# An Introduction to Entanglement Measures in Integrable Quantum Field Theory


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# Background:

These talks will often refer to results from my papers. They can all be accessed at:

http://olallacastroalvaredo.weebly.com/publications.html

Two good introductions to the topic of twist fields and entanglement measures are:

John L. Cardy, O.C.-A. and Benjamin Doyon, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, J. Stat. Phys. 130 (2008) 129-168, arXiv:0706.3384.

O.C.-A. and Benjamin Doyon, *Bi-partite entanglement entropy in massive 1+1 dimensional quantum field theories*, J. Phys. A42 (2009) 504006, arXiv:0906.2946 (Review Article).

A good source are also the lectures we gave last year in Bologna, which were much more extensive. All the material can be found here: http://thebolognalectures.weebly.com

# Collaborators:

I would like to thank all my collaborators on this area of research:

Davide Bianchini, Former PhD Student
Olivier Blondeau-Fournier, Université Laval (Quèbec)
John L. Cardy, University of California, Berkeley
Benjamin Doyon, King's College London
Emanuele Levi, Former PhD Student
Francesco Ravanini, Università di Bologna

# Disclaimers:

- My approach to this topic is that of 1+1 dimensional many-body quantum systems/quantum integrable systems.
- I will not directly discuss the contributions form information theory/holography approach to entanglement even if these are also important.
- I will talk about 1+1 quantum systems only.
- I will mostly talk about 1+1 dimensional integrable systems with diagonal scattering:
  (integrable) two-body scattering matrices determine the whole scattering theory and
  (diagonal) all processes are of the form $a + b \to a + b$ with scattering matrix $S_{ab}(\theta)$ where $\theta$ is the rapidity difference between particles $a$ and $b$.
- Recall [Joao's talk] for analytic structure of such $S$-matrices and [Benjamin's talk] for special features of integrability such as the description of $n$-particle states.

# 1. Entanglement in quantum mechanics

- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.
  A typical example: a pair of opposite-spin electrons:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\,, \quad \langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

- What is special: Bell's inequality says that this cannot be described by local variables.
- A situation that looks similar to $|\psi\rangle$ but without entanglement is a factorizable state:

$$|\hat{\psi}\rangle = \frac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$

$$= \frac{1}{2}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \left(|\uparrow\rangle + |\downarrow\rangle\right)$$

- These examples are extremely simple but what happens in extended many-body quantum systems?
- First of all, what provides a good measure of entanglement? [Plenio & Virmani'05]
  1. Entanglement monotone: no increase under LOCC
  2. Invariant under unitary transformations
  3. Zero for separable states
  4. Non-zero for non-separable states
- Among others, the bi-partite entanglement entropy and the logarithmic negativity are good measures of entanglement according to these properties.

# 2. Bi-partite (von Neumann) Entanglement Entropy

Let us consider a spin chain of length $N$, subdivided into regions $A$ and $\bar{A}$ of lengths $L$ and $N-L$ [Bennett et al.'96]

$$\cdots s_{i-1} \otimes \underbrace{s_i \otimes s_{i+1} \otimes \quad \cdots \quad \otimes s_{i+L-1}}_{\mathrm{A}} \otimes s_{i+L} \quad \cdots$$

then we define

**Von Neumann Entanglement Entropy**

$$S_A = -\mathrm{Tr}_{\mathcal{A}}(\rho_A \log(\rho_A)) \quad \text{with} \quad \rho_A = \mathrm{Tr}_{\bar{\mathcal{A}}}(|\Psi\rangle\langle\Psi|)$$

$|\Psi\rangle$ ground state and $\rho_A$ the reduced density matrix.
Other entropies may also be defined such as

**Other Entropies**

$$S_A^{\text{Rényi}} = \frac{\log(\mathrm{Tr}_{\mathcal{A}}(\rho_A^n))}{1-n}, \quad S_A^{\text{Tsallis}} = \frac{1-\mathrm{Tr}_{\mathcal{A}}(\rho_A^n)}{n-1}$$

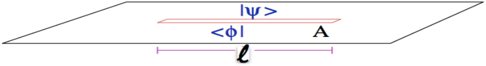
# 3. Replica Trick I

The object $\mathrm{Tr}_{\mathcal{A}}\rho_A^n$ with $n$ integer is also a partition function [Callan & Wilczek'93; Holzhey, Larsen & Wiczek'94; Calabrese & Cardy'04]:

$$\mathrm{Tr}_{\mathcal{A}}\rho_A^n = \frac{Z_n}{Z_1^n},$$

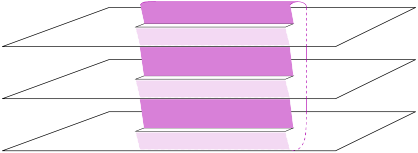
but now it is defined on an $n$-sheeted Riemann surface:

$${}_A\langle\phi|\rho_A|\psi\rangle_A \sim$$




$$\mathrm{Tr}_{\mathcal{A}}(\rho_A^n) \sim Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_3 =$$



