

Exact results for KPZ universality in 1+1 dimension (2/2)



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Exact methods in low dimensional statistical physics

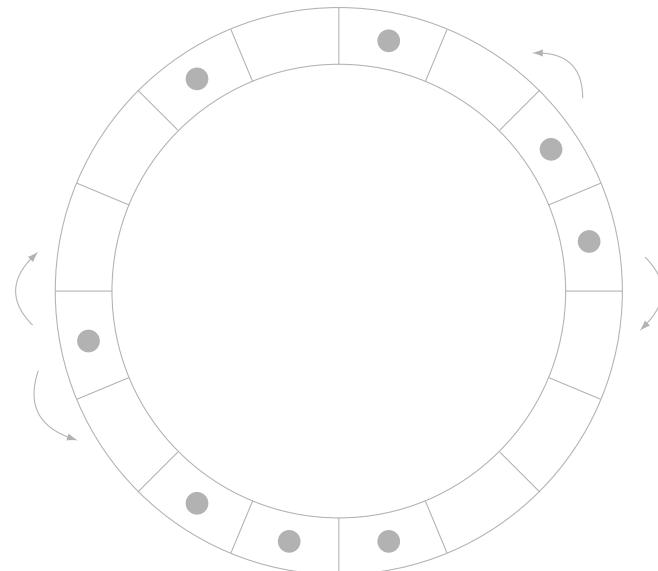
I TASEP, ASEP, WASEP & KPZ

II Infinite system



III Finite volume

$$S_\tau[\varphi] = \int_{-\infty}^{\varphi^{-1}(s)} dv \left(\varphi'(v)^2 + \tau \varphi(v) \right)$$



The asymmetric simple exclusion process (ASEP)

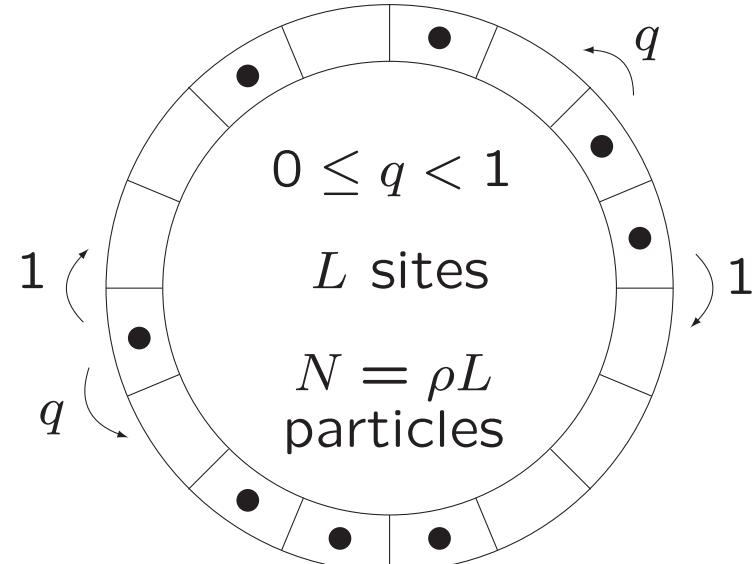
Continuous time Markov process

Detailed balance broken, **irreversible**

$$|P_t\rangle = e^{tM}|P_0\rangle$$

M Markov matrix, **non-Hermitian**

$$\sum_{\mathcal{C}} \langle \mathcal{C} | M = 0 \quad \Rightarrow \quad \sum_{\mathcal{C}} \langle \mathcal{C} | P_t \rangle = 1$$



TASEP $q = 0$

$$\text{XXZ spin chain: } -\frac{2}{\sqrt{q}} M - L\Delta \mathbb{1} \sim H_{\text{XXZ}}$$

$$H_{\text{XXZ}} = - \sum_{i=1}^L \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right)$$

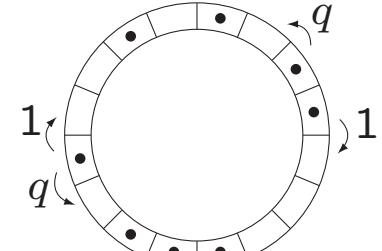
$$\Delta = \frac{q^{1/2} + q^{-1/2}}{2} > 1$$

$$S_{L+1}^\pm = q^{\mp L/2} S_1^\pm$$

Transfer matrix $T(z) = \text{tr}_a [\mathcal{L}_{aL}(z) \dots \mathcal{L}_{a2}(z) \mathcal{L}_{a1}(z)]$

$$\langle T(z) \rangle_t = \sum_{\mathcal{C}} \langle \mathcal{C} | T(z) | P_t \rangle = 1 + q^N z^L$$

WASEP and KPZ



WASEP $q = 1 - \frac{4\lambda}{\sqrt{L}}$ diffusive scaling $\begin{cases} \text{site } i = xL \\ \text{time } t = \tau L^2/2 \end{cases}$

$$Q_i(t) - \frac{(1-q)t}{4} \underset{L \rightarrow \infty}{\simeq} -\frac{\sqrt{L}}{2} \left(h_\lambda(x, \tau) - h_\lambda(x, 0) + \frac{\lambda^3 \tau}{3} \right) \quad \rho = 1/2 \text{ otherwise moving frame}$$

- $Q_i(t)$ net time-integrated current WASEP between sites i and $i+1$
- h_λ solution of the KPZ equation $\partial_\tau h = \frac{1}{2} \partial_x^2 h + \lambda (\partial_x h)^2 + \xi$

Scaling properties

$$\xi\left(\frac{x}{a}, \frac{\tau}{b}\right) \equiv \sqrt{ab} \xi(x, \tau) \Rightarrow h_\lambda(x, \tau/\lambda) \equiv \frac{h_1(\lambda^2 x, \lambda^3 \tau)}{\lambda}$$

