump lengths $d(v,w)$ in any direction**. Quasi-particles are now seen as the actual particles on which we put a dynamics, and at collisions they jump, forward or backward, by a distance $d(v,w)$. This is what we call the **“flea gas”**.



9. Molecular dynamics

Actual distance travelled in a macroscopic time Δt:

$$\Delta x = \Delta t \, v_{\rm cl}^{\rm eff}(v).$$

Δx results from **linear displacements at velocity** v, $\Delta t \, v$, plus accumulation of **jumps the quasi-particle undergoes as it travels through the gas**.

Oriented distance jumped due to hitting a quasi-particle with velocity w is $\mathrm{sign}(v-w) d(v,w)$.

Average number of quasi-particles of velocity between w and $w + \mathrm{d}w$ crossed, is the total number $\mathrm{d}w \, \rho_{\rm cl}(w) \Delta x$ within Δx, times probability $\Delta t / \Delta x \times |v_{\rm cl}^{\rm eff}(v) - v_{\rm cl}^{\rm eff}(w)|$ crossing occurs in time Δt.

Thus

$$\Delta x = \int \mathrm{d}w \, d(v,w) \, \rho_{\rm cl}(w) \, \Delta t \, (v_{\rm cl}^{\rm eff}(v) - v_{\rm cl}^{\rm eff}(w)).$$

9. Molecular dynamics

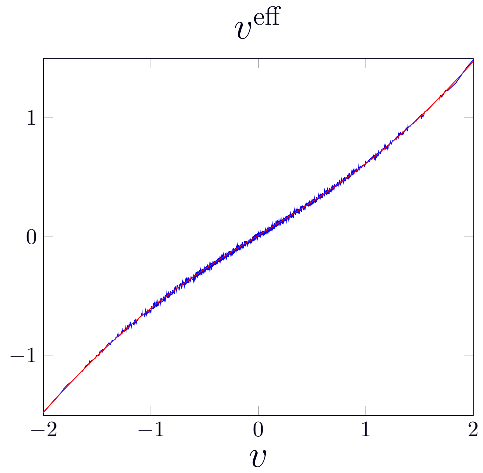
Thus we find

$$v_{\rm cl}^{\rm eff}(v) = v + \int {\rm d}w\, d(v,w)\, \rho_{\rm cl}(w)\, (v_{\rm cl}^{\rm eff}(v) - v_{\rm cl}^{\rm eff}(w))$$

and the exact correspondence is

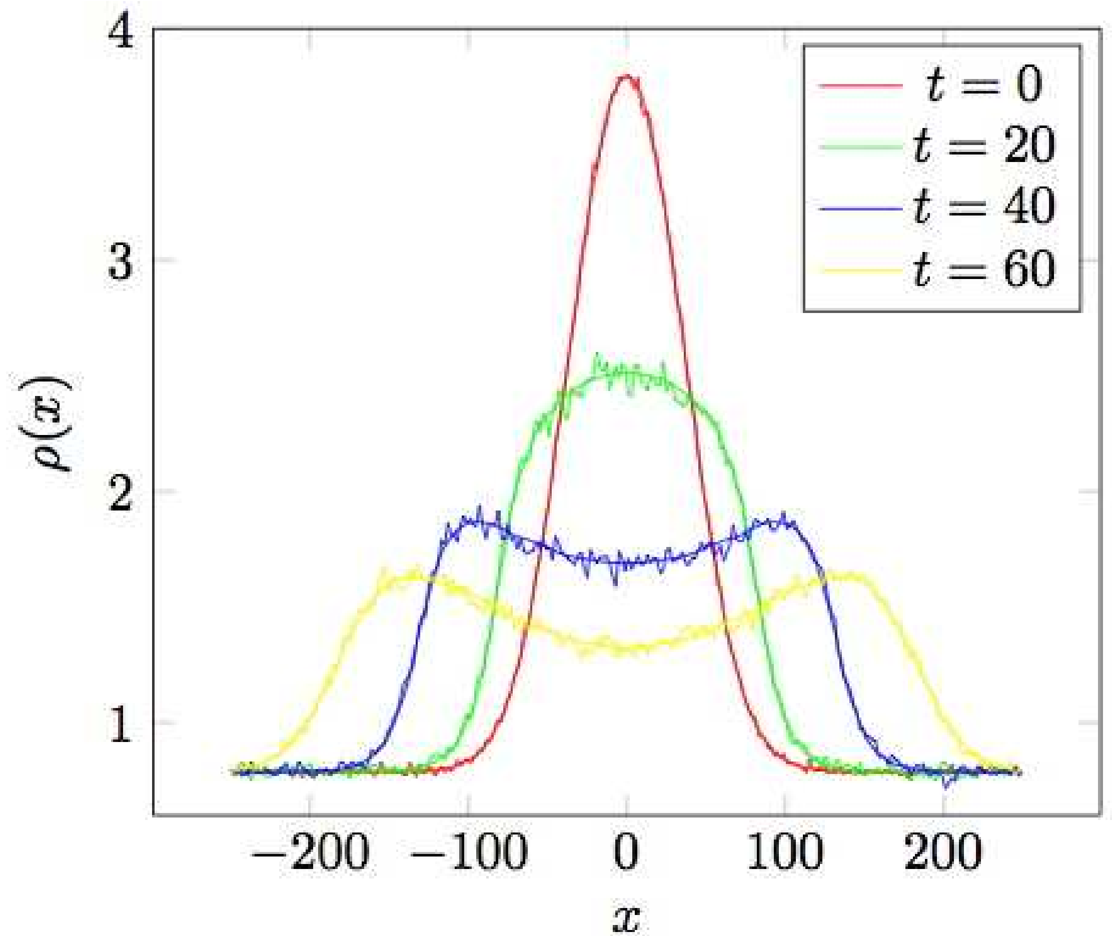
$$\rho_{\rm cl}(v){\rm d}v = \rho_{\rm p}(\theta){\rm d}\theta, \quad v = v^{\rm gr}(\theta), \quad v_{\rm cl}^{\rm eff}(v) = v^{\rm eff}(\theta), \quad d(v,w) = -\varphi(\theta,\alpha)/p'(\theta).$$

One can check numerically that it works (here Lieb-Liniger model):



9. Molecular dynamics

Here is a numerical check: comparing the **exact integral-equation solution** with **the flea gas** ($\rho(x) = \mathtt{n}(x)$ is the actual particle density at the point x).



