Fractional quantum Hall effect
Conformal Field Theory
and Matrix Product States

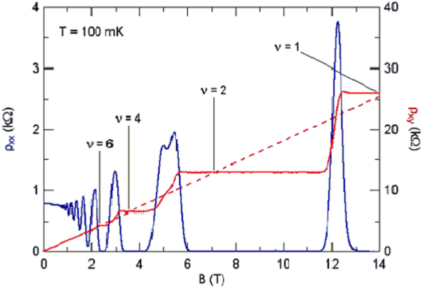
Benoit Estienne (LPTHE, Paris)

Exact methods in low dimensional statistical physics

Cargèse

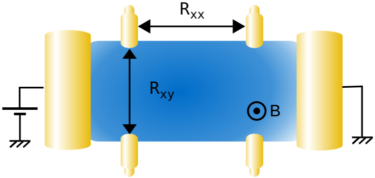
1. Integer quantum Hall effect
   - Landau levels
2. Fractional quantum Hall effect
   - Laughlin state
3. The chiral boson
   - and the Laughlin state
4. Conformal field theory...
   - as an ansatz for FQH states
5. Matrix Product States

Integer quantum Hall effect



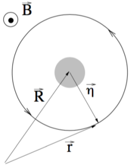
Landau levels

Classical Hall effect



Hall effect : a 2D electron gas in a perpendicular magnetic field.

$\Rightarrow$ **current $\perp$ voltage**
& transverse resistivity $\rho_{xy} \propto B$



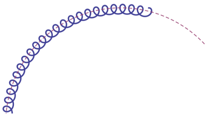
$$\vec{r} = \vec{R} + \vec{\eta}$$

R_μ : guiding center
No electric field :

$$\dot{R}_\mu = 0$$

Cyclotron motion at frequency

$$\omega_c = \frac{|eB|}{m}$$



With electric field

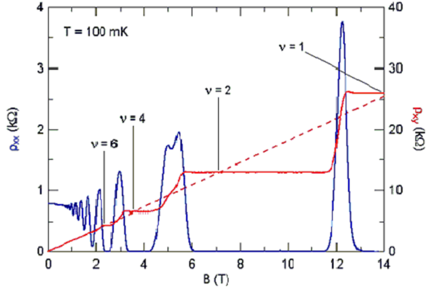
$$\omega_c \dot{R}_\mu = \epsilon_{\mu\nu} E_\nu$$

Electron classical equation of motion :

- $\vec{\eta}$: fast cyclotron motion (ω_c)
- $\vec{R}$: slow drift along equipotentials

Integer Quantum Hall effect (IQHE)

At low temperature and high magnetic field however :
ρ_{xy} is no longer linear in B (plateaux) !



IQHE : von Klitzing (1980)

Quantized Hall conductance

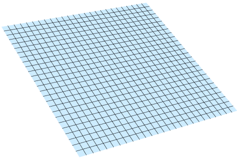
$$\sigma_{xy} = \nu \frac{e^2}{h}$$

ν is an integer up to $O(10^{-9})$
Used in metrology

This is a manifestation of quantum mechanics on macroscopic scales ! !

A single electron in 2D and in a $\perp$ magnetic field B.

Uniform $\perp$ magnetic field : gauge choice



$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2, \qquad \vec{A} = \frac{B}{2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

$$H = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} + \frac{eB}{2}y\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{eB}{2}x\right)^2$$

- **energy scale** cyclotron frequency $\omega_c = \frac{|eB|}{m}$,
- **length scale** : magnetic length $l_B = \sqrt{\frac{\hbar}{|eB|}}$

$$H = \frac{1}{2}\hbar\omega_c\left[\left(-il_B\frac{\partial}{\partial x} + \frac{y}{2l_B}\right)^2 + \left(-il_B\frac{\partial}{\partial y} - \frac{x}{2l_B}\right)^2\right]$$

Landau levels

In (dimensionless) complex coordinate $z = (x + iy)/l_B$, and setting

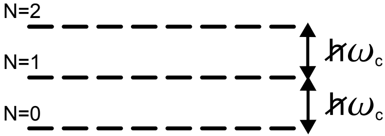
$$a = \sqrt{2}\left(\frac{\partial}{\partial \bar{z}} + \frac{z}{2}\right), \qquad a^\dagger = -\sqrt{2}\left(\frac{\partial}{\partial z} - \frac{\bar{z}}{2}\right)$$

Familiar form of the Hamiltonian

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2}\right) \qquad\qquad [a, a^\dagger] = 1$$

$(N+1)^{\mathrm{th}}$ Landau level :

$$E_N = \hbar\omega_c \left(N + \frac{1}{2}\right)$$



Discrete spectrum, large **degeneracy**
(translation invariance/guiding center).

Lowest Landau Level ($N = 0$)

Since $a = \sqrt{2}\left(\frac{\partial}{\partial \bar{z}} + \frac{z}{2}\right)$, ground states are of the form

$$\Psi(z, \bar{z}) = f(z)\, e^{-\frac{z\bar{z}}{4 l_B^2}}$$

with $f(z)$ is any holomorphic function ($\partial_{\bar{z}} f = 0$).

$$\Rightarrow \textbf{chirality} : (x, y) \to z = (x + iy)$$

Ground states, a.k.a. Lowest Landau level (LLL) states

$$\Psi(x, y) = f(x + iy)\, e^{-(x^2+y^2)/4 l_B^2}$$

Projection to the LLL : x and y no longer commute $[\hat{x}, \hat{y}] = i\, l_B^2$

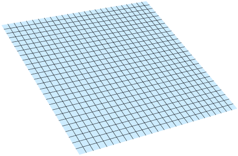
$$\Delta_x\, \Delta_y \geq l_B^2/2$$

$\Rightarrow$ **each electron occupies an area** $2\pi l_B^2$

magnetic flux through this area = quantum of flux $\Phi = h/e$

LLL degeneracy $\sim$ number N_Φ of flux quanta through the surface

Magnetic translations



translation invariance : $\vec{x}$ and $\vec{x}+\vec{u}$ are equivalent

up to a gauge transformation (since $\vec{A}=\vec{A}(x,y)$)

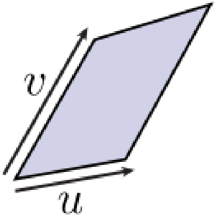
$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \qquad \text{and} \qquad \Psi \to \tilde{\Psi} = e^{i\Lambda}\Psi$$

Magnetic translations

$T(\vec{u}) = \exp[\vec{u}.(\vec{\nabla} - i\vec{A}) - i\vec{u}\times\vec{r}]$

Aharonov-Bohm effect :

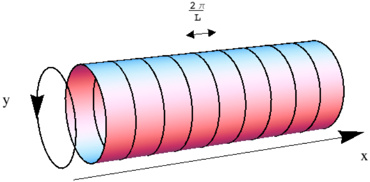
$$T_{\vec{u}}T_{\vec{v}} = e^{i\frac{\vec{u}\wedge\vec{v}}{l_B^2}}T_{\vec{v}}T_{\vec{u}}$$