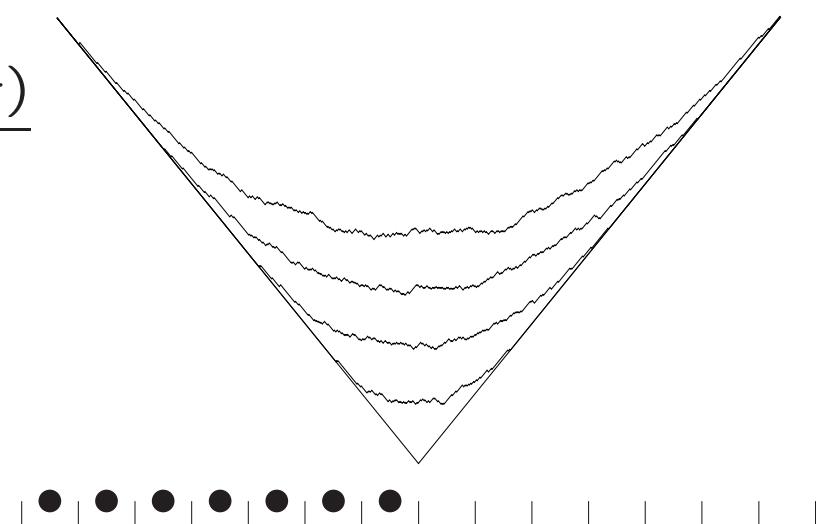
Universal renormalization group flow

Edwards-Wilkinson \rightarrow KPZ fixed point

$$\lambda = 0$$

$$\lambda \rightarrow \infty$$

ASEP \equiv TASEP



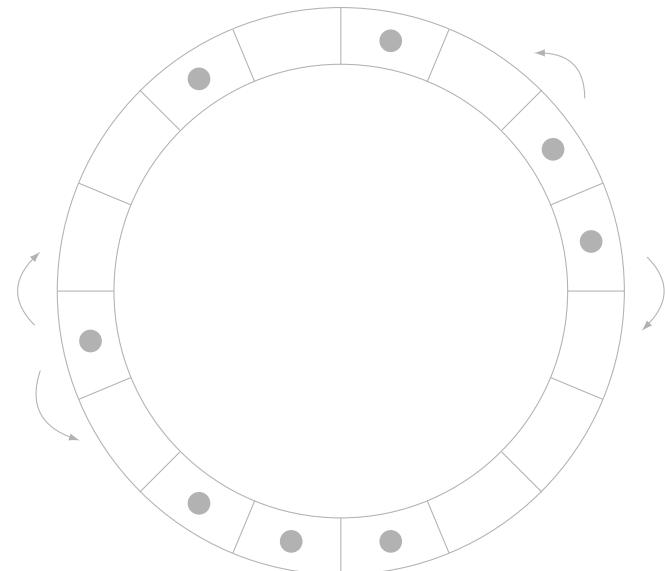
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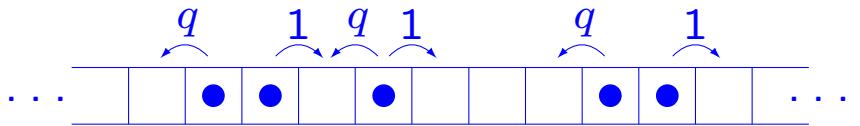


III Finite volume

$$S_\tau[\varphi] = \int_{-\infty}^{\varphi^{-1}(s)} dv \left(\varphi'(v)^2 + \tau \varphi(v) \right)$$



Bethe ansatz for ASEP on \mathbb{Z}



N particles on \mathbb{Z} , initial positions x_j^0 , final positions x_j at time t

Solution of the master equation $\mathbb{P}_t(\mathbf{x}|\mathbf{x}^0)$ must verify

- $\frac{d}{dt} \mathbb{P}_t(\mathbf{x}|\mathbf{x}^0) = \sum_{j=1}^N \left(\mathbb{P}_t(\mathbf{x}|\mathbf{x}^0)_{x_j \rightarrow x_{j-1}} + q \mathbb{P}_t(\mathbf{x}|\mathbf{x}^0)_{x_j \rightarrow x_{j+1}} - (1+q) \mathbb{P}_t(\mathbf{x}, \mathbf{x}^0) \right)$
- $\mathbb{P}_t(\dots, x, x, \dots | \mathbf{x}^0) = 0$
- $\mathbb{P}_0(\mathbf{x}|\mathbf{x}^0) = \delta_{\mathbf{x}, \mathbf{x}^0}$

Solution (C.Tracy and H. Widom 2008)

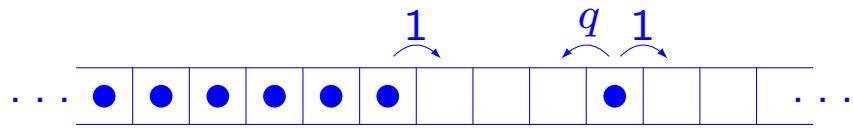
$$\mathbb{P}_t(\mathbf{x}|\mathbf{x}^0) = \sum_{\sigma \in \mathcal{S}_N} \oint_{\mathcal{C}_r} \frac{dz_1 \dots dz_N}{(2i\pi)^N z_1 \dots z_N} \prod_{\substack{i < j \\ \sigma(i) > \sigma(j)}} \frac{1 - (1+q)z_i + qz_i z_j}{1 - (1+q)z_j + qz_i z_j} \prod_{j=1}^N z_j^{x_j - x_{\sigma(j)}^0} e^{t(z_j^{-1} + qz_j - 1 - q)}$$

Infinite system: no Bethe equations

TASEP $q = 0$: $\sum_{\sigma \in \mathcal{S}_N} \rightarrow \det$ (G.M. Schütz 1997)

Fredholm determinants

C.Tracy and H. Widom 2008



$$\mathbb{P}_t(\mathbf{x}|\mathbf{x}^0) = \sum_{\sigma \in \mathcal{S}_N} \oint_{\mathcal{C}_r} \frac{dz_1 \dots dz_N}{(2i\pi)^N} \prod_{\substack{i < j \\ \sigma(i) > \sigma(j)}} \frac{1 - (1+q)z_i + qz_iz_j}{1 - (1+q)z_j + qz_iz_j} \prod_{j=1}^N z_j^{x_j - x_{\sigma(j)}^0 - 1} e^{t(z_j^{-1} + qz_j - 1 - q)}$$

Probability distribution m -th particle

- geometric sum over positions $x_1, \dots, x_{m-1}, x_{m+1}, x_N$
- go to large contours of integration: residues
- step initial condition $N \rightarrow \infty$: algebraic identities $\Rightarrow \prod_{i \neq j} \frac{z_j - z_i}{1 - (1+q)z_i + qz_iz_j}$
- Double product \rightarrow Cauchy determinant $\det \left(\frac{1}{1 - (1+q)z_i + qz_iz_j} \right)_{i,j=1,\dots,k}$
- $\sum_{k=0}^{\infty} \frac{1}{k!} \oint_{\mathcal{C}_R} dz_1 \dots dz_k \det(K(z_i, z_j)) \Rightarrow$ Fredholm determinant

$$\mathbb{P}(x_m(t) \leq x) = \oint \frac{d\lambda}{\lambda} \frac{\det(\mathbb{1} - q\lambda \mathbb{K})_{L^2(\mathcal{C}_R)}}{(\lambda; q^{-1})_m}$$

$$K(z, z') = \frac{1}{2i\pi} \frac{z^{-x} e^{t(z^{-1} + qz - 1 - q)}}{1 - (1+q)z + qzz'}$$

Height fluctuations for droplet growth

Long time KPZ fluctuations: KPZ fixed point

TASEP $q = 0$: Johansson 2000

ASEP $0 < q < 1$: Tracy-Widom 2009

Long time asymptotic analysis kernel \Rightarrow GUE Tracy-Widom

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{x_m\left(\frac{t}{1-q}\right) - c_1 t}{c_2 t^{1/3}} \leq s \right) = F_{\text{GUE}}(s) \quad m = \sigma t \quad c_1 = 1 - 2\sqrt{\sigma} \quad c_2 = \sigma^{-1/6} (1 - \sqrt{\sigma})^{2/3}$$

Finite-time KPZ fluctuations: universal renormalization group flow

WASEP: Sasamoto-Spohn 2010, Amir-Corwin-Quastel 2011

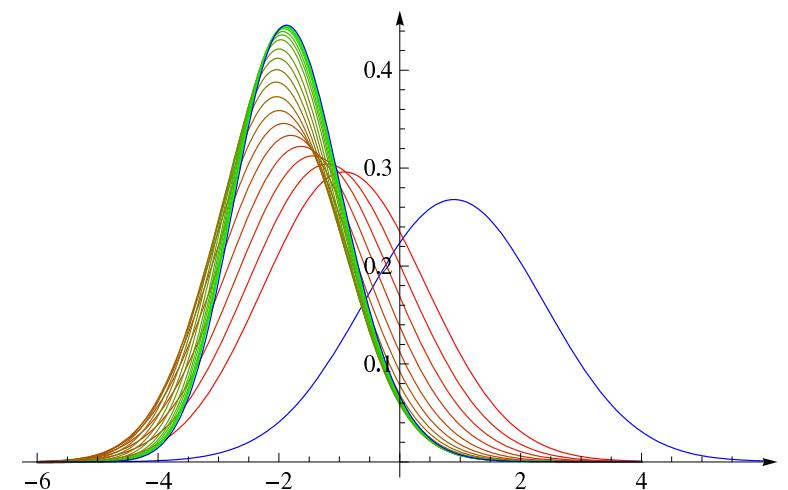
Replica KPZ equation: Dotsenko 2010, Calabrese-Le Doussal-Rosso 2010