# 4. Replica Trick II

- We can express the bi-partite entanglement entropy directly in terms of this partition function as

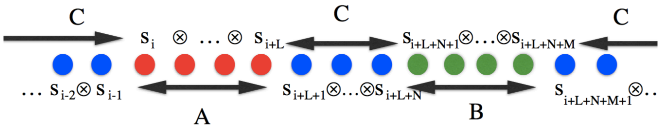
**Replica Trick**

$$S_A = -\mathrm{Tr}_{\mathcal{A}}(\rho_A \log(\rho_A)) = -\lim_{n\to 1^+} \frac{d}{dn}\mathrm{Tr}_{\mathcal{A}}(\rho_A^n)$$

- However, when computing this limit we need to extend our notion of "replica" to $n \geq 1$ and $n \in \mathbb{R}$.
- The analytic continuation problem is not solved in general although existence and uniqueness are expected and may be established under certain natural conditions.
- Note that this is only a difficult problem when trying to obtain analytical results. Numerically, if the eigenvalues of $\rho_A$ are known then any Rényi entropy can be computed.

# 5. Logarithmic Negativity (LN)

- A good measure of entanglement in pure and mixed states for non-complementary regions such as $A$ and $B$ [Zyczkowski et al.'98; Vidal'00; Eisert'01;Vidal & Werner'02; Plenio'05].




**Logarithmic Negativity of a Pure State**

$$\mathcal{E} = \log \mathrm{Tr}_{\mathcal{A}\cup\mathcal{B}} |\rho_{A\cup B}^{T_B}| \quad \text{with} \quad \rho_{\mathcal{A}\cup\mathcal{B}} = \mathrm{Tr}_{\mathcal{C}}(|\Psi\rangle\langle\Psi|)$$

- It involves the trace norm: $\mathrm{Tr}_{\mathcal{A}\cup\mathcal{B}} |\rho_{A\cup B}^{T_B}| = \sum_i |\lambda_i|$ where $\lambda_i$ are the eigenvalues of $\rho_{A\cup B}^{T_B}$.
- $T_B$ represents partial transposition in sub-system $B$, that is, let $e_i^A, e_i^B$ be bases in $\mathcal{A}$ and $\mathcal{B}$ then: $\langle e_i^A e_j^B | \rho_{A\cup B}^{T_B} | e_k^A e_l^B \rangle = \langle e_i^A e_l^B | \rho_{A\cup B} | e_k^A e_j^B \rangle$. The LN is basis-independent.
- $|\Psi\rangle$ is the state of the whole system (for pure states).

# 6. Logarithmic Negativity (LN): Replica Approach

- There is also a "replica" approach to the LN [Calabrese, Cardy & Tonni'12]:

**Logarithmic Negativity from the Replica Trick**

$$\mathcal{E}[n] = \log \mathrm{Tr}_{\mathcal{A}\cup\mathcal{B}}(\rho_{A\cup B}^{T_B})^n \quad \text{then} \quad \mathcal{E} = \lim_{n\to 1} \mathcal{E}[n_e]$$

where $\mathcal{E}[n_e]$ means the function $\mathcal{E}[n]$ for $n$ even. This limit requires analytic continuation from $n$ even to $n = 1$.

- There is also a partition function picture in this case. However, the $n$-sheeted Riemann surface is more complicated:

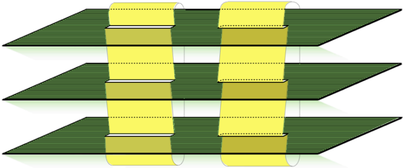
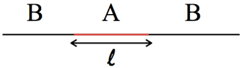


Fig. from [Calabrese, Cardy & Tonni'12].

# 7. Measures of Entanglement in QFT: Why?

- The EE of the ground state of 1+1 dimensional QFTs satisfies an area law: it grows proportionally to the number of boundary points and this has important implications on the efficiency of numerical simulations.
- Both the EE and the LN display universal behaviour near critical points, after a quantum quench and in many-body localized (MBL) states (recall [Joel's talk]).
- They can be used to classify critical points in a numerically very efficient way, to describe the dynamics after a quantum quench and to identify MBL states.

# 8. Rényi Entropies at and near Critical Points




At criticality ($\xi \to \infty$):

Universal scaling: [Holzhey, Larsen & Wilczek'94; Vidal, Latorre, Rico & Kitaev'03; Calabrese & Cardy'04; Bianchini et al.'15]:

$$S_n(\ell) \sim \frac{c_{\rm eff}(n+1)}{6n} \log \frac{\ell}{\epsilon}$$

$c_{\rm eff}$ is the effective central change which uniquely characterises the CFT.

$\epsilon$ is a non-universal cut-off.

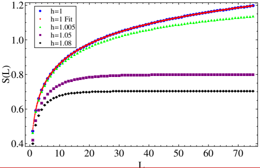
For more than one interval: information about operator content of CFT.