9. Molecular dynamics

Remarks:

- In the hard rod case, a well known solution method is to **collapse all rods to points**. We then gets freely evolving point particles without interaction. After this collapse, **the new space coordinate is the coordinate q introduced above**. That is, the metric can be seen as coming from the jumps of the quasi-particles in classical gases.

- It is known that **wave packets in integrable quantum models behave as classical solitons**. As classical solitons, they undergo space shifts after collisions. **These shifts are exactly distances $d(v, w)$ of the flea gas corresponding to the quantum model**! Thus, we see that quantum models, at the Euler scale, are equivalent to their corresponding classical soliton gas [...].

- One can also add acceleration, simply by accelerating the quasi-particles between collisions (works at least for Galilean single-particle spectrum).

10. Zero-entropy GHD

BD, J. Dubail, R. Konik, T. Yoshimura, arXiv:1704.04151 (2017)

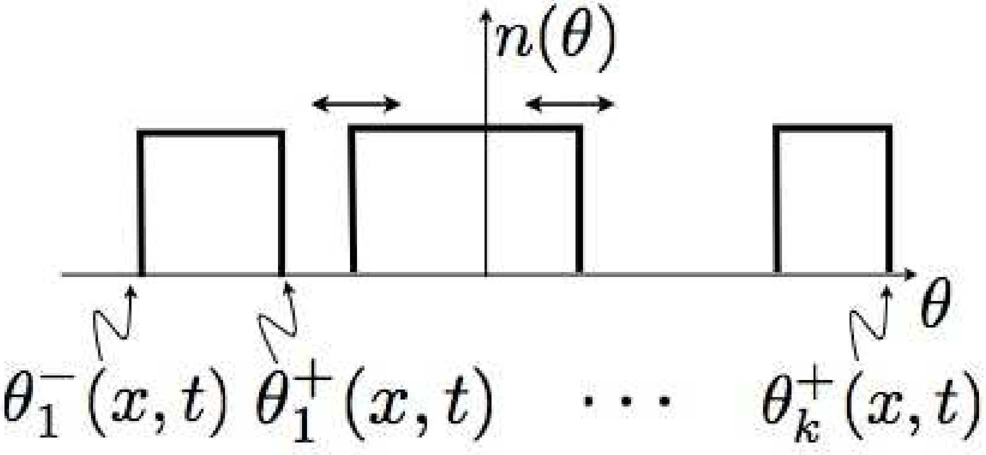
10. Zero-entropy GHD

GHD simplifies drastically in the zero-entropy subspace of fermionic systems

Let the occupation function $n(\theta)$ be formed of k disjoint, filled Fermi seas. Such occupation functions have zero entropy. It turns out that the GHD equations simplify to a **finite-component hydrodynamics for the Fermi seas' endpoints**:

$$\partial_t \theta_j^\pm + v^{\rm eff}_{\{\theta\}}(\theta_j^\pm)\partial_x \theta_j^\pm = 0.$$

This is obtained from using e.g. $\partial_{t,x}\Theta(\theta - \theta_j^-) = \delta(\theta - \theta_j^-)$.
We refer to this as "2kHD", for $k = 1, 2, 3, \ldots$



10. Zero-entropy GHD

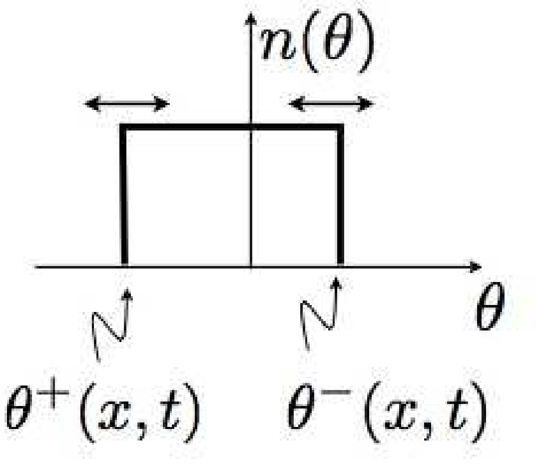
Consider 2HD. A single Fermi sea: **this is what we get in a zero temperature initial state**.

This is a **2-component, Galilean invariant hydrodynamics**. Hence the only possibility is **conventional hydrodynamics**! That is,

2HD = conventional hydrodynamics (CHD)