Long time asymptotic analysis kernel:

weak asymmetry $1 - q \simeq \beta/t^{1/4}$

F_β interpolating between Gaussian and F_{GUE}

F_β also appear for KPZ droplet time $\tau = \beta^3$
trapped fermions



Multiple-point and multiple-time statistics

Multiple-point statistics: Airy processes

GUE Dyson's Brownian motion $\frac{dA(u)}{du} = -\frac{A(u)}{N} + W(u)$ A Hermitian $N \times N$
Stationary distribution GUE

Eigenvalues $\lambda_1 \leq \dots \leq \lambda_N$: $\frac{d\lambda_j(u)}{du} = -\frac{\lambda_j(u)}{2N} + \frac{1}{N} \sum_{k \neq j} \frac{1}{\lambda_j(u) - \lambda_k(u)} + \frac{\xi_j(u)}{\sqrt{N}}$
 λ_j non-crossing Wiener processes

Airy₂ process: $\mathcal{A}_2(u) \equiv \lim_{N \rightarrow \infty} \frac{\lambda_N(2uN^{2/3}) - 2\sqrt{N}}{N^{-1/6}}$

KPZ droplet growth: $\lim_{t \rightarrow \infty} \frac{h(t^{2/3}x, t) - ct}{at^{1/3}} \equiv \mathcal{A}_2(x) - x^2$

Multiple-time statistics: ?

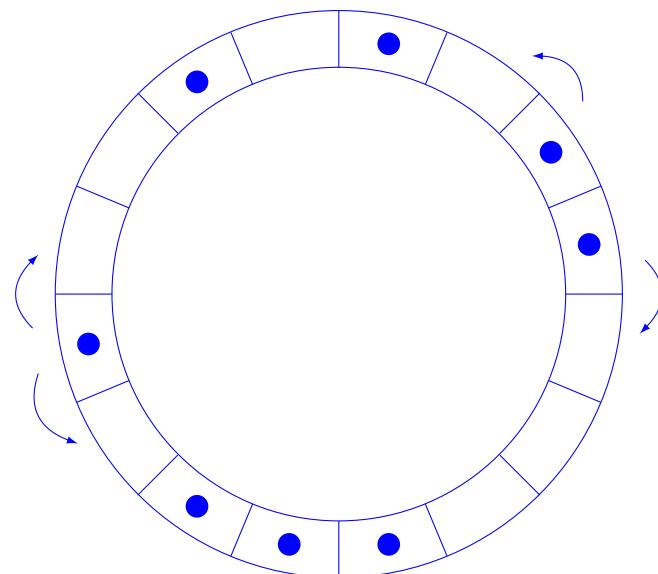
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Scaling properties finite volume KPZ equation

$$\text{KPZ equation } \partial_\tau h = \nu \partial_x^2 h + \lambda (\partial_x h)^2 + \sqrt{D} \xi \quad \frac{1}{a} \xi\left(\frac{x}{b}, \frac{\tau}{c}\right) \equiv \frac{\sqrt{bc}}{a} \xi(x, \tau)$$

$$\text{Periodic interface } x = x + \ell \Rightarrow h_{\nu, \lambda, D, \ell}(x, \tau) \equiv \frac{\nu}{\lambda} h_{1, 1, 1, \frac{\lambda^2 D}{\nu^3} \ell} \left(\frac{\lambda^2 D}{\nu^3} x, \frac{\lambda^4 D^2}{\nu^5} \tau \right)$$

$$\text{One parameter family } h_\ell(x, \tau) = \frac{h_{1, 1, 1, \ell}(\ell x, \ell^{3/2} \tau)}{\sqrt{\ell}} \quad x = x + 1$$

$$\text{Edwards-Wilkinson fixed point } \lim_{\ell \rightarrow 0} h_\ell(x, \sqrt{\ell} \tau)$$

$$\text{KPZ fixed point } h_\infty(x, \tau)$$

- growth $\tau \rightarrow 0$: $\delta h \sim \tau^{1/3}$ and $\delta x \sim \tau^{2/3}$ as in infinite system
- saturation $\tau \rightarrow \infty$: non-equilibrium steady state $\delta h \sim 1$ and $\delta x \sim 1$

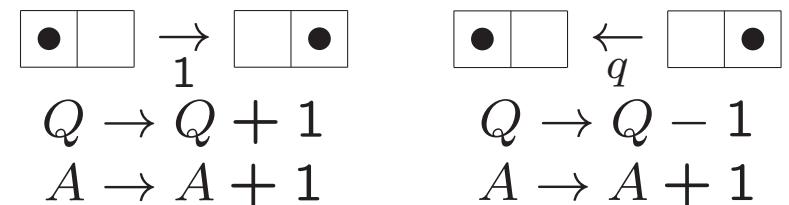
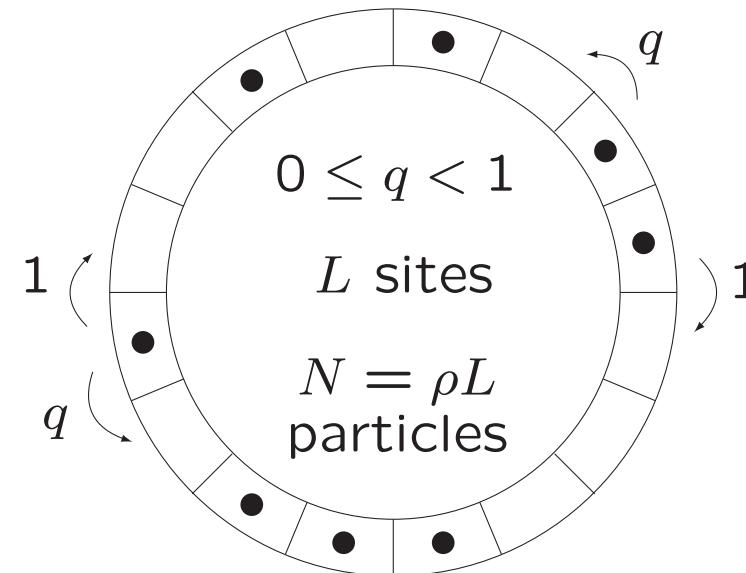
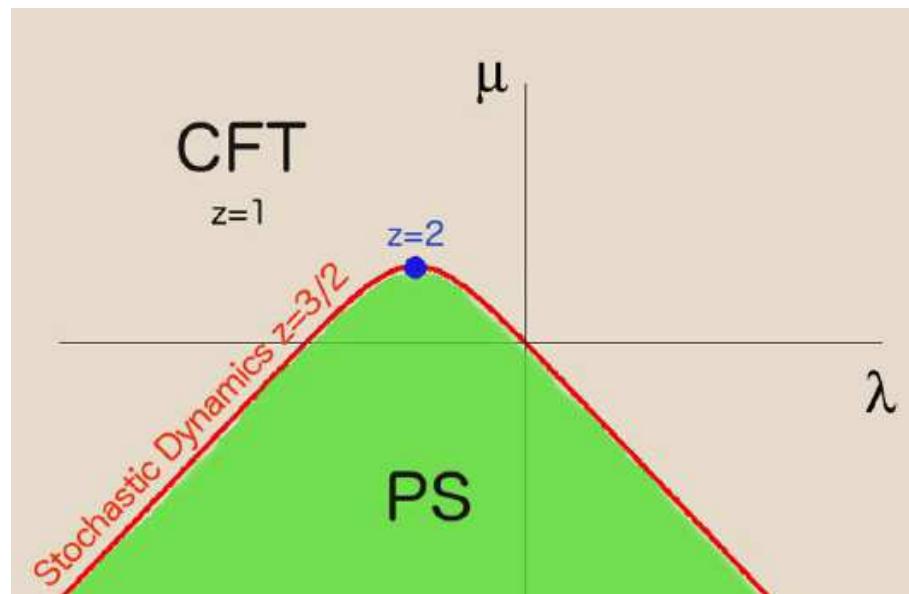
Statistics h_∞ : finite volume analogues Tracy-Widom distributions ?