Near criticality ($\xi$ finite):

Universal saturation [Calabrese & Cardy'04] and decay [Cardy, OC-A & Doyon'08; Doyon'09]

$$S(\ell) - \lim_{\ell\to\infty} S(\ell) = -\frac{1}{8} K_0(2m\ell) + \cdots$$

where $m$ is the mass of the lightest particle in the spectrum.



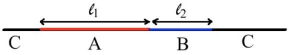
# 9. LN near Critical Points

At criticality:

Universal scaling: For "adjacent regions" [Calabrese, Cardy & Tonni'12'13'14]:

$$\mathcal{E}^{\perp}(\ell_1,\ell_2) \sim \frac{c}{4}\log\frac{\ell_1\ell_2}{\epsilon(\ell_1+\ell_2)}$$

$c$ is the central change.




For general confs: information about operator content of CFT. Best known for compactified free Boson.

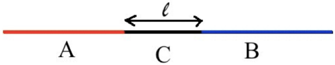
Near criticality:

Universal saturation and decay: For adjacent regions ($\ell_1 := \ell$, $\ell_2 \to \infty$) [Blondeau-Fournier, OC-A & Doyon'16]

$$\mathcal{E}^{\perp}(\ell) - \lim_{\ell\to\infty}\mathcal{E}^{\perp}(\ell) = -\frac{2}{3\sqrt{3}\pi}K_0(\sqrt{3}m\ell)+\cdots$$

where $m \propto \xi^{-1}$ is the mass of the lightest particle and $\mathcal{E}^{\perp}(\infty)$ is a constant which has a universal part.

For semi-infinite non-adjacent regions:




$$\mathcal{E}^{\dashv\vdash}(\ell) = \frac{(m\ell)^2}{2\pi^2}\left[K_0(m\ell)^2 + \frac{K_0(m\ell)K_1(m\ell)}{m\ell} - K_1(m\ell)^2\right]+\cdots$$

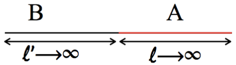
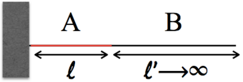
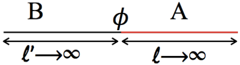
# 10. Branch Point Twist Fields

- In the context of entanglement, the idea of quantum fields associated with branch points of the Riemann surfaces $\mathcal{M}_n$ appeared first in [Calabrese & Cardy'04].
- The interpretation of these fields as symmetry fields of a QFT replica model $S_{\mu_1\mu_2}(\theta) = (S_{ab}(\theta))^{\delta_{ij}}$, with $\mu_1 = (a,i)$ and $\mu_2 = (b,j)$ was first given in [Cardy, OC-A & Doyon'08]:

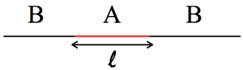
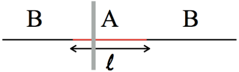
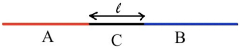
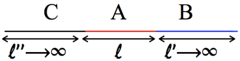
$$\begin{aligned} \Phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_{i+1}(y) \qquad x^1 > y^1, \\ \Phi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\Phi_i(y) \qquad x^1 < y^1, \end{aligned}$$

  for $i = 1, \ldots, n$ and $n+i \equiv i$. Similarly $\tilde{\mathcal{T}} = \mathcal{T}^\dagger$ implements the symmetry $i \mapsto i-1$.
- Twist fields have a quantum spin chain counterpart [OC-A & Doyon'11] as product of local operators on replica chains.
- Branch point twist fields were studied earlier in the context of orbifold CFT [Knizhnik'87; Dixon et al.'87] and their conformal dimension was known: $\Delta_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$.

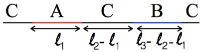
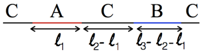
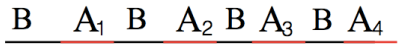
# 11. Entanglement-One Point Functions Dictionary

| | | |
|---|---|---|
| $\epsilon^{2\Delta_n}\langle\mathcal{T}\rangle_n$ |  | EE of semi-infinite region. By CTM approach [Calabrese, Cardy, Peschel, Bianchini, Ercolessi, Franchini, Evangelisti, Ravanini,...], Töplitz determinants [Its, Jin & Korepin'04], QFT techniques [Cardy,OC-A & Doyon'08, Blondeau-Fournier & Doyon'16] or numerically [Vidal et al.'03] |
| $\epsilon^{2\Delta_n}\langle 0\vert\mathcal{T}(\ell)\vert B\rangle_n$ |  | EE of finite interval with boundary. Boundary entropy from twist fields [OC-A & Doyon'09] or by CFT techniques [Cornfeld & Sela'17]. In impurity problems [Saleur et al.'13; Vasseur et al.'17]. |
| $\epsilon^{2(\Delta_n+\frac{\Delta}{n}-n\Delta)}$ $\times\ \frac{\langle:\phi\mathcal{T}:\rangle_n}{\langle\phi\rangle^n}$ |  | EE of semi-inifinite region in non-unitary QFT where $\phi(0)\vert 0\rangle$ is the "conformal" ground state and $\Delta$ is the dimension of $\phi$ [Bianchini et al.'15'16] |