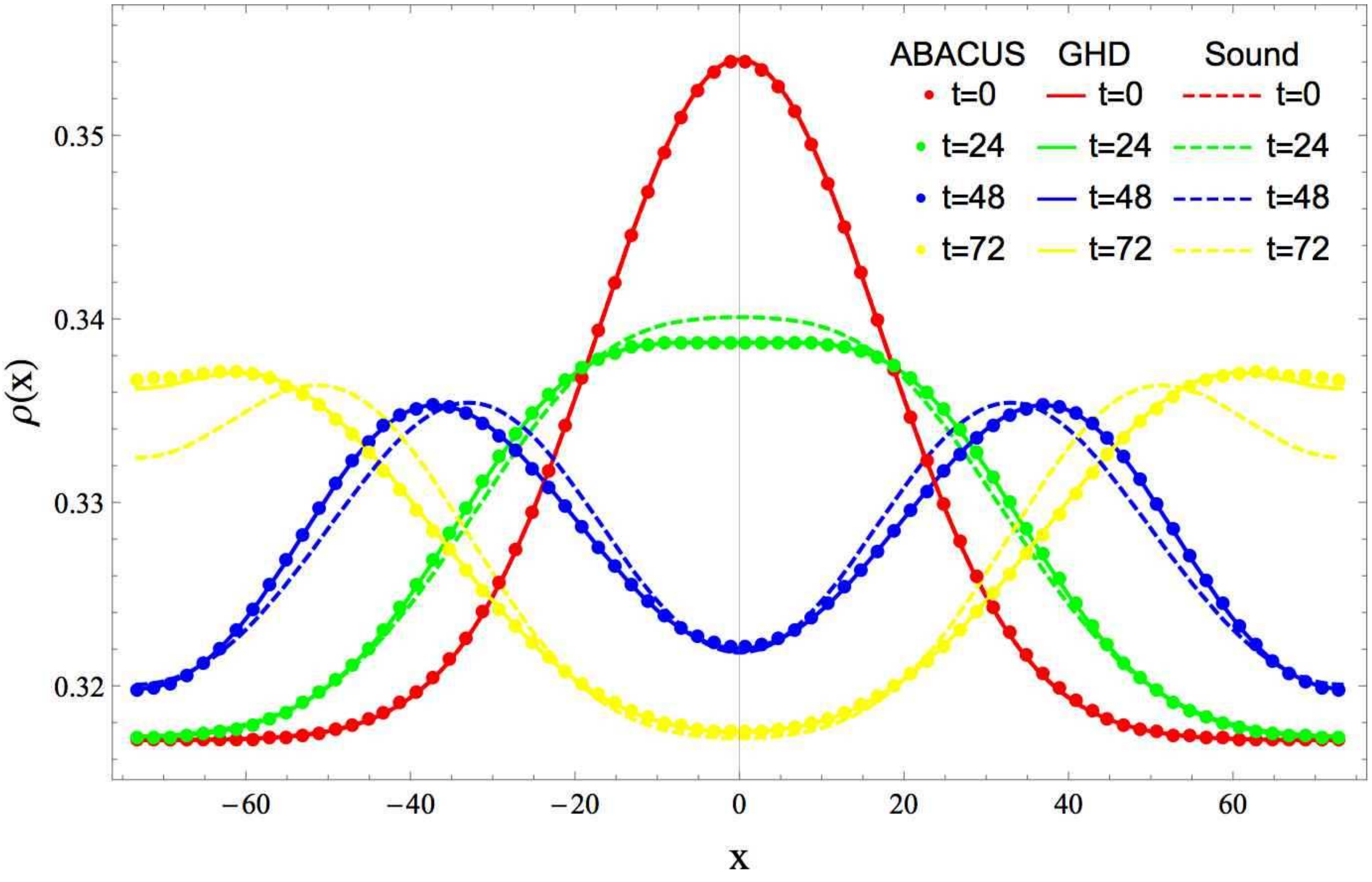
$$\partial_t \mathtt{n} + \partial_x (v\mathtt{n}) = 0, \quad \partial_t v + v\partial_x v = -\frac{1}{m\mathtt{n}}\partial_x \mathcal{P}, \qquad \mathcal{P} = \mathcal{P}(v,\mathtt{n}).$$

Indeed a single Fermi sea is determined by fixing the **density of particles** and the **boost velocity**. So the local density matrices have the form $e^{-\beta(H-\mu N-\nu P)}|_{\beta\to\infty}$.



10. Zero-entropy GHD

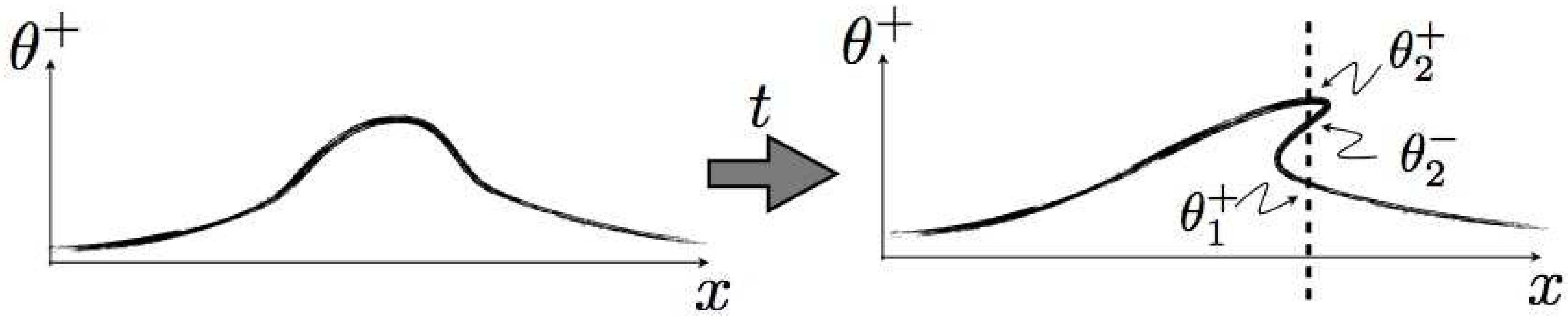
Comparison with ABACUS quantum evolution:



10. Zero-entropy GHD

In CHD, one usually encounters **shocks** that are sustained over time. At shocks, entropy is continuously produced (the production of entropy at shocks is described by higher-derivative terms neglected in Euler hydrodynamics).

However, **in GHD, shocks are not sustained. Gradient catastrophes arise in 2HD, but immediately "dissolve," thanks to the availability of a large fluid space, into 4HD**.



10. Zero-entropy GHD

Remarks:

- CHD has been used in the past many times in order to describe the Lieb-Liniger model in inhomogeneous situations. It was quite successful. However it has been observed that at both collision of clouds and in propagation of density waves, shocks appear, beyond which CHD is unable to reproduce numerics. Here we have a full explanation of **why CHD works** at zero temperature based on hydrodynamic principle only (local entropy maximization), and exact hydrodynamic equations for describing **what happens after the gradient catastrophes**.
- The 2kHD equations are very simple to solve, and do not necessitate integral equations of molecular dynamics (and agree with them). **They fully describe zero-temperature problems**.
- In real systems, gradient catastrophes do not ever occur: higher derivative terms (beyond Euler hydrodynamics) become important before. However, at very large scales (Euler scales), gradients will look very sharp. GHD captures well what happens at such scales.