KPZ, CFT and phase separated regimes of ASEP

D. Karevski and G.M. Schütz 2017, G.M. Schütz 2017

Total current Q and activity A

$$\langle e^{\lambda Q_t + \mu A_t} \rangle = \sum_C \langle C | e^{tM(\lambda, \mu)} | C_0 \rangle$$



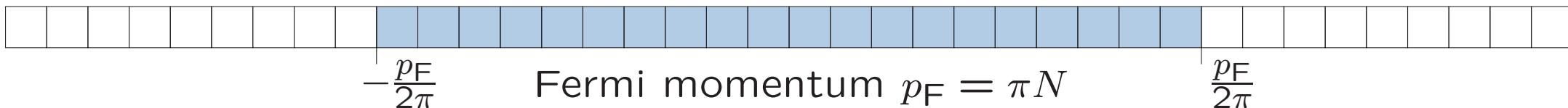
$\mu = 0$: only current

- KPZ fluctuations $\lambda \sim L^{-3/2}$
- Conformal regime $\lambda \sim L^0$, $\lambda > 0$: large deviations **higher current**
- Phase Separated $\lambda \sim L^0$, $\log q < \lambda < 0$: large deviations **lower current**

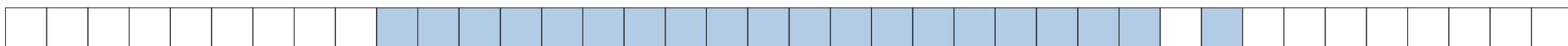
Bethe ansatz for ASEP

Eigenstates characterized by sets $\{n_1, \dots, n_N\}$ of integers / half-integers

Stationary state Fermi sea $\{n_j^{(0)}\} = \{-\frac{N-1}{2}, -\frac{N-3}{2}, \dots, \frac{N-3}{2}, \frac{N-1}{2}\}$



Spectral gap (Gwa-Spohn 1992, Kim 1995, Golinelli-Mallick 2004)



$$\frac{E(\lambda)}{1-q} = \sum_{j=1}^N \left(\frac{1}{1-y_j} - \frac{1}{1-qy_j} \right) \quad \text{and} \quad |\psi(\lambda)\rangle = B(y_1) \dots B(y_N) |0\rangle$$

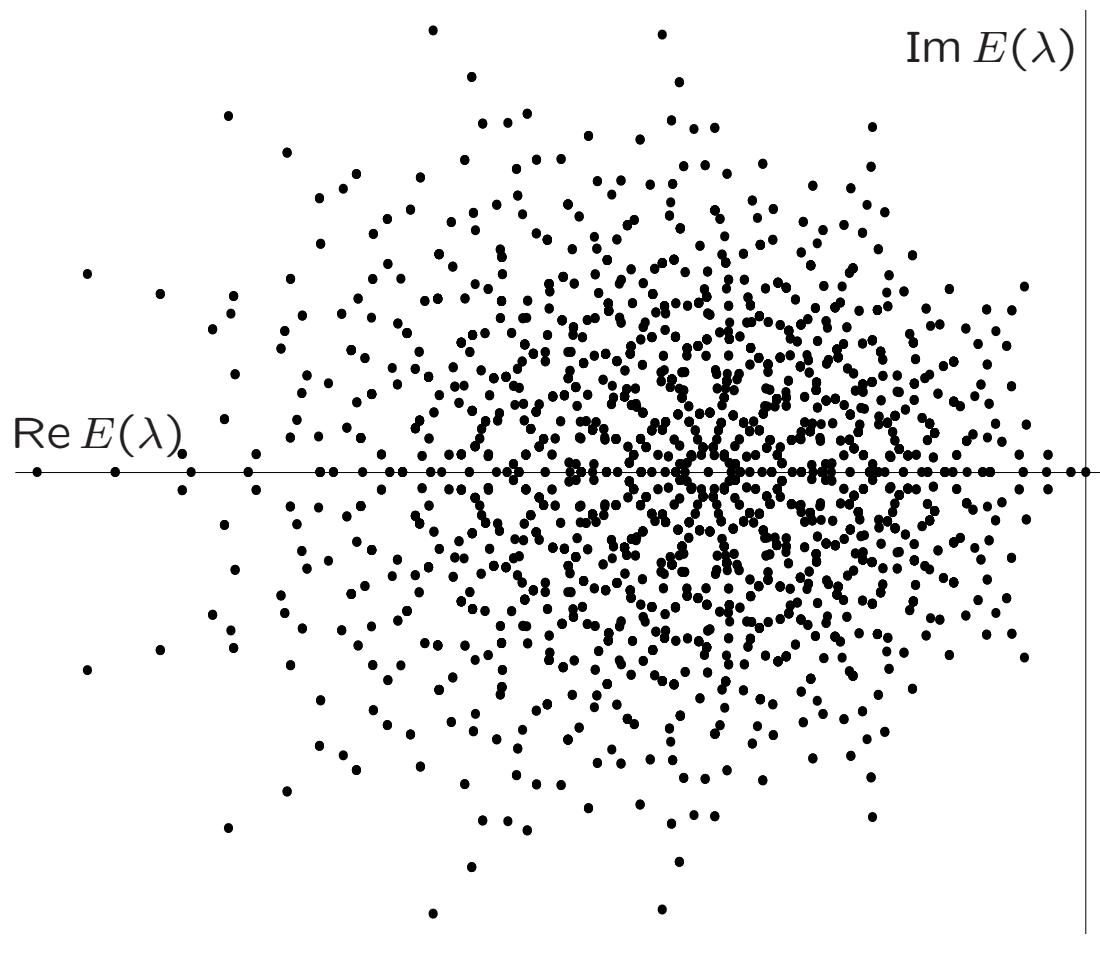
with $\{y_1, \dots, y_N\}$ solution of Bethe equations $f(y_j) = \frac{2i\pi n_j}{L}$

TASEP $q = 0$ $f(y) = \log \left(\frac{1-y}{y^\rho} \right) + b$ with $b - \lambda = \frac{1}{L} \sum_{\ell=1}^N \log y_\ell$ “mean field”