# 12. Entanglement-Two Point Functions Dictionary

<table>
<tr>
<td>$\epsilon^{4\Delta_n}\langle\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\rangle_n$</td>
<td>
</td>
<td>EE of finite interval. In CFT [Callan & Wilczek'93; Holzhey et al.'94; Vidal et al.'03;Calabrese & Cardy'04...]. In massive models [Cardy, OC-A, Doyon, Levi, Bianchini...]</td>
</tr>
<tr>
<td>$\epsilon^{4\Delta_n}\langle\mathcal{T}(0)D\tilde{\mathcal{T}}(\ell)\rangle_n$</td>
<td>
</td>
<td>EE of finite interval in the presence of a defect [Jiang'17].</td>
</tr>
<tr>
<td>$\epsilon^{4\Delta_n}\langle\mathcal{T}(0)\mathcal{T}(\ell)\rangle_n$</td>
<td>
</td>
<td>LN of disjoint semi-infinite intervals [Blondeau-Fournier, OC-A & Doyon'16]. For the free non-compactified massive Boson [Bianchini & OC-A'16]</td>
</tr>
<tr>
<td>$\epsilon^{2(\Delta_n+\Delta_{n'})}$<br>$\times\langle\mathcal{T}(0)\tilde{\mathcal{T}}^2(\ell)\rangle_n$</td>
<td>
</td>
<td>LN of adjacent intervals (one of them semi-infinite) [Calabrese, Cardy & Tonni'12'13'14; Blondeau-Fournier, OC-A & Doyon'16].</td>
</tr>
</table>

# 13. Entanglement-Multipoint Functions Dictionary

| | | |
|---|---|---|
| $\epsilon^{8\Delta_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell_1)\mathcal{T}(\ell_2)\tilde{\mathcal{T}}(\ell_3)\rangle_n$ |   | EE of two disconnected regions [Calabrese, Cardy, Tonni, Casini, Huerta, Furukawa, Pasquier, Shiraishi, Caraglio, Gliozzi, Igloi, Peschel, Alba, Tagliacozzo, Fagotti, Rajabpour, Datta, David...] |
| $\epsilon^{8\Delta_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell_1)\tilde{\mathcal{T}}(\ell_2)\mathcal{T}(\ell_3)\rangle_n$ |   | LN of connected regions [Calabrese, Cardy, Tonni, de Nobile, Ruggiero, Alba, Coser...] |
| $\epsilon^{4k\Delta_n} \langle \mathcal{T}(\ell_1)\tilde{\mathcal{T}}(\ell_2)\cdots\mathcal{T}(\ell_{2k-1})\tilde{\mathcal{T}}(\ell_{2k})\rangle_n$ |   | EE of multiple disconnected regions. Little known yet except for free Fermions in CFT [Calabrese & Cardy'04] |

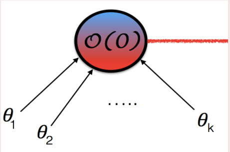
# 14. Form Factors of Local Fields: Definition

- Let $|\theta_1, \ldots, \theta_k\rangle_{\mu_1 \ldots \mu_k}$ a $k$-particle *in*-state. The particles have rapidities $\theta_1 > \cdots > \theta_k$ and quantum numbers $\mu_1 \ldots \mu_k$. Let $\mathcal{O}(0)$ be a local field located at the origin of space-time. Let $|0\rangle = (\langle 0|)^\dagger$ be the ground state (vacuum).

**Form Factor**

$$F_k^{\mathcal{O}|\mu_1 \ldots \mu_k}(\theta_1, \ldots, \theta_k) \quad := \quad \langle 0|\mathcal{O}(0)|\theta_1, \ldots, \theta_k\rangle_{\mu_1 \ldots \mu_k}$$

$$=$$




- Form factors are the building blocks of correlation functions. If all FFs of local fields are known then all correlators of the QFT are known (at least formally).

# 15. Form Factors of Local Fields: Properties

- It is easy to "shift" operators away from the origin by using:

$$\langle 0|\mathcal{O}(\mathbf{x})|\theta_1,\ldots,\theta_k\rangle_{\mu_1\ldots\mu_k} = \left(\prod_{j=1}^{k} e^{ip^\nu(\theta_j)x_\nu}\right) F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k).$$

- Note that $p^0(\theta_j) = m_{\mu_j}\cosh\theta_j$ and $p^1(\theta_j) = m_{\mu_j}\sinh\theta_j$.
- Under Hermitian conjugation:

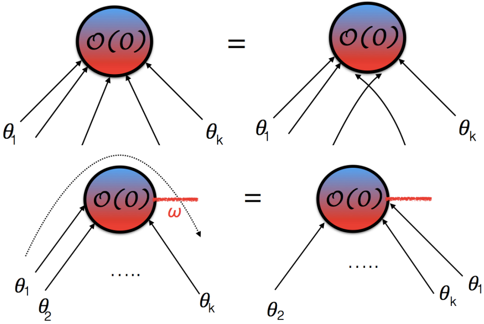
$${}_{\mu_1\ldots\mu_k}\langle\theta_k\ldots\theta_1|\mathcal{O}(0)|0\rangle = (F_k^{\mathcal{O}^\dagger|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k))^*$$