Infinitesimal generators of translations commute with H, but

$$[t_x, t_y] = -i \neq 0$$

Cylinder with perimeter L (we identify $y \equiv y + L$)



Natural gauge choice : $\vec{A} = B \begin{pmatrix} 0 \\ x \end{pmatrix}$

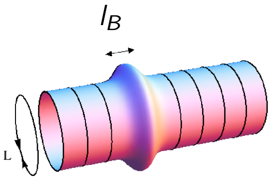
$$t_y|\Psi_{k_y}\rangle = k_y|\Psi_{k_y}\rangle, \qquad k_y = \frac{2\pi n}{L}$$

LLL $\qquad \Psi_{k_y}(x,y) = e^{iyk_y} e^{-\frac{(x-l_B^2 k_y)^2}{2l_B^2}}$

Momentum k_y and position x are locked :

$$x \sim l_B^2 k_y$$

- $[\hat{x}, \hat{y}] = i l_B^2$ implies that $\hbar \hat{x} = l_B^2 \hat{p}_y$.
- localized in $\hat{x}$ and delocalized in $\hat{y}$
- the interorbital distance is $\frac{2\pi}{L} l_B^2$



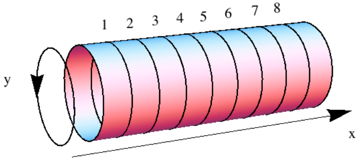
Density profile of the LLL orbital $\Psi_{k_y}(x,y)$.

Projection to the LLL : dimensional reduction

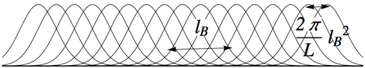
Projection to the LLL : x and y no longer commute $[\hat{x}, \hat{y}] = i\, l_B^2$ (link with non-commutative geometry).

4 dimensional phase space $\Rightarrow$ 2 dimensional phase space

A **basis** of LLL states



looks like a one-dimensional chain

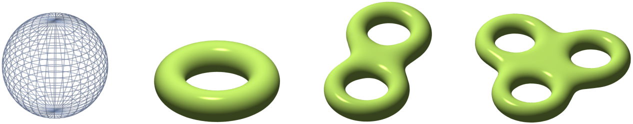


But !

Physical short range interactions become long range in this description (distance of order l_B means $\sim L/l_B$ sites).

Landau problem on arbitrary surfaces

Lowest Landau Level on arbitrary surface :



The magnetic flux has to be quantized $\int d^2x\, B = N_\Phi \frac{h}{e}$, with N_Φ integer.

The ground state degeneracy on a surface of genus g is

$$N_\Phi + (1 - g)$$

provided N_ϕ is not too small, namely $N_\phi > 2g - 2$.

- it depends on the topology (genus).
- it does NOT depend on the geometry (metric)

For instance on the torus : boundary conditions

- The (flat) torus is

$$\mathbb{T}^2 = \mathbb{C}/(L_1 + e^{i\theta} L_2)\mathbb{Z}$$

- Boundary conditions

$$T(\vec{L}_\alpha) \left|\Psi\right\rangle = e^{i\phi_\alpha} \left|\Psi\right\rangle, \qquad \alpha = 1, 2$$

ϕ_α : solenoid fluxes passing through the torus cycles.

- Consistency of two b.c. requires quantized magnetic field

$$[T(\vec{L}_1), T(\vec{L}_2)] = 0 \qquad \Leftrightarrow \qquad |\vec{L}_1 \times \vec{L}_2| = 2\pi N_\Phi, \qquad N_\Phi \in \mathbb{Z}$$

- discrete translations $T(\vec{u})$ with

$$\vec{u} = \frac{n}{N_\Phi}\vec{L}_1 + \frac{m}{N_\Phi}\vec{L}_2$$

Let's work in the Landau gauge $\vec{A} = (-y, 0)$.

$$\Psi(x, y) = e^{-y^2/2} f(w)$$

where f has boundary conditions

$$\boxed{f(w + L_1) = e^{i\phi_1} f(w), \qquad f(w + e^{i\theta} L_2) = e^{i\phi_2} e^{-i2\pi N_\Phi \left(\frac{w}{L_1} + \frac{\tau}{2}\right)} f(w)}$$

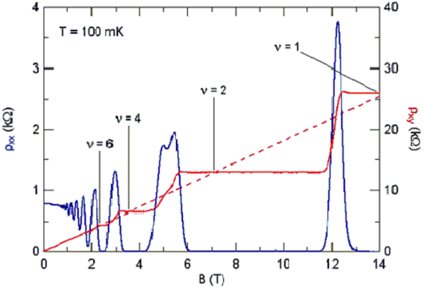
(holomophic sections of degree N_ϕ)

$$\text{where} \qquad N_\Phi = \frac{L_1 L_2 \sin\theta}{2\pi}, \qquad \tau = \frac{L_2}{L_1} e^{i\theta}$$

The number of independent solutions is N_ϕ, for instance

$$f_m(w) = \frac{1}{\sqrt{L_1\sqrt{\pi}}} \vartheta \left[\begin{array}{c} \frac{m}{N_\Phi} + \frac{\phi_1}{2\pi N_\Phi} \\ -\frac{\phi_2}{2\pi} \end{array} \right] \left(N_\Phi \frac{w}{L_1} \middle| N_\Phi \tau \right)$$

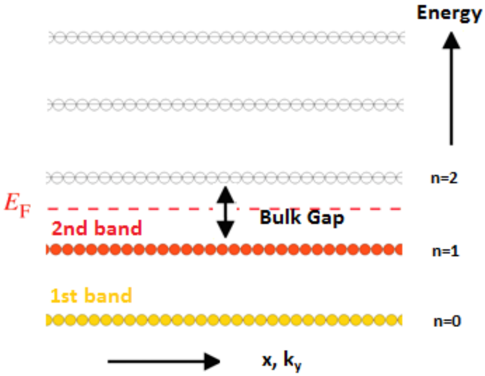
Integer quantum Hall effect

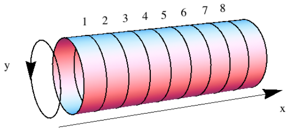


a band insulator

The IQHE : bulk insulator

Cartoon picture : no interactions, no disorder

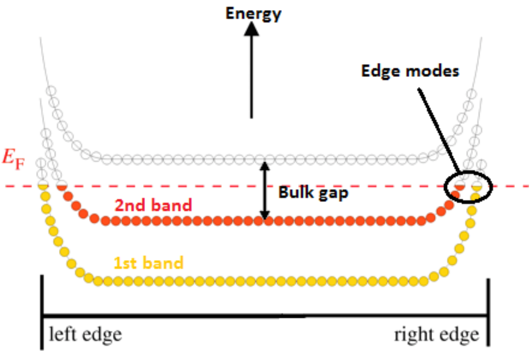




- Landau Levels = flat bands
- Integer filling with fermions
  $\Rightarrow$ **Bulk insulator.**

How come we have $I \propto V$ then ?