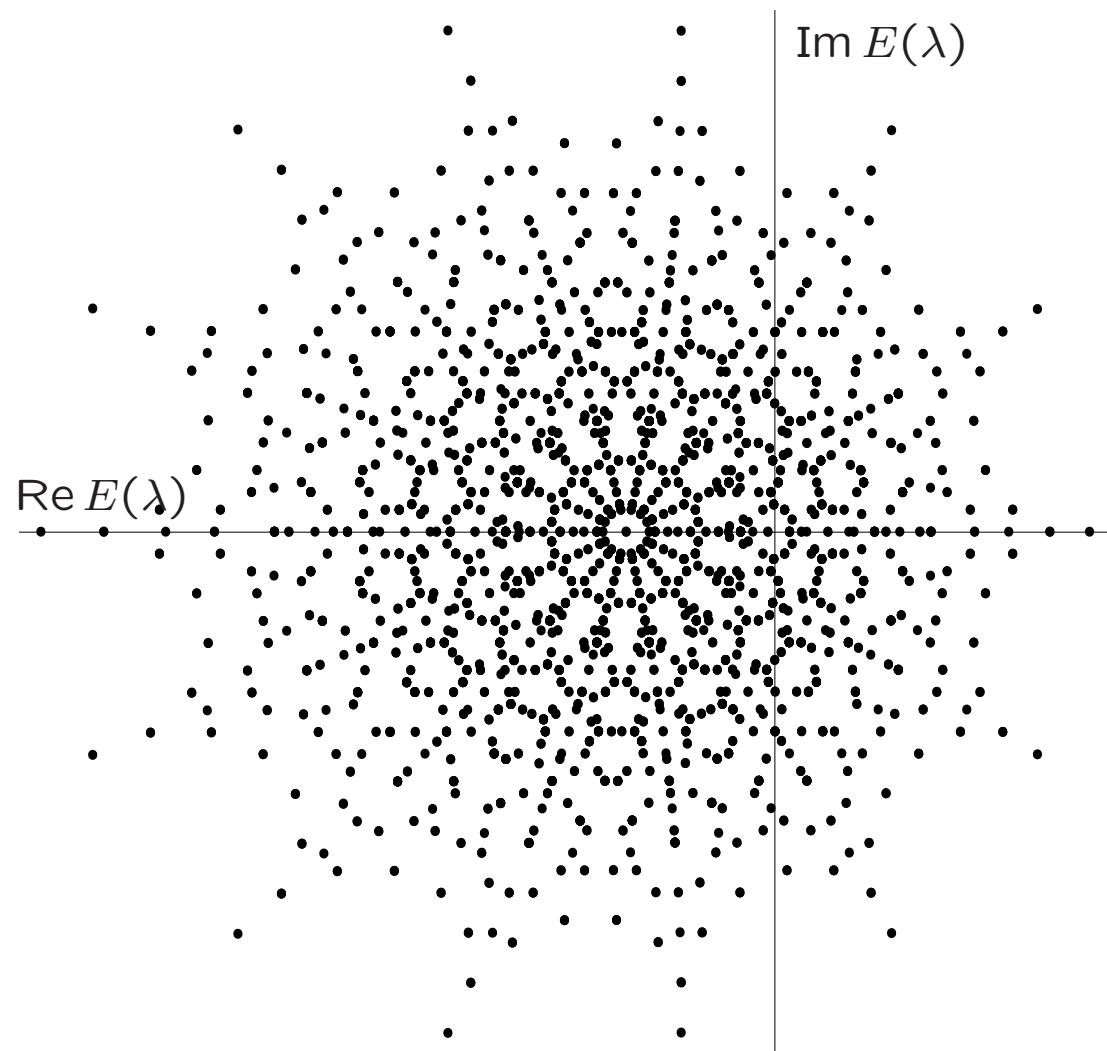
ASEP $f(y) = \lambda + \log \left(\frac{1}{y^\rho} \frac{1-y}{1-qy} \right) - \frac{1}{L} \sum_{\ell=1}^N \log \left(\frac{1}{y} \frac{y-qy_\ell}{y_\ell-qy} \right)$ KPZ universality
Extrapolation

Spectrum of TASEP

KPZ $\lambda = 0$



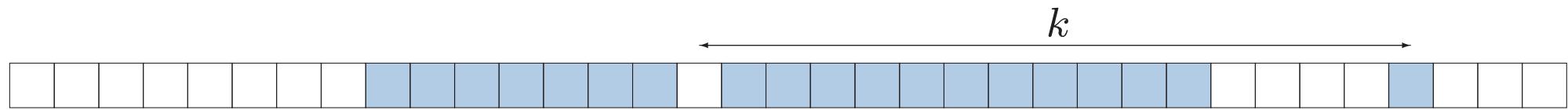
CFT $\lambda = 1$



TASEP $q = 0$ $L = 14$ sites
 $N = 7$ particles

$\lambda \rightarrow \infty$: free Fermions

One particle-hole spectrum of TASEP

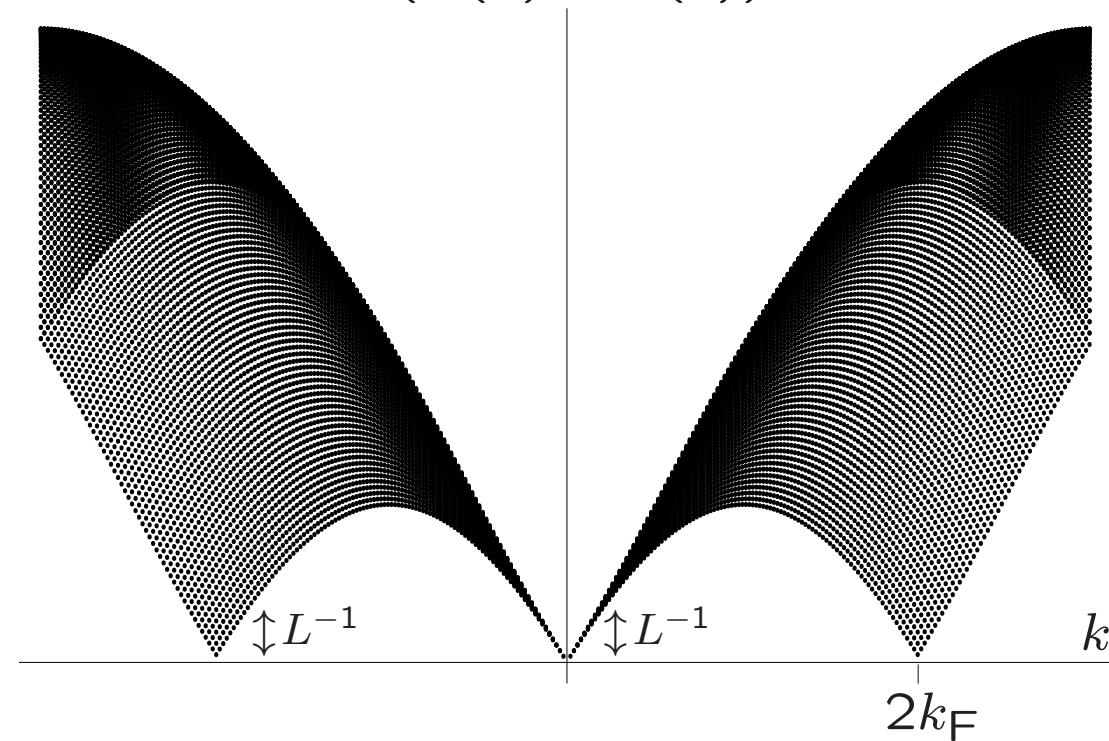
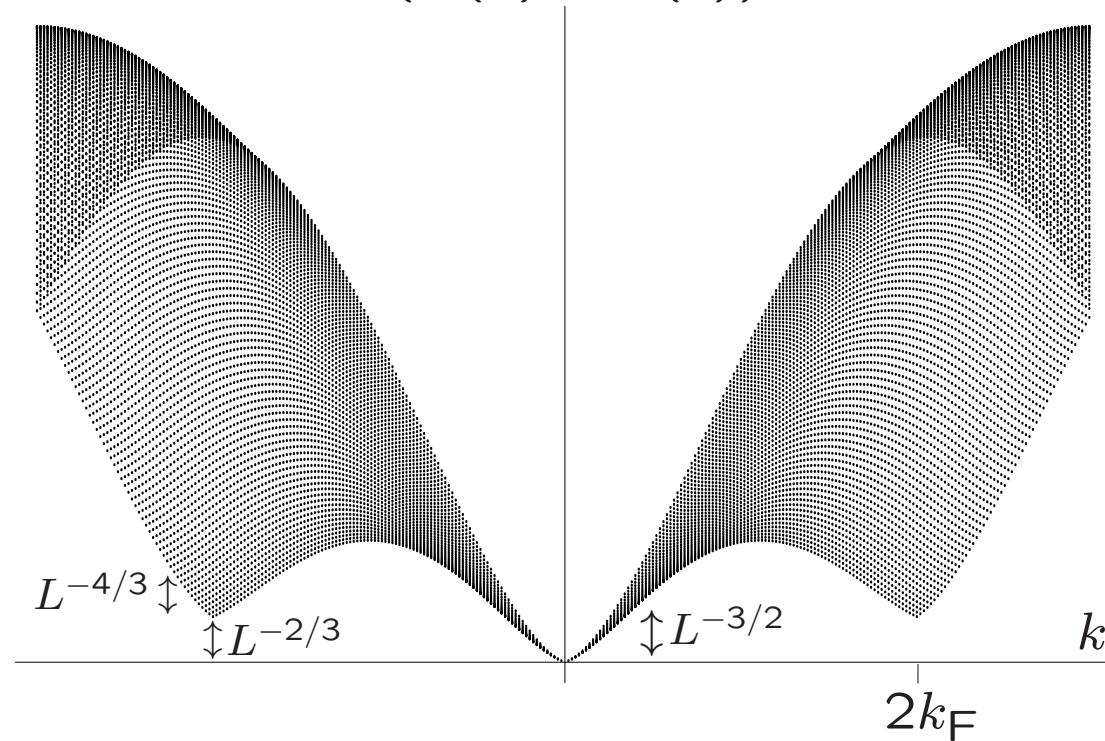