- For local fields in integrable quantum field theory, FFs are known to be the solutions of a Riemann-Hilbert problem and have been computed for many models [Karowski & Weisz'78; Smirnov'90s]

# 16. Standard Watson's Equations

$$F_k^{\mathcal{O}|\ldots\mu_p\mu_{p+1}\cdots}(\ldots\theta_p,\theta_{p+1}\ldots) = S_{\mu_p\mu_{p+1}}(\theta_{p,p+1})F_k^{\mathcal{O}|\ldots\mu_{p+1}\mu_p\cdots}(\ldots,\theta_{p+1},\theta_p,\ldots)$$

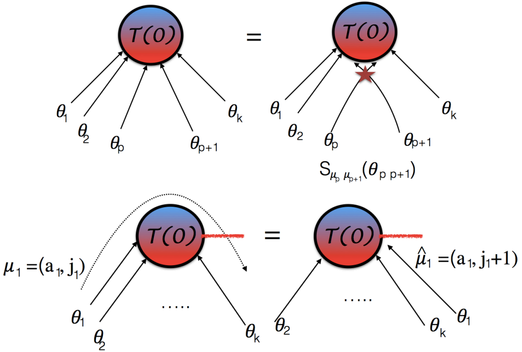
$$F_k^{\mathcal{O}|\mu_1\cdots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = \omega F_k^{\mathcal{O}|\mu_2\cdots\mu_k\mu_1}(\theta_2,\ldots,\theta_k,\theta_1)$$



# 17. Twist Field Watson's Equations

$$F_k^{\mathcal{T}|\ldots\mu_p\mu_{p+1}\ldots}(\ldots\theta_p,\theta_{p+1}\ldots) = S_{\mu_p\mu_{p+1}}(\theta_{p,p+1})F_k^{\mathcal{T}|\ldots\mu_{p+1}\mu_p\ldots}(\ldots,\theta_{p+1},\theta_p,\ldots)$$

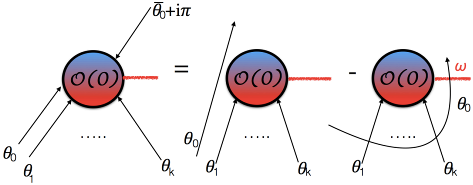
$$F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = F_k^{\mathcal{T}|\mu_2\ldots\mu_k\hat{\mu}_1}(\theta_2,\ldots,\theta_k,\theta_1)$$



# 18. Standard Kinematic Residue Equations

$$\lim_{\bar{\theta}_0 \to \theta_0} (\bar{\theta}_0 - \theta_0) F_{k+2}^{\mathcal{O}|\bar{\mu}\mu\mu_1\ldots\mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1, \ldots, \theta_k)$$

$$= i \left( 1 - \omega \prod_{j=1}^{k} S_{\mu\mu_j}(\theta_{0j}) \right) F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1, \ldots, \theta_k)$$

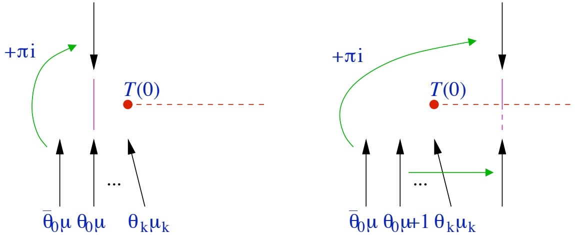


# 19. Twist Field Residue Equations

- For twist fields, the kinematic residue equation splits into two equations:

$$\lim_{\bar{\theta}_0 \to \theta_0} (\bar{\theta}_0 - \theta_0) F_{k+2}^{\mathcal{T}|\bar{\mu}\mu\mu_1\ldots\mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1, \ldots, \theta_k) = i F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1, \ldots, \theta_k)$$

$$\lim_{\bar{\theta}_0 \to \theta_0} (\bar{\theta}_0 - \theta_0) F_{k+2}^{\mathcal{T}|\bar{\mu}\hat{\mu}\mu_1\ldots\mu_k}(\bar{\theta}_0 + i\pi, \theta_0, \theta_1, \ldots, \theta_k) = -i \prod_{j=1}^{k} S_{\hat{\mu}\mu_j}(\theta_{0j}) F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1, \ldots, \theta_k)$$




- The bound state residue equations remain unchanged.