The IQHE : conducting edges



$\Rightarrow$ **Conducting edges**
each channel contributes e^2/h to the Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

Chiral (and therefore protected) massless edges

Topological insulator

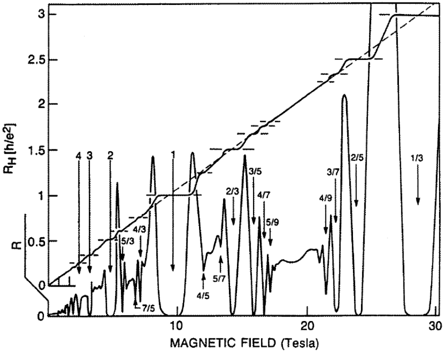
This quantization is insensitive to disorder or strong periodic potential :

topological invariant : the Chern number

Disclaimer : this is just a cartoon picture. Does not explain plateaux.

Fractional filling

the many-body problem



FQHE trial wavefunctions

Fractional filling : the role of electron-electron interactions

Partially filled band $\Rightarrow$ conventional **metallic (i.e. gapless) bulk**.

Yet, experimentally, emergence of exotic gapped states :

- insulating bulk,
- metallic chiral edge modes,
- bulk excitations with fractional charges.

How is this possible ? thanks to electron-electron interaction

Technical problem : the interaction cannot be treated perturbatively.

N fermions in N_Φ states $\Rightarrow$ **macroscopic degeneracy** $\binom{N_\phi}{N}$.

So what can we do ?

Numerics (*e.g.* exact diagonalization), effective field theories (theories of anyons), model wavefunctions.

What are model states/wave functions ?

• Typically **an idealized hamiltonian/interaction**
for which the ground state, quasihole, and edge excitations can be found exactly (as zero energy states)

• They are highly fine tuned and **non-generic** similar to integrable vs generic systems
(for instance they minimize quantum entanglement)

• A model state is merely a **representative of a universality class** characterised by some quantum numbers/symmetries (**topological order**).

The mother of all trial wave functions

The $\nu = 1/3$ Laughlin state.

filling fraction $\nu = 1/3$ + short range model interaction

$\Rightarrow$ exact ground-state :

$$\Psi_{\frac{1}{3}}(z_1, \cdots, z_N) = \prod_{i<j} (z_i - z_j)^3 \, e^{-\sum_i |z_i|^2/4l_B^2}$$

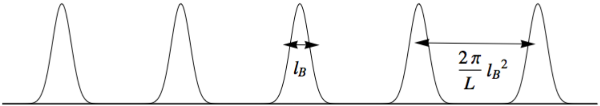
The model interaction is the short range part of Coulomb.

Extremely high overlap with Coulomb interaction !
(obtained by exact diagonalization)

First hints of a topological phase :

- excitations with fractional charge $e/3$
- topology dependent ground state degeneracy : 3^g exact ground states.

Cartoon picture : thin cylinder limit ($L \ll l_B$)



Very small cylinder perimeter L : **LLL orbitals no longer overlap**
1d problem

Laughlin's Hamiltonian $\rightarrow$ Haldane's exclusion statistics
no more than 1 particle in three orbitals

At filling fraction $\nu = 1/3$, we get three possible states

$$|\Psi_1\rangle = |\cdots 1\,0\,0\,1\,0\,0\,1\,0\,0 \cdots\rangle$$
$$|\Psi_2\rangle = |\cdots 0\,1\,0\,0\,1\,0\,0\,1\,0 \cdots\rangle$$
$$|\Psi_3\rangle = |\cdots 0\,0\,1\,0\,0\,1\,0\,0\,1 \cdots\rangle$$

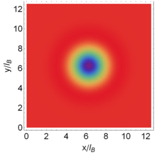
3-fold degenerate ground state on the cylinder (and torus).

Bulk excitations/defects : anyons

Adiabatic insertion of a flux quantum at position w
creates a hole in the electronic liquid :

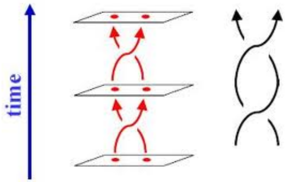
$$\Psi_w = \prod_i (w - z_i) \prod_{i<j} (z_i - z_j)^3$$

Cartoon picture : $|\cdots 1\,0\,0\,1\,0\,0\,0\,1\,0\,0 \cdots\rangle$



Electronic density around a quasihole
(N. Regnault)

fractionalization : the missing electronic charge is $e/3$
these excitations are called **quasi-holes**.



under adiabatic exchange of two quasi-holes

$\Rightarrow$ phase $e^{2i\pi/3}$
non trivial braiding !

$\Rightarrow$ **quasi-holes = abelian anyons**

Anyons of the $\nu = 1/m$ Laughlin state

- There are m **types of quasi-holes/anyons**

$$\Psi_w = \prod_i (w - z_i)^a \prod_{i<j}(z_i - z_j)^m, \qquad a = 0, \cdots, m-1$$

Indeed a is defined mod m ($a = m$ is simply a hole, *i.e.* a missing electron).

- Two anyons at positions w_1 and w_2 can be fused ($w_1 \to w_2$)

$$\Psi_{w_1,w_2} = \prod_i (w_1 - z_i)^a (w_2 - z_i)^b \prod_{i<j}(z_i - z_j)^m$$

fusion rules : $$a \times b = \sum_c N_{ab}^c c, \qquad N_{ab}^c = \delta_{a+b,c \mod m}$$

- Braiding anyons of type a and b gives a phase $e^{\frac{2i\pi}{m}ab}$ (plasma argument).

Laughlin $\nu = 1/m$ on the torus

The vanishing properties as $z_i \to z_j$ dictate

$$\Psi(z_1, \cdots, z_N) = F(Z) \prod_{i<j} \theta_1 \left(\frac{z_i - z_j}{L_1} \middle| \tau \right)^m$$

where $Z = \sum_i z_i$ is the center of mass. We recover the correct b.c. iff

$$F(Z + L_1) = (-1)^{(N-1)m} e^{i\phi_1} F(Z),$$

$$F(Z + L_2 e^{i\theta}) = (-1)^{(N-1)m} e^{i\phi_2} e^{-i2\pi m\left(\frac{Z}{L_1} + \frac{\tau}{2}\right)} F(Z)$$

m ground-states on the torus

$$\Psi_a(z_1, \cdots, z_N) = \vartheta \begin{bmatrix} \frac{a}{m} + \frac{\phi_1}{2\pi m} + \frac{N-1}{2} \\ -\frac{\phi_2}{2\pi} - \frac{m(N-1)}{2} \end{bmatrix} \left(m \frac{Z}{L_1} \middle| m\tau \right) \prod_{i<j} \theta_1 \left(\frac{z_i - z_j}{L_1} \middle| \tau \right)^m$$

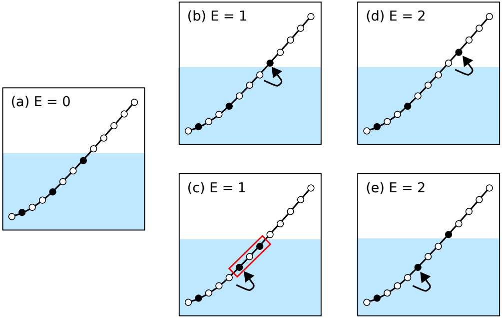
Metallic boundary : massless edge modes

$$\Psi_u = P_u(z_1, \cdots, z_N) \prod_{i<j} (z_i - z_j)^3$$

where P_u is any symmetric, homogeneous polynomial.