KPZ $\lambda = 0$

CFT $\lambda = 1$

$\text{Re}(E(0) - E(k))$

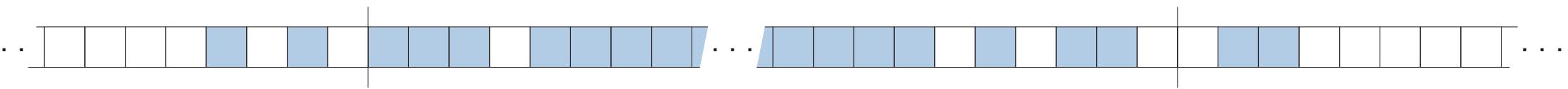
$\text{Re}(E(0) - E(k))$



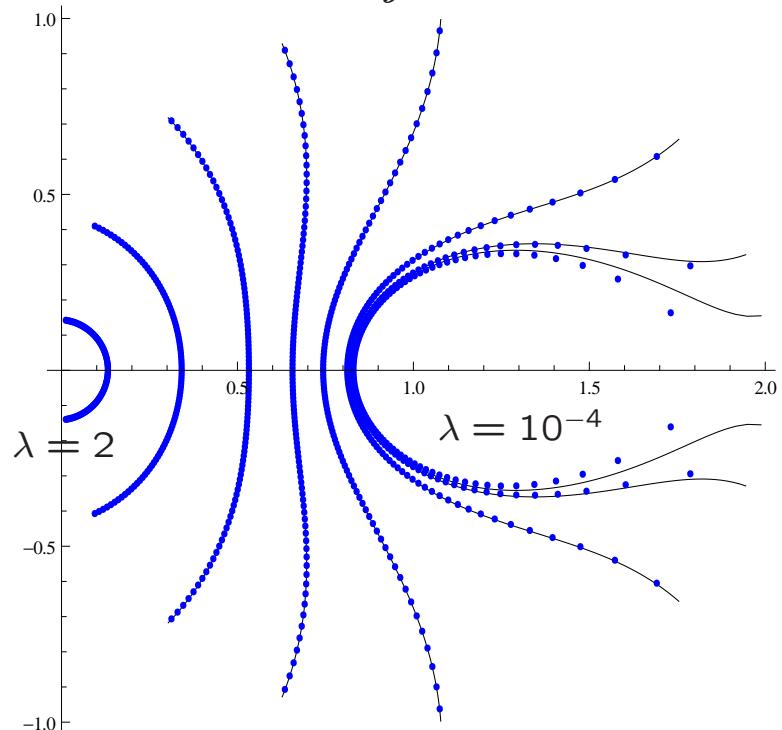
Luttinger liquid (imaginary time)

TASEP $q = 0$ $L = 300$ sites
 $N = 100$ particles

Spectrum in the CFT regime



Bethe roots $z_j = e^{iq_j} = 1 - y_j$



$\lambda \rightarrow \infty \Rightarrow q_j \in$ one string

$$E_1 = \left(\frac{\pi^2}{6} + 2\pi^2 \Delta^2 - \pi(p_R - p_L) + i\eta(1 - 2\rho)(p_R + p_L) \right) \frac{\partial_\lambda^2 H(-e^{-b(\lambda)})}{b''(\lambda)}$$

Imbalance on right side of Fermi sea
 $\Delta = (\text{nb quasi-particles}) - (\text{nb holes})$

Euler-Maclaurin formula

$$F(z) = \sum_{m=1}^{\infty} \binom{\rho m + 1}{m+1} \frac{(m+1) \operatorname{sinc}(m\pi\rho)}{m(\rho m + 1)} z^m$$

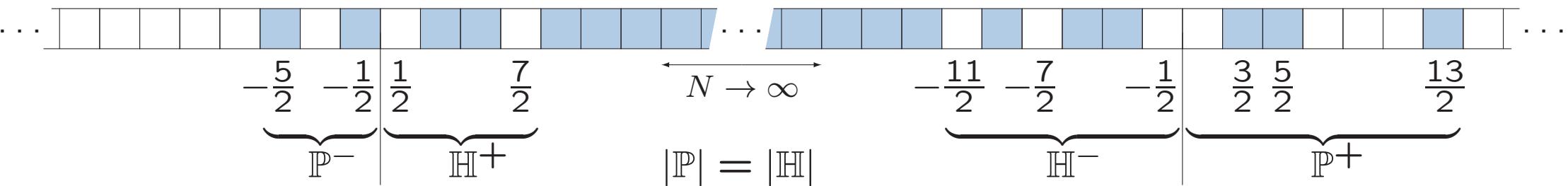
$$H(z) = \sum_{m=-1}^{\infty} \binom{\rho m + 1}{m+1} \frac{\operatorname{sinc}(m\pi\rho)}{\rho m + 1} z^m$$

$$-\frac{E\left(\lambda + \frac{2i\pi\Delta}{L}\right)}{\rho(1-\rho)} \underset{L \rightarrow \infty}{\simeq} H\left(-e^{-b(\lambda)}\right)L + \frac{E_1}{L}$$

$$b(\lambda) - \lambda = F\left(-e^{-b(\lambda)}\right)$$

$$\frac{\partial_\lambda^2 H(-e^{-b(\lambda)})}{b''(\lambda)}$$

Spectrum at the KPZ fixed point



No imbalance $|\mathbb{P}^\pm| = |\mathbb{H}^\mp|$

$$\text{Total momentum } p = \sum_{k \in 2\pi\mathbb{P}} k - \sum_{k \in 2\pi\mathbb{H}} k$$

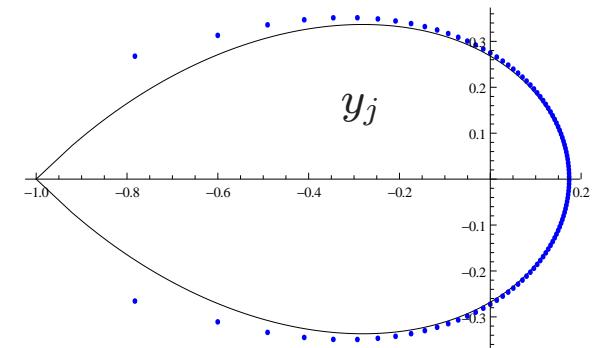
Euler-Maclaurin with square root singularities

$$\Xi(v) = -\frac{\text{Li}_{5/2}(-e^v)}{\sqrt{2\pi}} + \sum_{k \in 2\pi\mathbb{P}} \frac{(2ik - 2v)^{3/2}}{3} + \sum_{k \in 2\pi\mathbb{H}} \frac{(2ik - 2v)^{3/2}}{3}$$