11. Exact solution to domain wall problems

- O. A. Castro-Alvaredo, BD, T. Yoshimura, Phys. Rev. X 6, 041065 (2016);
- B. Bertini, M. Collura, J. De Nardis, M. Fagotti, Phys. Rev. Lett. 117, 207201 (2016);

See also:

- Review of works pre-2016: D. Bernard, BD, J. Stat. Mech 2016, 064005 (2016);
- BD, H. Spohn, J. Stat. Mech. 2017, 073210 (2017);
- L. Piroli, J. De Nardis, M. Collura, B. Bertini, M. Fagotti, arXiv: 1706.00413

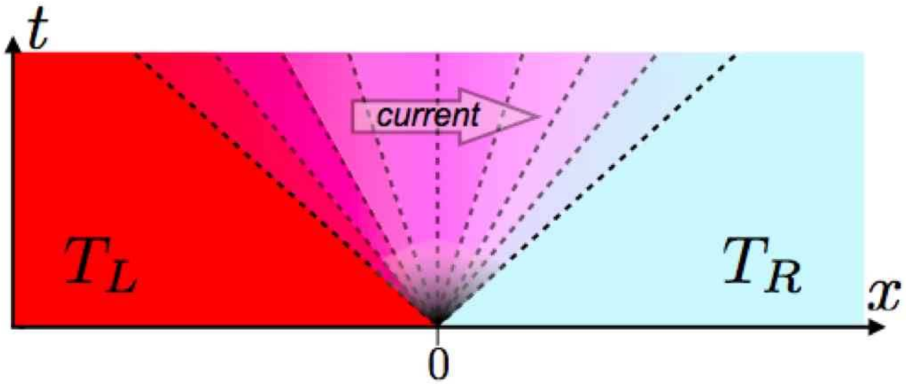
and more...

11. Exact solution to domain wall problems

In the domain wall problem, the initial state is prepared to be a juxtaposition of two homogeneous states, one on the left, one on the right.

The two halves play the role of baths, able to furnish unbounded energy, particles, etc. At large times, a **current-carrying steady state is expected to emerge if there is ballistic transport.** This has been referred to as **the partitioning protocol** for generating non-equilibrium currents.

[Caroli et. al. 1971; Rubin et. al. 1971; Spohn, Lebowitz 1977; Ruelle 2000; Tasaki 2000; Araki, Ho 2000; Aschbacher, Pillet 2003; Bernard, BD 2012]



11. Exact solution to domain wall problems

For instance, in a purely thermal case, the initial state is $\langle\cdots\rangle_{\rm ini}$ with

$$\rho_{\rm ini} = e^{-\beta_L H_L - \beta_R H_R}.$$

Unitary evolution is performed with the full homogeneous hamiltonian

$$H = H_L + H_R + \delta H_{LR}$$

and stationary state is defined by

$$\mathtt{O}^{\rm sta} := \lim_{t\to\infty} \langle e^{{\rm i}Ht} O e^{-{\rm i}Ht}\rangle_{\rm ini}, \qquad O \text{ local observable.}$$

As an example, in the Heisenberg chain, one could take

$$H_L = \sum_{i=-\infty}^{-1} \vec{\sigma}_i\cdot\vec{\sigma}_{i+1}, \quad H_R = \sum_{i=1}^{\infty} \vec{\sigma}_i\cdot\vec{\sigma}_{i+1}, \quad \delta H_{LR} = \vec{\sigma}_0\cdot\vec{\sigma}_1.$$

11. Exact solution to domain wall problems

The hydrodynamic description for the initial conditions is

$$n(x,0;\theta) = n^L(\theta)\Theta(-x) + n^R(\theta)\Theta(x).$$

Since the **evolution is invariant under scaling** $(x,t) \mapsto \lambda(x,t)$ and the **initial condition has the same invariance**, we may expect the **solution to be scaling invariant:**

$$n(x,t;\theta) = n(\xi;\theta), \qquad \xi = x/t.$$

Thus the GHD equations in the quasi-particle formulation simplifies to

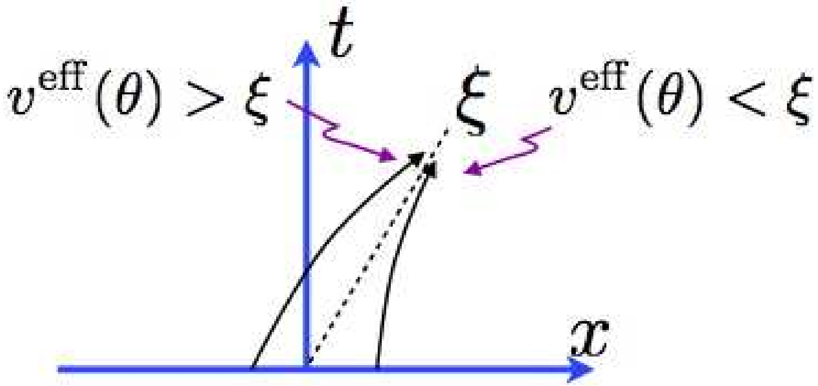
$$\big(\xi - v^{\rm eff}(\xi;\theta)\big)\partial_\xi n(\xi;\theta) = 0, \qquad n(\pm\infty;\theta) = n^{R,L}(\theta).$$

11. Exact solution to domain wall problems

The derivative is only nonzero where $\xi = v^{\text{eff}}(\xi;\theta)$. Therefore we get a **self-consistent system of integral equations**:

$$n(\xi;\theta) = n^L(\theta)\Theta(\theta - \theta_\star(\xi)) + n^R(\theta)\Theta(\theta_\star - \theta(\xi)), \qquad v^{\text{eff}}(\xi;\theta_\star(\xi)) = \xi.$$

Interpretations: Particles arrive either from the left reservoir or the right reservoir depending on their dressed velocity at the ray ξ.