Cartoon picture : no more than 1 electron in 3 orbitals.

- dispersion relation : $E \propto P$
  chiral and **gapless** edge
- Number of edge states :
  - $E = 0$: 1 state
  - $E = 1$: 1 state
  - $E = 2$: 2 states
  - $E = 3$: 3 states
  - $E = 4$: 5 states
  - $E = 5$: 7 states
  - $\cdots$



(cartoon picture)

spectrum of a massless chiral boson.

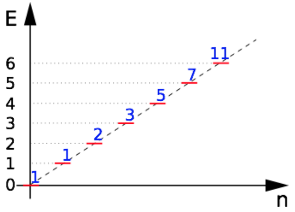
Metallic boundary : massless edge modes

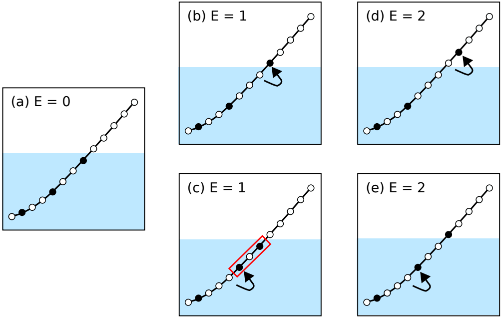
$$\Psi_u = P_u \prod_{i<j} (z_i - z_j)^3$$

where P_u is any symmetric, homogeneous polynomial.

Cartoon picture : no more than 1 electron in 3 orbitals.

- dispersion relation : $E \propto P$
  chiral and **gapless** edge
- Number of edge states :





(cartoon picture)

spectrum of massless chiral boson.

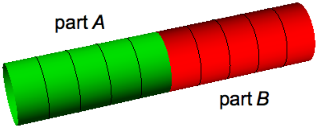
Entanglement entropy

Cut the system in two parts A and B (the boundary has length L)

The **entanglement entropy** is

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$$

with ρ_A the reduced density matrix.

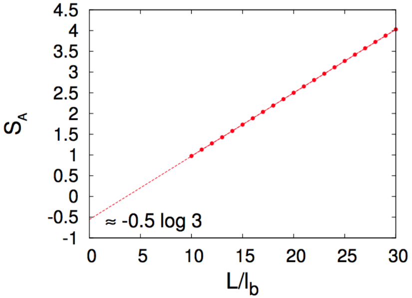


For a topological phase :

$$S_A \sim \alpha L - \log \mathcal{D}$$

where $\mathcal{D}$ is the quantum dimension.

For $\nu = 1/3$ Laughlin : $\mathcal{D} = \sqrt{3}$



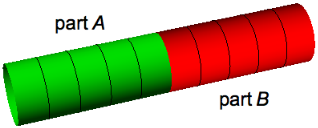
Entanglement entropy of the $\nu = 1/3$ Laughlin state as a function of the cylinder perimeter L (N. Regnault)

Entanglement spectrum

Schmidt decomposition

$$|\Psi\rangle = \sum_{\alpha} \exp(-\xi_\alpha/2)\, |{\color{green}A}, \alpha\rangle \otimes |{\color{red}B}, \alpha\rangle$$

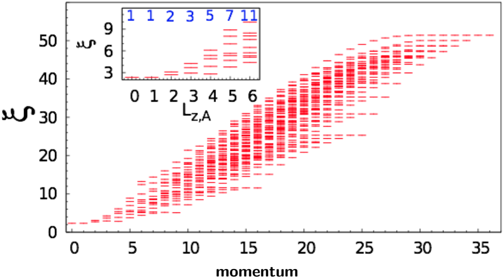
$$\rho_a = \sum_{\alpha} \exp(-\xi_\alpha)\, |A, \alpha\rangle \langle A, \alpha|$$



Entanglement spectrum

Li and Haldane (2008) :
spectrum of $\xi = -\log \rho_A$
(plot ξ vs momentum)

$\Rightarrow$ **Reproduces the physical edge spectrum !**



Entanglement spectrum of the $\nu = 1/3$ Laughlin state on the sphere

Chiral boson and Laughlin

using the edge theory to describe the bulk

The free boson a.k.a. U(1) CFT

Massless gaussian field in $1+1$ dimensions

$$S = \int \mathrm{d}^2 z \, \partial\phi \, \bar{\partial}\phi$$

The mode decomposition of the **chiral** free boson is

$$\phi(z) = \mathbf{\Phi_0} - i\mathbf{a_0}\log(z) + i\sum_{n\neq 0}\frac{1}{n}\mathbf{a_n} z^{-n}$$

$$[\mathbf{a_n}, \mathbf{a_m}] = n\delta_{n+m,0}, \qquad [\mathbf{\Phi_0}, \mathbf{a_0}] = i$$

$U(1)$ symmetry : $\phi(z) \to \phi(z) + \theta$

conserved current :

$$J(z) = i\partial\phi(z) = \sum_n a_n z^{-n-1}$$

Vertex operators :

$$V_Q(z) =: e^{iQ\varphi(z)} := \exp\left(Q\sum_{n>0}\frac{a_{-n}}{n}z^n\right)\exp\left(-Q\sum_{n>0}\frac{a_n}{n}z^{-n}\right)e^{iQ\varphi_0}z^{Qa_0}$$

Primary states/ vacua $|Q\rangle$ are defined by their U(1) charge $Q \in \frac{1}{\sqrt{3}}\mathbb{Z}$

$$a_0|Q\rangle = Q|Q\rangle, \qquad a_n|Q\rangle = 0 \text{ for } n > 0$$

The Hilbert space is simply a Fock space

Descendants are obtained with the lowering operators $a_n^\dagger = a_{-n}$, $n > 0$

- $\Delta E = 0$: 1 state : $|Q\rangle$
- $\Delta E = 1$: 1 state : $a_{-1}\,|Q\rangle$
- $\Delta E = 2$: 2 states : $a_{-1}^2\,|Q\rangle$, $a_{-2}\,|Q\rangle$
- $\Delta E = 3$: 3 states : $a_{-1}^3\,|Q\rangle$, $a_{-2}a_{-1}|Q\rangle$, $a_{-3}\,|Q\rangle$
- $\Delta E = 4$: 5 states : $a_{-1}^4\,|Q\rangle$, $a_{-2}a_{-1}^2\,|Q\rangle$, $a_{-2}^2\,|Q\rangle$, $a_{-3}a_{-1}\,|Q\rangle$, $a_{-4}\,|Q\rangle$
- $\Delta E = 5$: 7 states : $\cdots$

The Laughlin state written in terms of a $U(1)$ CFT

Ground state wavefunction

$$\prod_{i<j}(z_i - z_j)^m = \langle 0|\mathcal{O}_{\text{b.c.}} V(z_1)\cdots V(z_N)|0\rangle, \qquad V(z) =: e^{i\sqrt{m}\varphi(z)} :$$

where $\mathcal{O}_{\text{b.c.}} = e^{-i\sqrt{m}N\varphi_0}$ is just a neutralizing background charge.