Fugacity KPZ regime $s = \sqrt{\rho(1-\rho)} \lambda L^{3/2}$ $\nu(s)$ solution of $\Xi'(\nu(s)) = s$

$$E(\lambda) - \rho(1-\rho)\lambda L \simeq -\frac{(1-2\rho)ip}{L} + \sqrt{\rho(1-\rho)} \frac{\Xi(\nu(s))}{L^{3/2}}$$

Universal scaling functions independent of density ρ : no parameter



Asymptotics of sums: Euler-Maclaurin formula

No singularity for f at 0 and 1: asymptotic expansion

$$\sum_{j=1}^N f\left(\frac{j+d}{N}\right) \simeq N \left(\int_0^1 du f(u) \right) - \sum_{\ell=1}^{\infty} \frac{B_{\ell}(d+1) f^{(\ell-1)}(0)}{\ell! N^{\ell-1}} + \underbrace{\sum_{\ell=1}^{\infty} \frac{B_{\ell}(d+1) f^{(\ell-1)}(1)}{\ell! N^{\ell-1}}}_{\mathcal{R}}$$

Logarithmic singularity: Stirling's formula

$$\sum_{j=1}^N \log\left(\frac{j+d}{N}\right) \simeq -N + (d + \frac{1}{2}) \log N + \log\left(\frac{\sqrt{2\pi}}{\Gamma(d+1)}\right) + \mathcal{R}$$

Non-integer power: Hurwitz ζ function

$$\sum_{j=1}^N \left(\frac{j+d}{N}\right)^{\nu} \simeq \frac{N}{\nu+1} + \frac{\zeta(-\nu, d+1)}{N^{\nu}} + \mathcal{R} \quad (\nu \neq -1) \quad \zeta(s, d) = \sum_{j=0}^{\infty} \frac{1}{(j+d)^s}$$

Triangle with non-integer powers: multi zeta functions

$$\sum_{j=1}^N \sum_{k=j+1}^N (j+d)^{\nu} (k+d')^{\nu'} = \text{triangle} = \text{triangle}_1 + \text{triangle}_2 - \text{triangle}_3$$

$$= \zeta(-\nu, -\nu'; d+1, d'+1) + \zeta(-\nu', -\nu; N+d'+1, N+d) - \zeta(-\nu, d+1) \zeta(-\nu', N+d'+1)$$

$$\downarrow \\ \text{Li}_{1-s}(e^{\pm 2i\pi d})$$

Accurate numerics: Richardson extrapolation

L.F. Richardson 1927

Integrable models: exact solutions & accurate numerics

Richardson extrapolation: extract a_∞ from sequence a_M

$$\text{expansion } a_M \underset{M \rightarrow \infty}{\simeq} a_\infty + \frac{b}{M^\theta} + \frac{c}{M^{2\theta}} + \frac{d}{M^{3\theta}} + \dots$$

“Deferred approach to the limit”: extrapolate $f(x) = a_{1/x}$ to $x = 0$
from few values of M instead of large M

Bethe ansatz: finite size expansion in integrable models: $\theta = 1$ (CFT)
 $\theta = 1/2$ (KPZ)

Integral equations: groundstate energy δ -Bose/ δ -Fermi/Toda weak coupling

$$e_B(\gamma) \simeq \gamma - \frac{4\gamma^{3/2}}{3\pi} + \left(\frac{1}{6} - \frac{1}{\pi^2}\right)\gamma^2 + b\gamma^{5/2} \Rightarrow b \approx -0.001587699865 \approx \left(\frac{3\zeta(3)}{8} - \frac{1}{2}\right)\frac{1}{\pi^3}$$

$$e_F(\gamma) \simeq \pi^2/12 - \gamma/2 - \gamma^2/12 + c\gamma^3 \Rightarrow c \approx -0.0123402948369572 \approx -\zeta(3)/\pi^4$$

$$e_{\text{Toda}}(\gamma) \simeq d\gamma^2 \Rightarrow d \approx 3.35236412938821853512903205 \approx ?$$

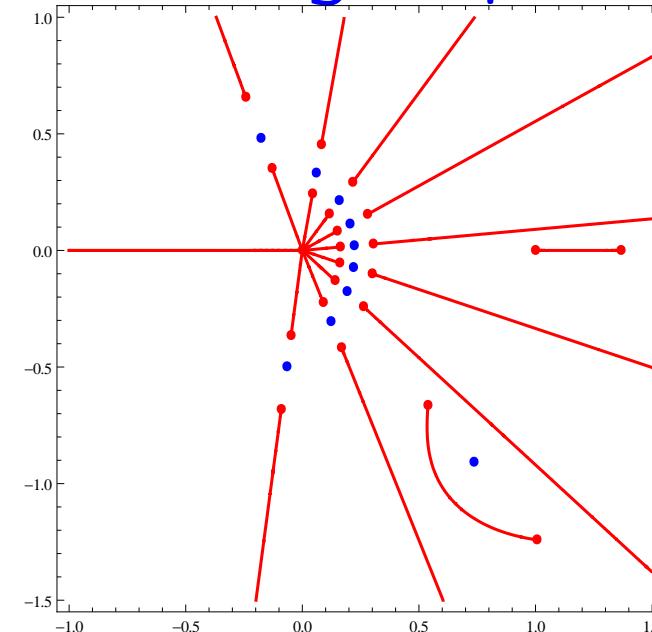
Spectral gap KPZ renormalization group flow

WASEP $1 - q = \frac{\mu}{\sqrt{L}} \Rightarrow E \simeq \frac{e(\mu)}{L^2}$

Reduced Bethe roots w_j

$$y_j \simeq -1 - \frac{2w_j}{\sqrt{N}} \quad y_{N-j} \simeq -1 - \frac{2w_{-j}}{\sqrt{N}}$$

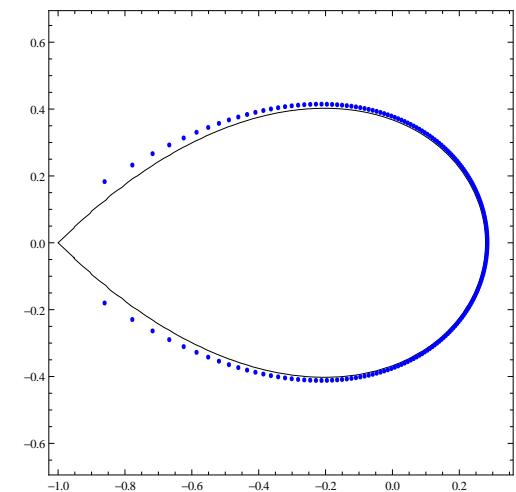
$w_j \xrightarrow[\mu \rightarrow 0]{} \text{zeroes of } \text{erfc}$



Truncated exponential $f_M(z) = \sum_{k=0}^M \frac{(-z)^k}{k!}$

Zeroes $z_j \Rightarrow \frac{z_j}{M}$ accumulate on Szegő curve

$\frac{z_j}{M} \simeq -1 - \frac{2v_j}{\sqrt{2M}}$ with v_j zero of erfc



Quantum Wronskian $Q(x)P(x - \mu) - Q(x - \mu)P(x) = C e^{\frac{x^2}{8}}$

Current fluctuations in the KPZ regime

$Q_i(t)$ time integrated current between sites i and $i + 1$

Generating function

$$\langle e^{L\lambda Q_i(t)} \rangle = \sum_{\mathcal{C}} \langle \mathcal{C} | e^{tM_i(\lambda)} | \mathcal{C}_0 \rangle \quad M_i(\lambda) = e^{-\lambda X_i} M(\lambda) e^{\lambda X_i} \quad X_i |\mathcal{C}\rangle = \sum_{j=1}^N x_j |\mathcal{C}\rangle$$