# 20. Properties

- Other properties of the twist field form factors, such as invariance under global rapidity shifts or large rapidity asymptotics are the same as for other local fields.
- For diagonal theories, the solution procedure is also similar as for other local fields.
- It is possible to make a general ansatz based on a minimal form factor satisfying:

$$F_{\mathbf{min}}^{\mathcal{T}|\mu_1\mu_2}(\theta_1,\theta_2) = S_{\mu_1\mu_2}(\theta_1-\theta_2)F_{\mathbf{min}}^{\mathcal{T}|\mu_2\mu_1}(\theta_2,\theta_1) = F_{\mathbf{min}}^{\mathcal{T}|\mu_2\hat{\mu}_1}(\theta_2,\theta_1-2\pi i,)$$

- $F_{\mathbf{min}}^{\mathcal{T}|\mu_1\mu_2}(\theta_1,\theta_2)$ is a (minimal) two-particle form factor without poles in the (extended) physical sheet $\mathrm{Im}(\theta_1-\theta_2) \in [0, 2\pi n)$ and $\hat{\mu} = (a, j+1)$ if $\mu = (a, j)$.
- A special feature of the twist field form factors is that they must all vanish at $n = 1$ (except for $\langle\mathcal{T}\rangle_n$).

- In the absence of bound state poles, it is possible to write a general expression for the two particle form factor, which follows directly from this ansatz:

**Two Particle Form Factor**

$$F_2^{\mathcal{T}|\mu_1\mu_2}(\theta_1,\theta_2)=\frac{\langle\mathcal{T}\rangle_n \sin\frac{\pi}{n}\left(\frac{F_{\mathbf{min}}^{\mathcal{T}|\mu_1\mu_2}(\theta_1,\theta_2)}{F_{\mathbf{min}}^{\mathcal{T}|\mu_1\mu_2}(i\pi,0)}\right)}{2n\sinh\left(\frac{\theta_1-\theta_2+i\pi(2(j_1-j_2)-1)}{2n}\right)\sinh\left(\frac{i\pi(2(j_2-j_1)-1)-\theta_1+\theta_2}{2n}\right)}$$

where $\mu_1=(a_1,j_1)$ and $\mu_2=(a_2,j_2)$.

- The $\sin\frac{\pi}{n}$ term guarantees that the FF is zero at $n=1$.
- Repeated use of the FF equations shows that every FF may be expressed in terms of FFs of particles in the same copy.
- Also, all FFs of $\tilde{\mathcal{T}}$ can be expressed in terms of FFs of $\mathcal{T}$.

# 22. Some Useful Identities

- Due to the relationship between branch point twist fields and cyclic permutations there are many symmetries that may be used to related FFs of different copies to each other as well as those of $\mathcal{T}$ and $\tilde{\mathcal{T}}$. Here are some examples (we use the fact that the FFs only depend on rapidity differences):

**Form Factor Properties**

$$F_2^{\mathcal{T}|(a,i)(b,j)}(\theta) = F_2^{\tilde{\mathcal{T}}|(a,n-i)(b,n-j)}(\theta)$$

$$F_2^{\mathcal{T}|(a,i)(b,i+k)}(\theta) = F_2^{\mathcal{T}|(a,j)(b,j+k)}(\theta)$$

$$F_2^{\mathcal{T}|(a,1)(b,j)}(\theta) = F_2^{\mathcal{T}|(a,1)(b,1)}(2\pi(j-1)i-\theta) \quad \text{for} \quad j>1$$

$$F_2^{\tilde{\mathcal{T}}|(a,1)(b,j)}(\theta) = F_2^{\mathcal{T}|(a,1)(b,1)}(2\pi(j-1)i+\theta)$$

# 23. Two-Point Functions

- The computation of measures of entanglement is reduced to the computation of multi-point functions of twist fields and of their analytic continuation in $n$.
- The two-point function of branch-point twist fields can be decomposed as follows, giving a *large-distance expansion*:

$$\begin{aligned}\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_n &= \langle 0|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|0\rangle \\ &= \sum_{\text{state } k} \langle 0|\mathcal{T}(0)|k\rangle\langle k|\tilde{\mathcal{T}}(r)|0\rangle\end{aligned}$$

where $\sum_k |k\rangle\langle k|$ is a sum over a complete set of states and $|0\rangle$ is the ground state

- The matrix elements $\langle 0|\mathcal{T}(0)|k\rangle$ are the form factors
- In 1+1 dimensions

$$\sum_k |k\rangle\langle k| \mapsto \int_{-\infty}^{\infty}\frac{d\theta_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{d\theta_k}{2\pi}|\theta_1,\ldots,\theta_k\rangle\langle\theta_k,\cdots,\theta_1|$$

# 24. Application: Exponential Corrections to EE

- Recall that

$$S(\ell) = -\lim_{n\to 1} \frac{\partial h(n)}{\partial n} \quad \text{with} \quad h(n) = \epsilon^{4\Delta_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\rangle_n$$