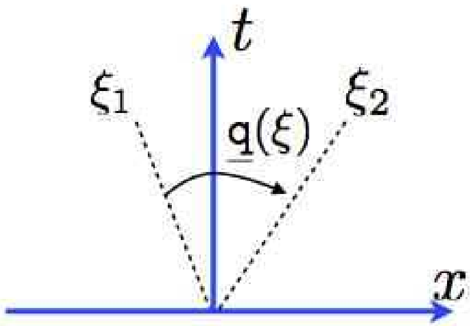
11. Exact solution to domain wall problems

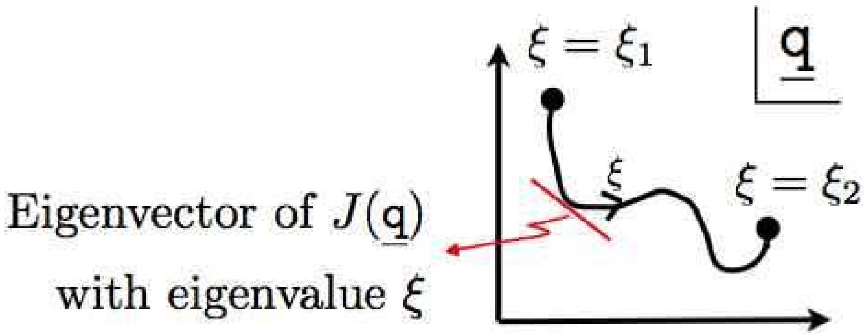
Parenthesis: in ordinary (finite-component) fluids, generically there are shocks developing in the domain wall problem. Here no shocks thanks to the infinite-dimensionality of the space of local fluid states.

In finite hydro, we write equations of state $\underline{\mathtt{j}} = \underline{\mathcal{F}}(\underline{\mathtt{q}})$ and we get

$$(J(\underline{\mathtt{q}}) - \xi \mathbf{1})\partial_\xi \underline{\mathtt{q}} = 0, \quad J(\underline{\mathtt{q}})_{ij} = \partial_j \mathcal{F}_i(\underline{\mathtt{q}}).$$

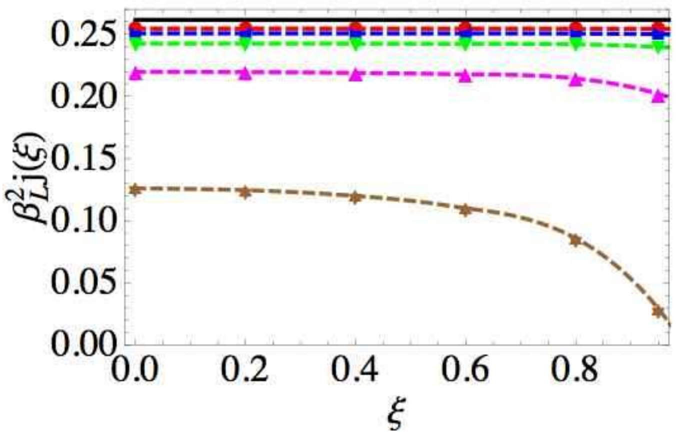
A shock is a jump in $\underline{\mathtt{q}}$ as a function of ξ. A rarefaction wave is a smooth solution between ξ_1 and ξ_2. With discrete spectrum: not enough freedom to bridge generic states with rarefaction waves $\Rightarrow$ need shocks.



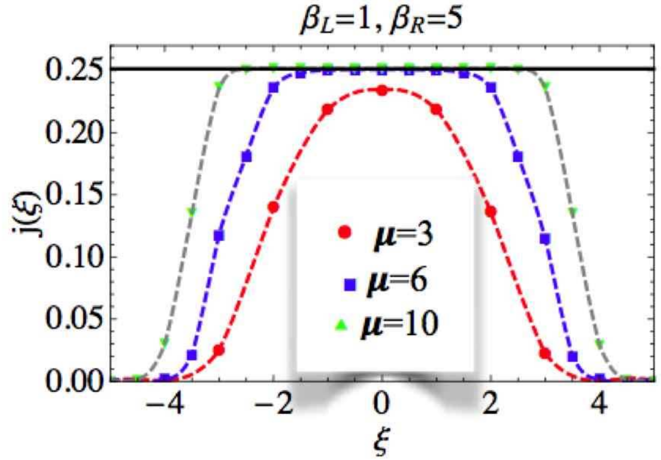


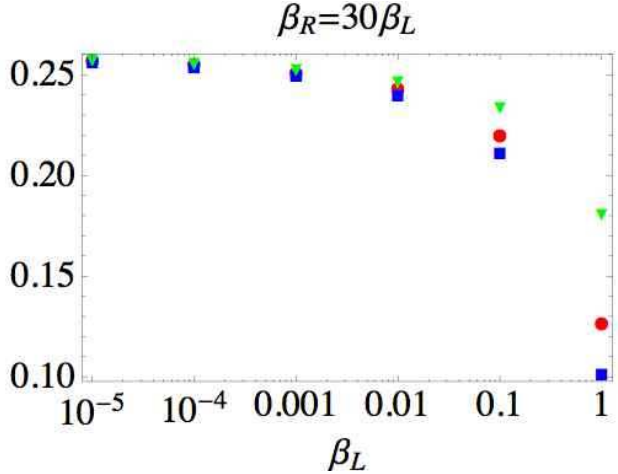
[Castro-Alvaredo, BD, Yoshimura 2016]

High-temperature in sinh-Gordon model: CFT.