Bulk excitations

Wavefunction for p quasiholes

$$\langle \mathcal{O}_{\text{b.c.}} V_{\text{qh}}(w_1)\cdots V_{\text{qh}}(w_p) V(z_1)\cdots V(z_N)\rangle$$

with

$$V_{\text{qh}}(w) =: e^{\frac{i}{\sqrt{m}}\varphi(w)} :$$

Edge excitations

$$\Psi_u = \langle u|\mathcal{O}_{\text{b.c.}} V(z_1)\cdots V(z_N)|0\rangle$$

- edge mode = CFT descendant
- we recover $1, 1, 2, 3, 5, 7, \cdots$

Laughlin's anyons

- The Hilbert space splits into m anyon sectors

$$\mathcal{H} = \bigoplus_{a=0}^{m-1} \mathcal{H}_a \qquad \mathcal{H}_a = \{\text{states with } \sqrt{m}Q = a \text{ mod } m\}$$

- Anyon of type a :

$$\Psi_w = \prod_i (w - z_i)^a \prod_{i<j} (z_i - z_j)^m, \qquad \Phi_a(w) =: e^{\frac{ia}{\sqrt{m}}\varphi(w)} :$$

- Fusion rules

$$\text{fusion rules :} \qquad a \times b = \sum_c N_{ab}^c c, \qquad N_{ab}^c = \delta_{a+b,c}$$

- m torus conformal blocks

$$\Psi_a(z_1, \cdots, z_N) = \mathrm{Tr}_{\mathcal{H}_a}\left(e^{i2\pi\tau L_0 - i\sqrt{\nu}N_\Phi\varphi_0} V(z_1)\cdots V(z_N)\right)$$

Laughlin's edge modes

$$\Psi_u = \langle u|\mathcal{O}_{\text{b.c.}} V(z_1)\cdots V(z_N)|0\rangle = P_u(z_1,\cdots,z_N) \prod_{i<j}(z_i - z_j)^m$$

<u>At level 0</u>

- $\langle 0|\mathcal{O}_{\text{b.c.}} V(z_1)\cdots V(z_N)|0\rangle = 1 \prod_{i<j}(z_i - z_j)^m$

<u>At level 1</u>

- $\langle 0|a_1\mathcal{O}_{\text{b.c.}} V(z_1)\cdots V(z_N)|0\rangle \propto \sum_i z_i \prod_{i<j}(z_i - z_j)^m$

<u>At level 2</u>

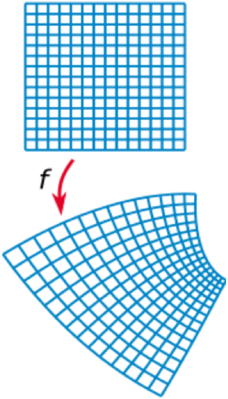
- $\langle 0|a_1^2\mathcal{O}_{\text{b.c.}} V(z_1)\cdots V(z_N)|0\rangle \propto \left(\sum_i z_i\right)^2 \prod_{i<j}(z_i - z_j)^m$
- $\langle 0|a_2\mathcal{O}_{\text{b.c.}} V(z_1)\cdots V(z_N)|0\rangle \propto \left(\sum_i z_i^2\right) \prod_{i<j}(z_i - z_j)^m$

<u>At level 3</u>

- $\cdots$

One-to-one map between edge modes and CFT states

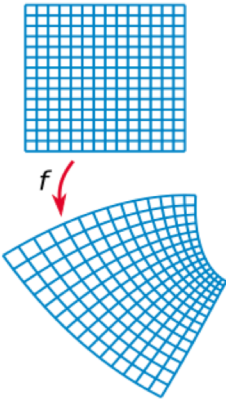
Conformal field theories (CFT)



CFT = Quantum Field Theory + conformal invariance

conformal = angle preserving

$$z \to f(z) = \sum_n f_n z^n$$



Symmetry generators $\{L_n, \; n \in \mathbb{Z}\}$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$$

In particular L_0 generates dilatations.

conformal invariance comes from **criticality**.

- 2D classical stat mech models : **scale invariance**
- 1+1 quantum models : **masslessness**

FQH model wave-function from CFT

Moore and Read (1990) proposed to write
FQH model wavefunctions as **CFT conformal blocks**

$$\Psi_\alpha(z_1, \cdots, z_N) = \langle V(z_1) \cdots V(z_N) \Phi_{a_1}(w_1) \cdots \Phi_{a_p}(w_p) \rangle_\alpha e^{-\frac{1}{4 l_B^2} \sum_i |z_i|^2}$$

with quasihole of type a_i at position w_i. α labels the different conformal blocks.

Underlying idea :

Universality classes of FQH states are to be distinguished solely by

- the quantum numbers of the ground state and excitations
- by the braiding and fusion algebras ;

in other words by the corresponding CFT.
This construction yields consistent anyon models (a.k.a. modular tensor categories)

Constraints on the CFT

- $\langle V(z_1)\cdots V(z_N)\Phi_{a_1}(w_1)\cdots\Phi_{a_p}(w_p)\rangle_\alpha$ must be (anti-)symmetric in z_i

$\Rightarrow$ $V(z)$ must be **bosonic** or **fermionic**.

- $\langle V(z_1)\cdots V(z_N)\Phi_{a_1}(w_1)\cdots\Phi_{a_p}(w_p)\rangle_\alpha$ must be a **polynomial** in z_i

$V(z)$ must be mutually local w.r.t. all fields
$\Rightarrow$ $V(z)$ is a chiral current generating an **extended chiral algebra**

- Charge conservation and density excitations

$\Rightarrow$ the extended chiral algebra must contain a U(1) current $J = i\partial\varphi$
$V(z)$ is assumed to be of the form $V(z) = \Psi(z) \otimes :\exp\left(i\frac{1}{\sqrt{\nu}}\varphi(z)\right):$

- Finitely many anyon types/ finite ground-state degeneracy on the torus

$\Rightarrow$ the CFT must be **rational** *w.r.t.* the extended chiral algebra

A few examples

- U(1) $\quad$ $\nu = 1/m$ **Laughlin state** $\quad$ $V(z) =: e^{i\sqrt{m}\varphi(z)}:$

$$\Psi_{\text{ground-state}} = \prod_{i<j} (z_i - z_j)^m$$

- $\mathrm{SU}(2)_2$ $\quad$ (bosonic) **Moore-Read state** $V(z) = \Psi(z) \otimes : e^{i\varphi(z)} :$

$$\Psi_{\text{ground-state}} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)$$