- So the basic object we need to compute is the two-point function:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\rangle_n = \langle \mathcal{T}\rangle_n^2 + \sum_{\mu} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} (F_1^{\mathcal{T}|\mu}(\theta))^* (F_1^{\tilde{\mathcal{T}}|\mu}(\theta)) e^{-\ell m_\mu \cosh\theta}$$

$$+\frac{1}{2}\sum_{\mu_1\mu_2} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta_2}{2\pi} (F_2^{\mathcal{T}|\mu_1\mu_2}(\theta_1,\theta_2))^* (F_2^{\tilde{\mathcal{T}}|\mu_1\mu_2}(\theta_1,\theta_2)) e^{-\ell m_{\mu_1}\cosh\theta_1 - \ell m_{\mu_2}\cosh\theta_2}$$

$$+\cdots$$

# 25. Some Simplifications

- Let us consider now a simple case: a theory with a single particle in the spectrum.
- In that case we can label particles just by the copy number $j = 1 \ldots n$.
- We also know the twist field is a spinless field: one-particle form factors are rapidity-independent and they are all equal because all copies are identical: $F_1^{\mathcal{T}|\mu}(\theta) := F_1(n)$.
- Two-particle form factors only depend on rapidity differences: $F_2^{\mathcal{T}|\mu_1\mu_2}(\theta_1, \theta_2) := F_2^{ij}(\theta, n)$ and $F_2^{\tilde{\mathcal{T}}|\mu_1\mu_2}(\theta_1, \theta_2) := \tilde{F}_2^{ij}(\theta, n)$ with $\theta = \theta_1 - \theta_2$.
- Finally, recall that all form factors are zero at $n = 1$.

# 26. First Term: Saturation

- The first term in the expansion of the two-point function is the expectation value of twist fields. This is a function of $n$ which is only known for free theories.
- This term characterizes saturation of EE for large sub-systems:

$$\lim_{\ell\to\infty} S(\ell) = -\lim_{n\to 1}\frac{\partial}{\partial n}\left(\epsilon^{4\Delta_n}\langle\mathcal{T}\rangle_n^2\right) = -\frac{c}{3}\log\epsilon - 2\lim_{n\to 1}\frac{\partial\langle\mathcal{T}\rangle_n}{\partial n}$$

$$= -\frac{c}{3}\log(\epsilon m) - U \quad \text{with} \quad \langle\mathcal{T}\rangle_n = m^{2\Delta_n} U_n$$

- and $U = 2\lim_{n\to 1}\frac{\partial U_n}{\partial n}$. Note that $U$ is a universal constant in the sense that it does not depend on the cut-off $\epsilon$, hence can be uniquely determined for each QFT.

- For a theory with a single particle the one-particle form factor contribution can be written simply as

$$n\,|F_1(n)|^2 \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-\ell m \cosh\theta} = \frac{n}{\pi}\,|F_1(n)|^2\, K_0(m\ell).$$

- This provides the leading correction to saturation of the two-point function, however it vanishes under differentiation w.r.t. $n$ and limit $n \to 1$.
- This is because $F_1(1) = F_1(1)^* = 0$.
- This means that the one-particle form factors (if they are non-vanishing) will provide the leading correction to the Rényi entropies but no contribution to the EE.

# 28. Third Term: Two-Particle Form Factor

- For a theory with a single particle two-particle form factor sum can be simplified as:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(F_2^{ij}(\theta,n))^*(\tilde{F}_2^{ij}(\theta,n)) = n\sum_{j=1}^{n}(F_2^{1j}(\theta,n))^*(\tilde{F}_2^{1j}(\theta,n))$$

because all copies are identical. Using the identities we saw in the previous lecture:

$$n\sum_{j=1}^{n}(F_2^{1j}(\theta,n))^*(\tilde{F}_2^{1j}(\theta,n)) = n\left|F_2^{11}(\theta,n)\right|^2 + n\sum_{j=2}^{n}\left|F_2^{11}(-\theta+2\pi i(j-1),n)\right|^2$$

$$= n(\left|F_2^{11}(\theta,n)\right|^2 - \left|F_2^{11}(-\theta,n)\right|^2) + n\sum_{j=0}^{n-1}\left|F_2^{11}(-\theta+2\pi i j,n)\right|^2$$

- The derivative at $n=1$ of the first term will be zero because $F_2^{11}(\theta,1) = F_2^{11}(\theta,1)^* = 0$. So it will contribute to the Rényi entropies but not to the EE.

# 29. In Summary: Leading Correction to EE

- In summary, we need to compute

$$-\frac{1}{4}\lim_{n\to 1}\frac{\partial}{\partial n}\left(\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\int_{-\infty}^{\infty}\frac{d\beta}{2\pi}n\sum_{j=0}^{n-1}\left|F_2^{11}(-\theta+2\pi i j,n)\right|^2 e^{-2m\ell\cosh\frac{\theta}{2}\cosh\frac{\beta}{2}}\right)$$

  with $\theta=\theta_1-\theta_2$ and $\beta=\theta_1+\theta_2$.

- The integral in $\beta$ can be carried out giving a Bessel function. So, we end up with:

$$-\lim_{n\to 1}\frac{\partial}{\partial n}\left(\int_{-\infty}^{\infty}\frac{d\theta}{(2\pi)^2}n\sum_{j=0}^{n-1}\left|F_2^{11}(-\theta+2\pi i j,n)\right|^2 K_0(2m\ell\cosh\frac{\theta}{2})\right)$$

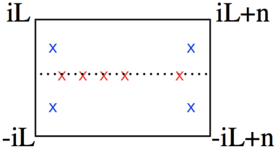
- In order to take the derivative, we need to somehow get rid of the sum up to $n-1$.
- A well-known way of doing this is to use the cotangent trick.