Low-temperature in Lieb-Liniger model with chemical potential: Luttinger liquid





Inequalities [BD 2015]

$$\frac{\mathtt{h}^L - \mathtt{h}^R}{2} \geq \mathtt{p}^{\rm sta} \geq \frac{\mathtt{j_p}^L - \mathtt{j_p}^R}{2}$$

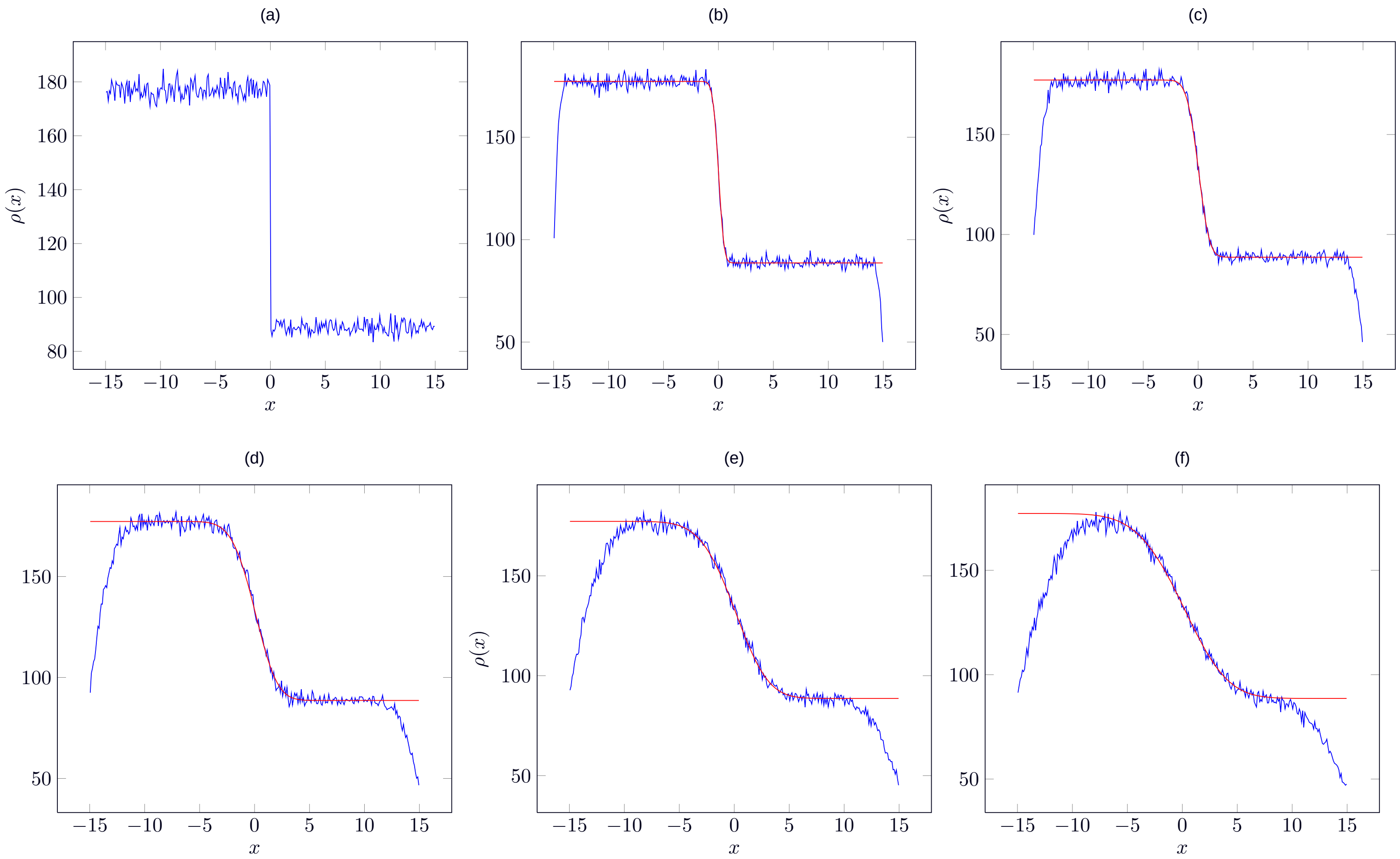


Figure 1: Hard rod density at times (a) $t = 0$, (b) $t = 0.5$, (c) $t = 1$, (d) $t = 2$, (e) $t = 3$, (f) $t = 4$, rod length $a = 0.001$. Simulation data in blue, exact solution in red. [BD, Spohn 2017]