- $\mathrm{SU}(2)_k$ $\quad$ (bosonic) **Read-Rezayi state**

$$V(z) = J^+(z) = \Psi_1(z) \otimes : e^{i\sqrt{2/k}\varphi(z)} :$$

$$\Psi_{\text{ground-state}} = \text{some complicated (Jack) polynomial}$$

$\nu = 1$ bosonic Moore-Read state ($\mathrm{SU}(2)_2 = \mathbb{Z}_2 \otimes U(1)$)

- 3 anyon types : $0, 1, \sigma$, with corresponding fields
  - a trivial quasi-hole $\Phi_0 = 1$ (spin 0)
  - an abelian quasi-hole $\Phi_1 =: e^{i\sqrt{\varphi}}$: (spin 1)
  - a non-abelian quasi-hole $\Phi_\sigma = \sigma \otimes : e^{i\frac{1}{2}\sqrt{\varphi}}$: (spin 1/2)
- fusion rules : $0 \times a = a$, $1 \times \sigma = \sigma$ and $\sigma \times \sigma = 0 + 1$
- Expected **non-abelian braiding !**

$$\langle \sigma(\infty)\sigma(1)\sigma(w)\sigma(0)\rangle_\pm = \frac{(1 \pm \sqrt{1-w})^{1/2}}{\sqrt{2}(w(1-w))^{1/8}}$$

- Exclusion principle : no more than 2 particles in 2 consecutive orbitals
- 3 ground-states on the torus : $\cdots 2020 \cdots$, $\cdots 0202 \cdots$ and $\cdots 1111 \cdots$

What about the torus ?

Holomorphic is no longer sufficient, we need the correct b.c.

$$f(z+1) = f(z), \qquad f(z+\tau) = e^{-i2\pi N_\Phi\left(z+\frac{\tau}{2}\right)} f(z)$$

Answer (for bosons) :

$$\Psi_a(z_1, \cdots, z_N) = \mathrm{Tr}_{\mathcal{H}_a}\left(e^{i2\pi\tau L_0 - i\sqrt{\nu}N_\Phi\varphi_0} V(z_1)\cdots V(z_N)\right)$$

easy to check using (setting $q = e^{2i\pi\tau}$ and $w = e^{2i\pi z}$)

$$q^{L_0} V(w) = V(qw) q^{L_0}, \qquad \beta^{\sqrt{\nu}a_0} V(w) = \beta V(w) \beta^{\sqrt{\nu}a_0}$$

$$e^{-i\sqrt{\nu}N_\Phi\varphi_0} V(w) = w^{N_\Phi} V(w) e^{-i\sqrt{\nu}N_\Phi\varphi_0}$$

RCFT $\rightarrow$ finitely many conformal blocks on the torus

degeneracy = number of anyon types

Topological sectors $\leftrightarrow$ primary fields Φ_a

Model wavefunctions from CFT

Extrapolating the **thermodynamic limit** of these model states is difficult.

- Gapped ?
- Well-defined quasi-holes ?
- Non-Abelian braiding ?
- Area law for the entanglement entropy ?
- Entanglement spectrum ?
- Quantum dimensions ?
- etc...

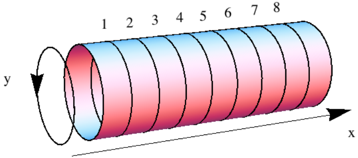
The natural conjecture is that they are described by the **anyon model** (TQFT) corresponding to the underlying CFT.

Matrix Product State (MPS)

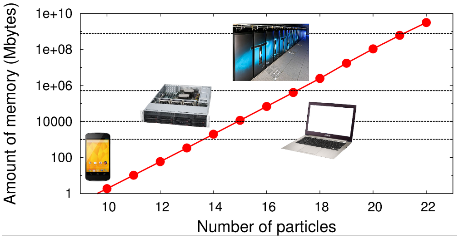
Limitations of exact diagonalizations and model wf

$\rightarrow$ decomposition of a state $|\Psi\rangle$ on a convenient occupation basis

$$|\Psi\rangle = \sum_{\{m_i\}} c_{\{m_i\}} \, |m_1, ..., m_{N_\Phi}\rangle$$



What is the amount of memory needed to store the Laughlin state ?



Can't store more than 21 particles !

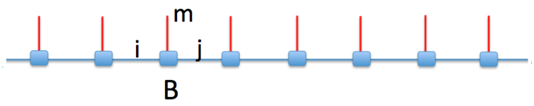
Matrix Product State : more compact and computationally friendly

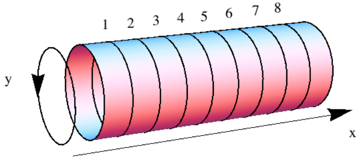
Matrix Product States

$$|\Psi\rangle = \sum_{\{m_i\}} c_{\{m_i\}} |m_1, ..., m_{N_\Phi}\rangle$$

replaced by

$$|\Psi\rangle = \sum_{\{m_i\}} \left(\langle u| A^{[m_1]} \cdots A^{[m_n]} |v\rangle \right) |m_1, ..., m_n\rangle$$

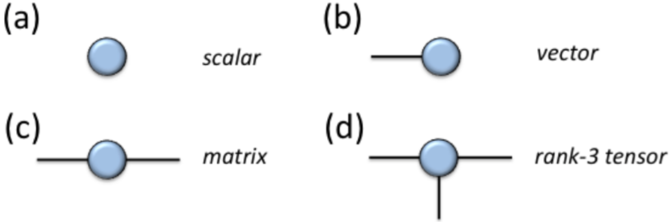




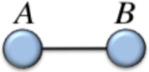
Why is this formalism interesting ?

Many quantities (correlation functions, entanglement spectrum, . . .) can be computed in the (relatively small) auxiliary space.

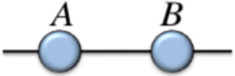
Tensor Networks diagrams (taken from Orus, arXiv :1306.2164)



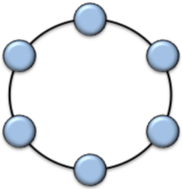
Contraction of indices = gluing links



(a) Scalar product



(b) Matrix product



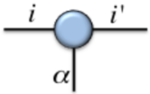
(c) Trace of the product of 6 matrices

MPS transfer matrix

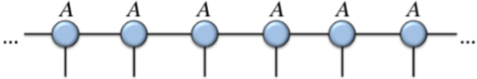
(taken from Orus, arXiv :1306.2164)

$$|\Psi\rangle = \sum_{\{m_i\}} \left(\langle u | A^{[m_1]} \dots A^{[m_n]} | v \rangle \right) \; | m_1, ..., m_n \rangle$$