# 30.Cotangent Trick I

- The idea is that the sum may be replaced by a contour integral

$$\frac{1}{2\pi i}\oint dz \pi \cot(\pi z) s(z,\theta,n)$$

with $s(z,\theta,n) = \left|F_2^{11}(-\theta+2\pi i z, n)\right|^2$, in such a way that the sum of the residues of poles of the cotangent enclosed by contour reproduces the original sum.




- Here the red crosses represent the poles of the cotangent at $z=1,2,\ldots,n-1$ and the blue crosses represent other poles in the contour due to the kinematic poles of the function $s(z,\theta,n)$ at $z=\frac{1}{2}\pm\frac{\theta}{2\pi i}$ and $z=n-\frac{1}{2}\pm\frac{\theta}{2\pi i}$.
- We shift $iL \to iL-\epsilon$ so as to avoid the pole at $z=n$. It includes $z=0$ but this does not affect the result.

# 31.Cotangent Trick II

- Since $s(z,\theta,n)$ decays exponentially as $\mathrm{Im}(z) \to \pm\infty$ so we can show that the contributions to the contour integral of the horizontal segments vanish.
- The contribution of the vertical segments can be written as:

$$-\frac{1}{4\pi i}\int_{-\infty}^{\infty}(S(\theta-\beta)S(\theta+\beta)-1)\coth\frac{\beta}{2}s(\beta,\theta,n)d\beta$$

  where $\beta = 2\pi i z$ and $S(\theta)$ is the $S$-matrix. Here we used the property $s(z+n,\theta,n) = S(\theta-2\pi i z)S(\theta+2\pi i z)s(z,\theta,n)$.
- Note that this is zero for free theories. Its derivative at $n=1$ is zero for similar reasons as before.
- Finally we are left with the contributions from the residues of the kinematic poles. They give:

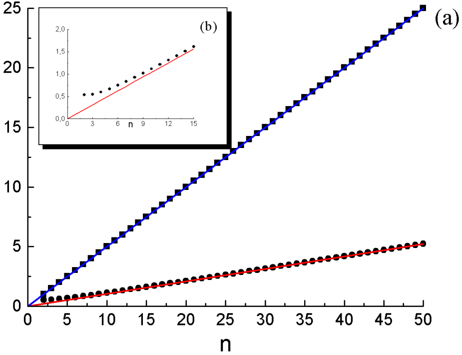
$$\tanh\frac{\theta}{2}\mathrm{Im}\left(F_2^{11}(-2\theta+i\pi,n) - F_2^{11}(-2\theta+2\pi i n - i\pi,n)\right)$$

# 32. Derivative

- The only two-particle contribution to the derivative comes from:

$$\mathrm{Im}\left(F_2^{11}(-2\theta+i\pi,n)-F_2^{11}(-2\theta+2\pi in-i\pi,n)\right)\tanh\frac{\theta}{2}$$

- Based on previous observations, it would seem that this should be zero as $F_2^{11}(\theta,1)=0$. However, something special happens to this function as $n\to 1$ and $\theta\to 0$ simultaneously.
- This can be best understood by doing some simple numerics ...




The sum $n\sum_{j=0}^{n-1}\left|F_2^{11}(-\theta+2\pi i j,n)\right|^2$ for $\theta=0$ in the Ising model (blue) and the sinh-Gordon model (red).

- Near $\theta = 0$ and $n = 1$:

$$\mathrm{Im}\left(F_2^{11}(-2\theta + i\pi, n) - F_2^{11}(-2\theta + 2\pi i n - i\pi, n)\right)\tanh\frac{\theta}{2}$$

$$\sim -\frac{1}{2}\left(\frac{i\pi(n-1)}{2(\theta + i\pi(n-1))} - \frac{i\pi(n-1)}{2(\theta - i\pi(n-1))}\right) \sim \frac{\pi^2(n-1)}{2}\delta(\theta).$$

- Putting this result back into the $\theta$ integral and differentiating w.r.t. $n$ we obtain the two-particle form factor contribution:

$$-\frac{1}{8}K_0(2m\ell)$$

- The result is striking for its simplicity. From the derivation we see that it follows from the pole structure of the FFs, which is universal.
- For this reason the same result can even be found for non-integrable 1+1 dimensional models [Doyon'09].

# Conclusions & Open Problems

- Branch point twist fields provide a powerful approach to the computation of partition functions in non-trivial geometries, which are related to measures of entanglement in QFT.
- There are many interesting open problems to be addressed within this approach:
  - Measures of entanglement in higher dimensions: can one define twist fields?
  - Measures of entanglement in excited states: more generic matrix elements of twist fields
  - Generic understanding of analytic continuations of replica partition functions: analytic structure of form factors and correlation functions
  - A more complete set of form factor solutions for integrable models: computation of higher-particle form factors for diagonal theories
  - The study of multi-point functions: are there any simple universal features?
  - Other twist fields: pentagonal amplitudes? [Basso, Sever & Vieira'14]


THE END