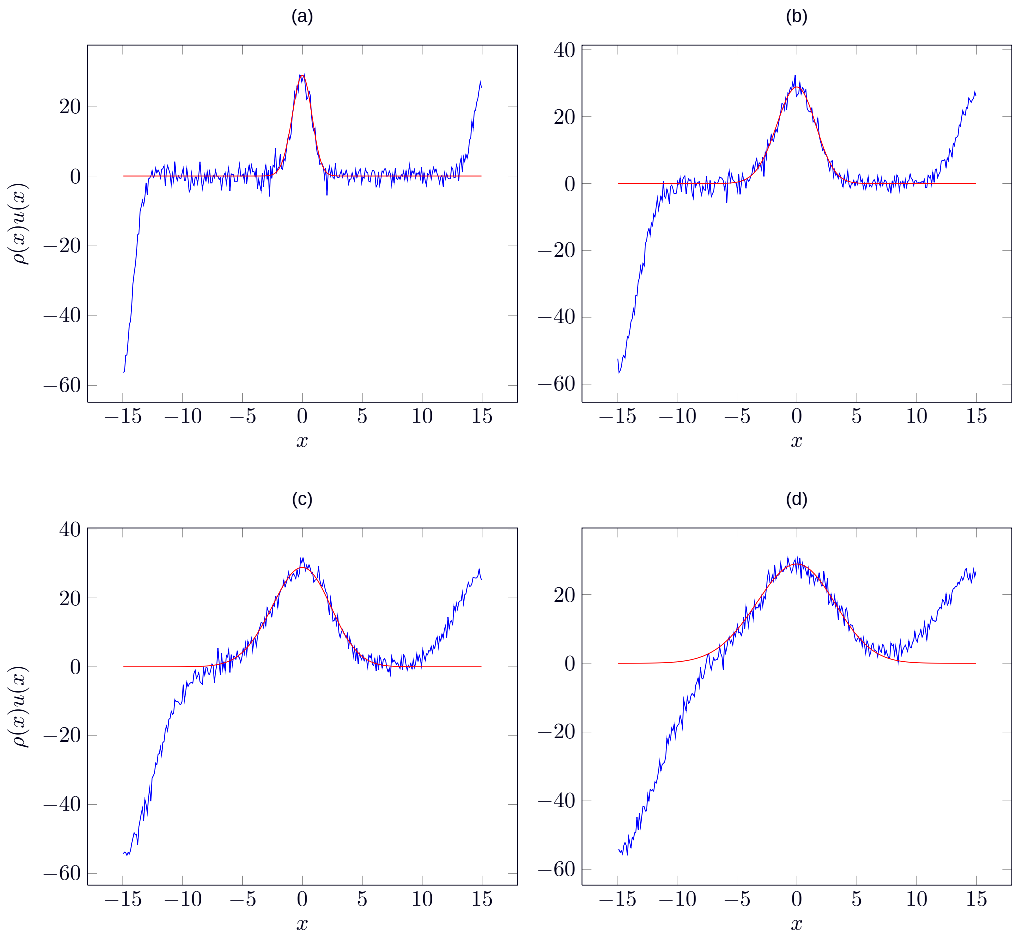


Figure 2: Current at times (a) $t = 1$, (b) $t = 2$, (c) $t = 3$, (d) $t = 4$. , rod length $a = 0.001$.
Simulation data in blue, exact solution in red. [BD, Spohn 2017]

(b) ρ

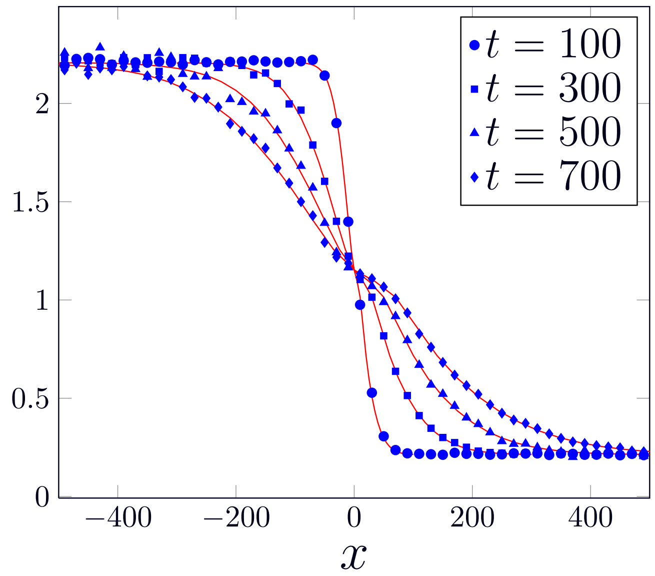


Figure 3: GHD for the LL model with $m = 1$, $c = 1$ is simulated using the classical flea gas. (a) Truncated Gaussian distribution $\rho_{\rm cl}(v) = 0.5e^{-v^2}\chi(-3 < v < 3)$. Effective velocity evaluated using approx. 1500 trajectories over a time of 1200 (blue); using the formula for effective velocity (red). (b) Density profile from domain wall initial condition, initial left and right temperatures 10 and $1/3$ (resp.), at times $t = 10, 30, 50, 70$. Simulation with approx. 2400 quasi-particles (initial baths of lengths 1000, open boundary condition) averaged over 1000 samples (blue); exact self-similar solution (red).[BD, Yoshimura, Caux 2017]

And very convincing numerical comparisons against tDMRG in

B. Bertini, M. Collura, J. De Nardis, M. Fagotti, Phys. Rev. Lett. 117, 207201 (2016).

12. Drude weights and correlations

General results within GHD found in: BD and H. Spohn, arXiv:1705.08141 (2017).

See also:

- S. Fujimoto and N. Kawakami, J. Phys. A **31**, 465 (1998);
- X. Zotos, Phys. Rev. Lett. **82**, 1764 (1999)
- A. Klümper and K. Sakai, J. Phys. A **35**, 2173 (2002);
- K. Sakai and A. Klümper, J. Phys. A **36**, 11617 (2003);
- T. Prosen, Phys. Rev. Lett. **106**, 217206 (2011);
- T. Prosen and E. Ilievski, Phys. Rev. Lett. **111**, 057203 (2013);
- E. Ilievski and T. Prosen, Commun. Math. Phys. **318**, 809-830 (2013);
- J. De Nardis and M. Panfil, SciPost Phys. **1**, 015 (2016);
- E. Ilievski and J. De Nardis, arXiv:1702.02930 (2017);
- V. B. Bulchandani, R. Vasseur, C. Karrasch and J. E. Moore, arXiv:1702.06146 (2017).

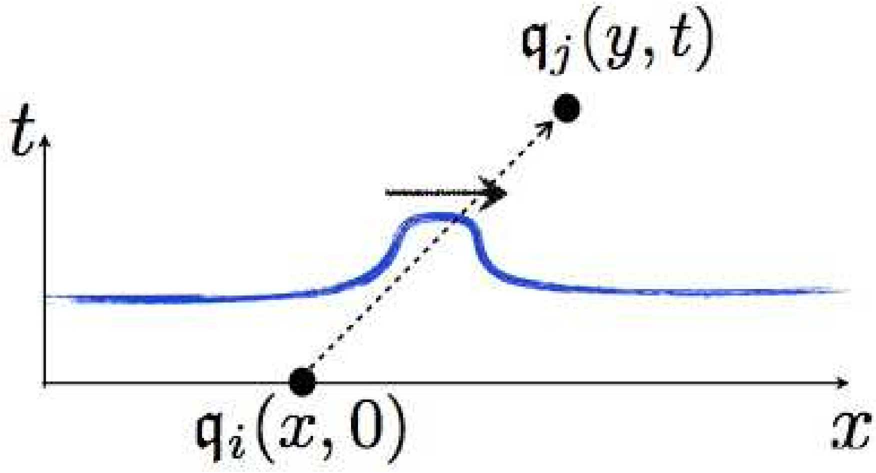
and more...