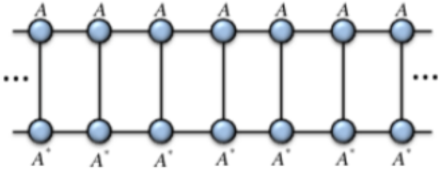
MPS matrices $A^{[\alpha]}_{i\,i'}$



$\langle u | A^{[m_1]} \dots A^{[m_n]} | v \rangle$



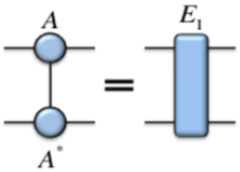
overlap $\langle \Psi | \Psi \rangle$



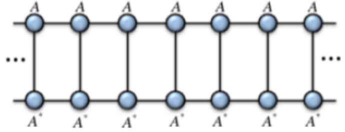
MPS transfer matrix

Everything can be computed in terms of

$$E_I = \sum_m A^{[m]} \otimes \bar{A}^{[m]}$$



For instance the overlap for N sites



now reads

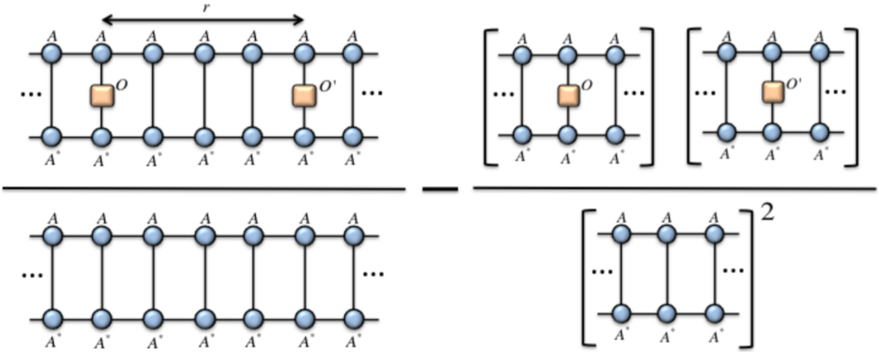
$$\langle \Psi | \Psi \rangle = \langle u, u | E_I^N | v, v \rangle \sim \lambda_1^N$$

We can work on the infinite cylinder !

for the FQHE : this means infinitely many electrons . . .

Correlation functions (taken from Orus, arXiv :1306.2164)

$$C(r) = \langle O(r)O'(0)\rangle - \langle O\rangle\langle O'\rangle$$



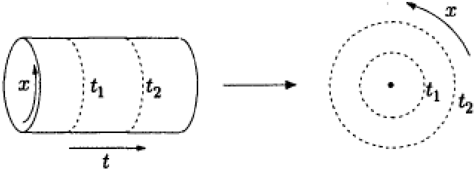
On the infinite cylinder :

$$C(r) = \langle GS|\,\mathcal{O}' E^r \mathcal{O}\,|GS\rangle - \langle GS|\,\mathcal{O}'\,|GS\rangle\,\langle GS|\,\mathcal{O}\,|GS\rangle \sim \left(\frac{\lambda_2}{\lambda_1}\right)^r$$

Where does this MPS come from ?

CFT : operator picture

From the $1 + 1$D perspective : cylinder of perimeter L.



$$\langle \phi_1(x_1, t_1)\phi_2(x_2, t_2) \cdots \phi_n(x_n, t_n)\rangle =$$
$$\langle 0|\hat{\phi}_n(x_n) \cdots \hat{\phi}_3(x_3)e^{-\hat{H}(t_3-t_2)}\hat{\phi}_2(x_2)e^{-\hat{H}(t_2-t_1)}\hat{\phi}_1(x_1)e^{-\hat{H}t_1}|0\rangle$$

Dilatations on the plane become translations in the *time* direction :

$$\hat{H} \sim \frac{2\pi}{L} L_0$$

The CFT ansatz $\Psi(z_1, \cdots, z_n) = \langle u | V(z_1) \cdots V(z_n) | v \rangle$
is a continuous MPS

Dubail, Read, Rezayi (2012)

Translation invariant MPS

$$|\Psi\rangle = \sum_{\{m_i\}} \left(\langle u | \, B^{[m_1]} B^{[m_2]} \dots B^{[m_n]} \, | v \rangle \right) | m_1 \cdots m_n \rangle$$

Zaletel, Mong (2012)

- the matrices $B^{[m]}$ are operators in the underlying CFT
- the auxiliary space is the (infinite dimensional) CFT Hilbert space ...
- ... which can be truncated while keeping arbitrary large precision

Starting from a model wavefunction given by a CFT correlator

$$\Psi(z_1, \cdots, z_N) = \langle u | \mathcal{O}_{\mathrm{b.c.}} V(z_1) \cdots V(z_N) | v \rangle$$

and expanding $V(z) = \sum_n V_{-n} z^n$, one finds (up to orbital normalization)

$$c_{(m_1, \cdots, m_n)} = \langle u | \, \mathcal{O}_{\mathrm{b.c.}} \frac{1}{\sqrt{m_n!}} V_{-n}^{m_n} \cdots \frac{1}{\sqrt{m_2!}} V_{-2}^{m_2} \frac{1}{\sqrt{m_1!}} V_{-1}^{m_1} \, | v \rangle$$

This is a site/orbital dependent MPS

$$c_{(m_1, \cdots, m_n)} = \langle u | \, \mathcal{O}_{\mathrm{b.c.}} B^{[m_n]}(n) \cdots B^{[m_2]}(2) B^{[m_1]}(1) \, | v \rangle$$

with matrices at site/orbital j (including orbital normalization)

$$\boxed{B^{[m]}(j) = \frac{e^{\left(\frac{2\pi}{L} j\right)^2}}{\sqrt{m!}} \left(V_{-j}\right)^m}$$

Translation invariant MPS

A relation of the form $B^{[m]}(j) = U^{-1} B^{[m]}(j-1) U$ yields

$$B^{[m]}(j) = U^{-j} B^{[m]}(0) U^{j}$$

and then

$$B^{[m_n]}(n) \cdots B^{[m_1]}(1) = U^{-n} \times B^{[m_n]}(0) U \cdots B^{[m_1]}(0) U$$

This is a **translation invariant MPS**, with matrices

$$A^{[m]} = B^{[m]}(0) U$$

Translation invariant MPS on the cylinder

Site independant MPS

$$B^{[m]}(j) = \frac{e^{\left(\frac{2\pi}{L} j\right)^2}}{\sqrt{m!}} \left(V_{-j}\right)^m \qquad \Rightarrow \qquad A^{[m]} = \frac{1}{\sqrt{m!}} \left(V_0\right)^m U$$

where U is the operator

$$U = e^{-\frac{2\pi}{L} H - i\sqrt{\nu}\varphi_0}$$

where

- φ_0 is the bosonic zero mode ($e^{-i\sqrt{\nu}\varphi_0}$ shifts the electric charge by ν)
- H is the cylinder Hamiltonian : $H = \frac{2\pi}{L} L_0$
- V_0 is the zero mode of $V(z)$

auxiliary space = CFT Hilbert space

infinite bond dimension :/

Truncation of the auxiliary space

The auxiliary space (i.e. the CFT Hilbert space) basis is graded by the conformal dimension Δ_α.

$$L_0 \, |\alpha\rangle = \Delta_\alpha |\alpha\rangle$$

But in the MPS matrices we have a term

$$A^{[m]} = \frac{1}{\sqrt{m!}} \, (V_0)^m \, e^{-\frac{i}{\sqrt{\nu}}\varphi_0} e^{-\left(\frac{2\pi}{L}\right)^2 L_0}$$

The conformal dimension provides a natural cut-off.
Truncation parameter P : keep only states with $\Delta_\alpha \leq P$.