12. Drude weights and correlations

One can go much further and calculate correlations between local densities and currents.

With hydrodynamics, one can evaluate large-scale correlations, which are due to propagating waves. One can look for the connected correlation functions

$$\langle \mathfrak{q}_i(x,0)\mathfrak{q}_j(y,t)\rangle^{\mathrm{c}}, \quad \langle \mathfrak{q}_i(x,0)\mathfrak{j}_j(y,t)\rangle^{\mathrm{c}}, \quad \langle \mathfrak{j}_i(x,0)\mathfrak{j}_j(y,t)\rangle^{\mathrm{c}}.$$



12. Drude weights and correlations

Consider **stationary, homogeneous states**.

It is simple to calculate

$$\int \mathrm{d}x\, \langle \mathfrak{q}_i(x,t)\mathfrak{q}_j(0,0)\rangle^{\mathrm{c}} = \frac{\partial}{\partial\beta_i}\mathtt{q}_j, \qquad \int \mathrm{d}x\, \langle \mathfrak{q}_i(x,t)\mathfrak{j}_j(0,0)\rangle^{\mathrm{c}} = \frac{\partial}{\partial\beta_i}\mathtt{j}_j.$$

It is less trivial to evaluate the **Drude weight**

$$D_{i,j} = \lim_{t\to\infty}\int \mathrm{d}x\, \langle \mathfrak{j}_i(x,t)\mathfrak{j}_j(0,0)\rangle^{\mathrm{c}}.$$

The techniques of **hydrodynamic projections** can be used. Formally, one writes

$$D_{i,j} = \sum_k \langle \mathfrak{j}_i|\mathfrak{q}_k\rangle C^{-1}_{k,l}\langle \mathfrak{q}_l|\mathfrak{j}_j\rangle, \qquad C_{i,j} = \langle \mathfrak{q}_i|\mathfrak{q}_j\rangle$$

with inner product $\langle \mathfrak{a}|\mathfrak{b}\rangle = \int \mathrm{d}x\, \langle \mathfrak{a}(x,t)\mathfrak{b}(0,0)\rangle^{\mathrm{c}}$. Similar projection methods also give the operator $A_{i,j}$ for generating space-time dependent correlations.

12. Drude weights and correlations

Results:

$$\begin{aligned}
\int \mathrm{d}x\, \langle \mathfrak{q}_i(x,0)\mathfrak{q}_j(0,0)\rangle^{\mathrm{c}} &= \int \mathrm{d}\theta \rho_{\mathrm{p}}(\theta)(1-\sigma n(\theta)) h_i^{\mathrm{dr}}(\theta) h_j^{\mathrm{dr}}(\theta),\\
\int \mathrm{d}x\, \langle \mathfrak{j}_i(x,0)\mathfrak{q}_j(0,0)\rangle^{\mathrm{c}} &= \int \mathrm{d}\theta \rho_{\mathrm{p}}(\theta)(1-\sigma n(\theta)) v^{\mathrm{eff}}(\theta) h_i^{\mathrm{dr}}(\theta) h_j^{\mathrm{dr}}(\theta),\\
\lim_{t\to\infty}\int \mathrm{d}x\, \langle \mathfrak{j}_i(x,t)\mathfrak{j}_j(0,0)\rangle^{\mathrm{c}} &= \int \mathrm{d}\theta \rho_{\mathrm{p}}(\theta)(1-\sigma n(\theta)) v^{\mathrm{eff}}(\theta)^2 h_i^{\mathrm{dr}}(\theta) h_j^{\mathrm{dr}}(\theta),\\
\int \mathrm{d}t\, \langle \mathfrak{j}_i(0,t)\mathfrak{j}_j(0,0)\rangle^{\mathrm{c}} &= \int \mathrm{d}\theta \rho_{\mathrm{p}}(\theta)(1-\sigma n(\theta)) |v^{\mathrm{eff}}(\theta)| h_i^{\mathrm{dr}}(\theta) h_j^{\mathrm{dr}}(\theta),
\end{aligned}$$

$$\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} \langle \mathfrak{q}_i(x,t)\mathfrak{q}_j(0,0)\rangle^{\mathrm{c}} = \int \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}ktv^{\mathrm{eff}}(\theta)} \rho_{\mathrm{p}}(\theta)(1-\sigma n(\theta)) h_i^{\mathrm{dr}}(\theta) h_j^{\mathrm{dr}}(\theta).$$

(the last relation being valid at $t\to\infty,\ k\to 0$ with kt fixed), where

$\sigma = 1,\ -1,\ 0$ for fermionic, bosonic, and classical gases respectively.

Some open questions

- Most important question of all: higher-derivative corrections. This includes viscosity terms and associated diffusive effects (analyzed in an extensive numerical study in XXZ [Ljubotina, Znidaric, Prosen (2017)]), and integrability terms when force field is present. Time scale of integrablity breaking?

- Second most important question of all: large deviation theory of charge transfer, fluctuation relations, macroscopic fluctuation theory. Our result for $\int \mathrm{d}t\, \langle \mathrm{j}_i(0,t)\mathrm{j}_j(0,0)\rangle^{\mathrm{c}}$ is the first “second cumulant” in any nontrivial integrable quantum model, but we need infinitely many more...

- Proving emergence of hydrodynamics using integrability techniques?

- Generalizing to time-dependent external field, GHD of classical field theory, ... (works in progress by various people).