- $P = 0$ recovers the thin-cylinder limit $|\cdots 100100100 \cdots\rangle$
- The correct 2d physics requires $L \gg$ bulk correlation length ζ
- For a cylinder perimeter L, we must take $P \sim L^2$
- Bond dimension $\chi \sim e^{\alpha L}$ $\cdots$ of course ! since $S_A \sim \alpha L$.

What about the torus ?

CFT ansatz : ground state $|\Psi\rangle_a$

$$\Psi_a(z_1, \cdots, z_N) = \mathrm{Tr}_a \left(e^{i2\pi\tau L_0 - i\sqrt{\nu}N\varphi_0} V(z_1) \cdots V(z_N) \right)$$

becomes

$$|\Psi\rangle_a = \sum_{\{m_i\}} \mathrm{Tr}_a \left(e^{i\pi(N-1)\sqrt{\nu}a_0} A^{[m_n]} \ldots A^{[m_1]} \right) |m_1, \cdots, m_n\rangle$$

where the blue term is only present for fermions (ensures antisymmetry). The MPS matrices are

$$A^{[m]} = q^{\frac{L_0}{2n}} e^{-i\frac{\sqrt{\nu}}{2}\varphi_0} \frac{1}{\sqrt{m!}} V_0^m e^{-i\frac{\sqrt{\nu}}{2}\varphi_0} q^{\frac{L_0}{2n}}, \qquad q = e^{2i\pi\tau}$$

Again χ grows exponentially with torus thickness.

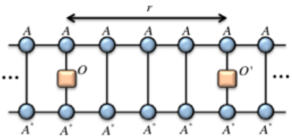
Matrix Product States :
a powerful numerical method

plots from collaborations with :

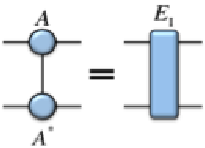
Y-L. Wu, Z. Papic, N. Regnault, B. A. Bernevig

Infinitely long cylinder, bulk correlation length

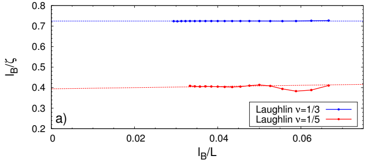
$\langle O(0)O'(r)\rangle \sim \exp(-r/\zeta)$

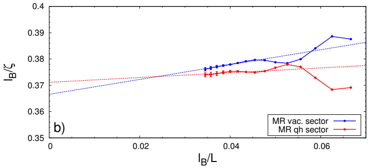


The **transfer matrix** $E_1 = \sum_m A^{[m]} \otimes \bar{A}^{[m]}$



$\Rightarrow$ correlation length $\zeta^{-1} \propto \log(\lambda_1/\lambda_2)$





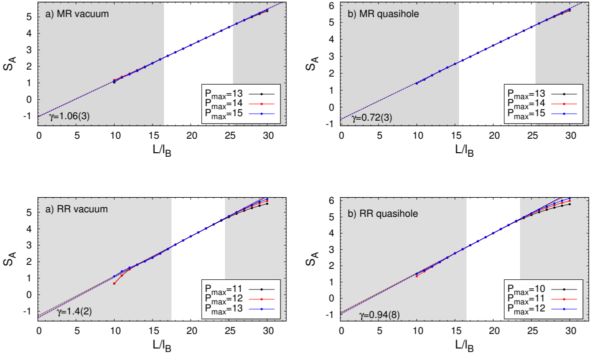
Model state	Laughlin 1/3	Laughlin 1/5	MR vac.	MR qh
ζ/l_B	1.381(1)	2.53(7)	2.73(1)	2.69(1)

Entanglement entropy (orbital cut)

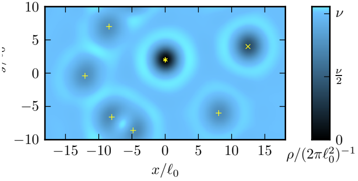
Area law $S_A = \alpha L - \gamma$, where the subleading term γ is universal

$$\gamma = \log \mathcal{D}/d_a$$

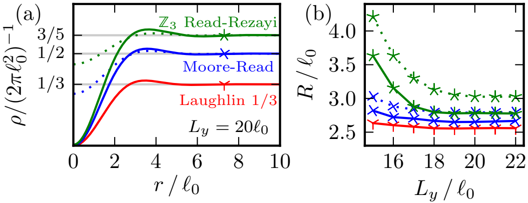
Model state	γ_{vac}	γ_{qh}	$\mathcal{D}$
MR	1.04	0.69	$2\sqrt{2}$
$\mathbb{Z}_3$ RR	1.45	0.97	$\frac{5}{2\sin\left(\frac{\pi}{5}\right)}$



Quasi-hole excitations

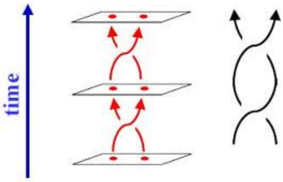


- Insert quasi-holes in the MPS
- Compute the density profile
- Measure the radius of the quasi-hole



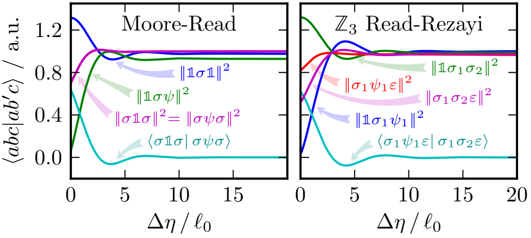
	ν	R/ℓ_0	
Laughlin	$\frac{1}{3}$	$\frac{e}{3}$: 2.6	
Moore-Read	$\frac{1}{2}$	$\frac{e}{4}$: 2.8	$\frac{e}{2}$: 2.7
$\mathbb{Z}_3$ Read-Rezayi	$\frac{3}{5}$	$\frac{e}{5}$: 3.0	$\frac{3e}{5}$: 2.8

Braiding non-Abelian quasi-holes



Instead of computing the Berry phase,
$\Rightarrow$ check the behavior of conformal block overlaps

$$\langle \Psi_a | \Psi_b \rangle = C_a \delta_{a\,b} + O\left(e^{-|\Delta\eta|/\xi_{ab}}\right)$$



Microscopic, quantitative verification of the non-Abelian braiding.

Conclusion

Conclusion

FQH model wavefunctions have been used for more than 30 years :

They are nothing but Matrix Product States in disguise

Numerically powerful

- **Bulk correlation length** ζ (or equivalently bulk gap)
- precision computation of the **topological entanglement entropy** γ (and the **quantum dimensions** d_a)
- Non-Abelian quasihole radius and **braiding**

CFT/MPS provide a strong link between microscopics and 3d TQFT

As conjectured by Moore and Read

Model states $\Rightarrow$ (non-Abelian) chiral topological phases.

Limitations : at the end of the day these states are model states
with the anyon data as an input. Similar to quantum-double models.

- Are they in the same universality class as the experimental states ?
- DMRG methods might help answer this question.