Expansion over Bethe eigenstates

$$\langle e^{L\lambda Q_i(t)} \rangle = \sum_r \frac{e^{tE_r(\lambda)}}{\langle \psi_r(\lambda) | \psi_r(\lambda) \rangle} \left(\sum_{\mathcal{C}} \langle \mathcal{C} | e^{-\lambda X_i} | \psi_r(\lambda) \rangle \right) \left(\langle \psi_r(\lambda) | e^{\lambda X_i} | \mathcal{C}_0 \rangle \right)$$

Large L asymptotics: universal KPZ regime $\rho = \frac{1}{2}$ $i = xL$ $\lambda = \frac{2s}{L^{3/2}}$

$$Q_i(t) \simeq \frac{t}{4} + \mathcal{R}L + \frac{\chi(x, \tau)\sqrt{L}}{2}$$

\mathcal{R} from deterministic hydrodynamics $t \sim L$
 $\mathcal{R}_{\text{flat}} = \mathcal{R}_{\text{stat}} = 0$ $\mathcal{R}_{\text{step}} = -|x|/2$

Asymptotics eigenstates $\Rightarrow \langle e^{s\chi(x, \tau)} \rangle = \sum_{\mathbb{P}, \mathbb{H}} \dots$

Effective field theory KPZ fixed point finite volume

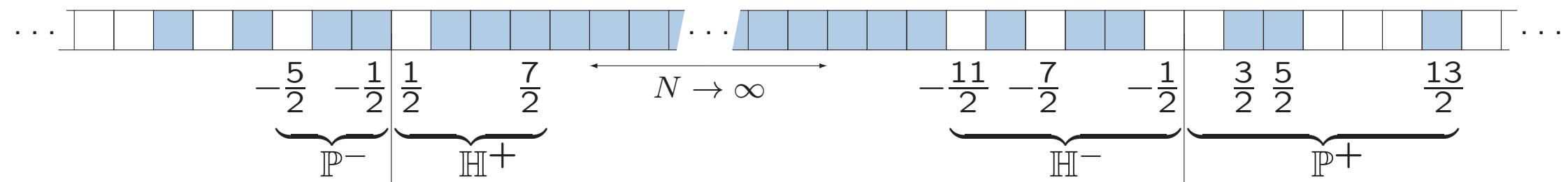
P. 2016 Phys. Rev. Lett. 116:090601

$$\langle e^{s\chi(x,\tau)} \rangle = \sum_{p,\varphi} \mathcal{A}_{x,s}[p, \varphi] e^{S_\tau[\varphi]} \quad S_\tau[\varphi] = \int_{-\infty}^{\varphi^{-1}(s)} dv (\varphi'(v)^2 + \tau\varphi(v))$$

Field φ conjugate to the height constructed from particle / hole excitations

$$\varphi(v) = -\frac{\text{Li}_{3/2}(-e^v)}{\sqrt{2\pi}} - \sum_{k \in 2\pi\mathbb{P}} \sqrt{2ik - 2v} - \sum_{k \in 2\pi\mathbb{H}} \sqrt{2ik - 2v} \quad \varphi = \Xi'$$

$$p = \sum_{k \in 2\pi\mathbb{P}} k - \sum_{k \in 2\pi\mathbb{H}} k \quad \text{total momentum}$$



Measure for the summation over p, φ :

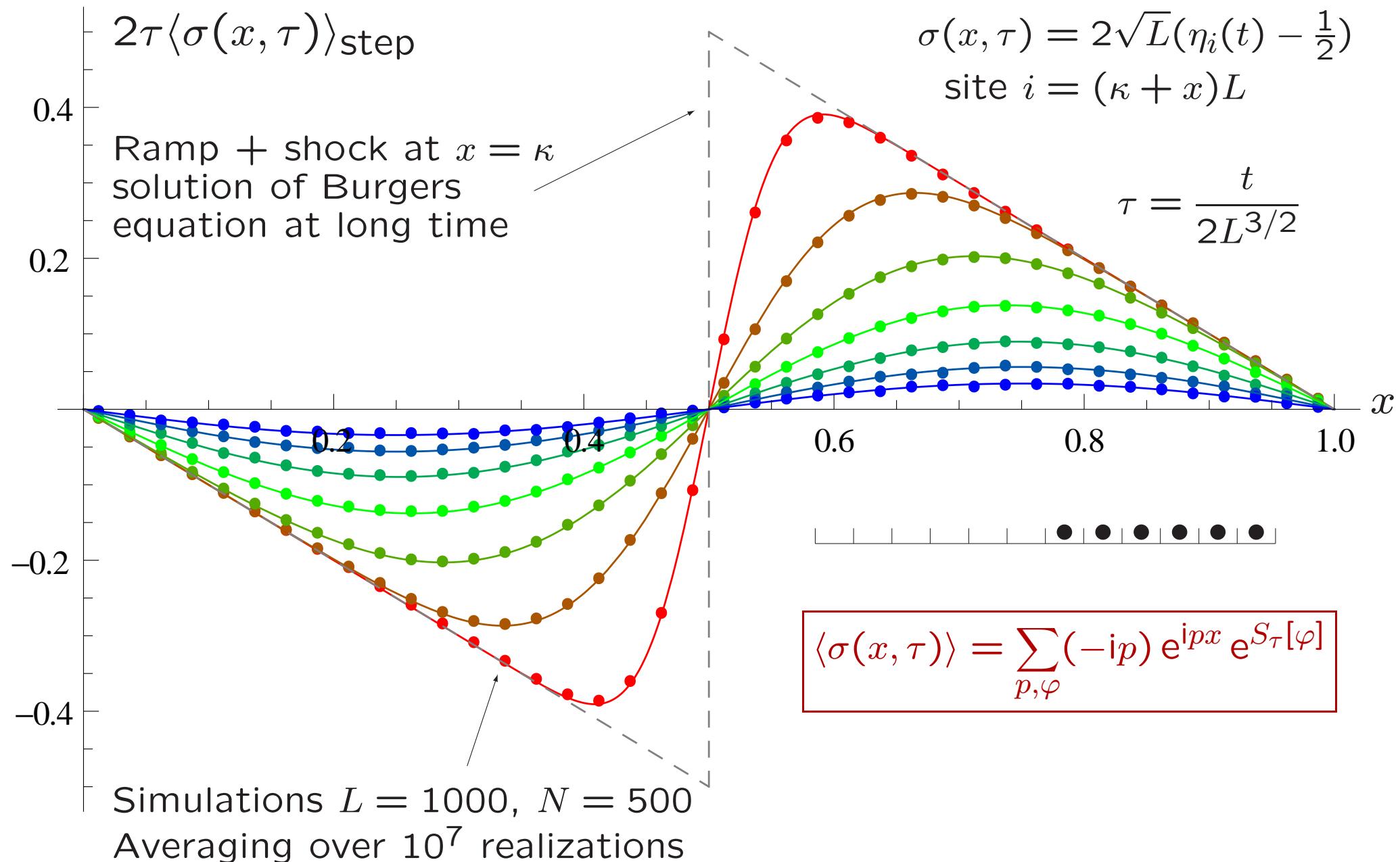
$$\sum_{p,\varphi} = \sum_{\substack{\mathbb{P}, \mathbb{H} \subset \mathbb{Z} + \frac{1}{2} \\ |\mathbb{P}| = |\mathbb{H}| \equiv m}} \mathbf{1}_{\{|\mathbb{P}^\pm| = |\mathbb{H}^\mp|\}} \frac{V(\mathbb{P})^2 V(\mathbb{H})^2}{(4i)^{2m} \varphi'(\varphi^{-1}(s))}$$

$$V(\mathbb{A}) = \prod_{k > k' \in 2\pi\mathbb{A}} (ik/4 - ik'/4)$$

Initial conditions: step and stationary

flat \rightarrow additional constraint $\mathbb{P} = \mathbb{H}$, extra factors of 1/2

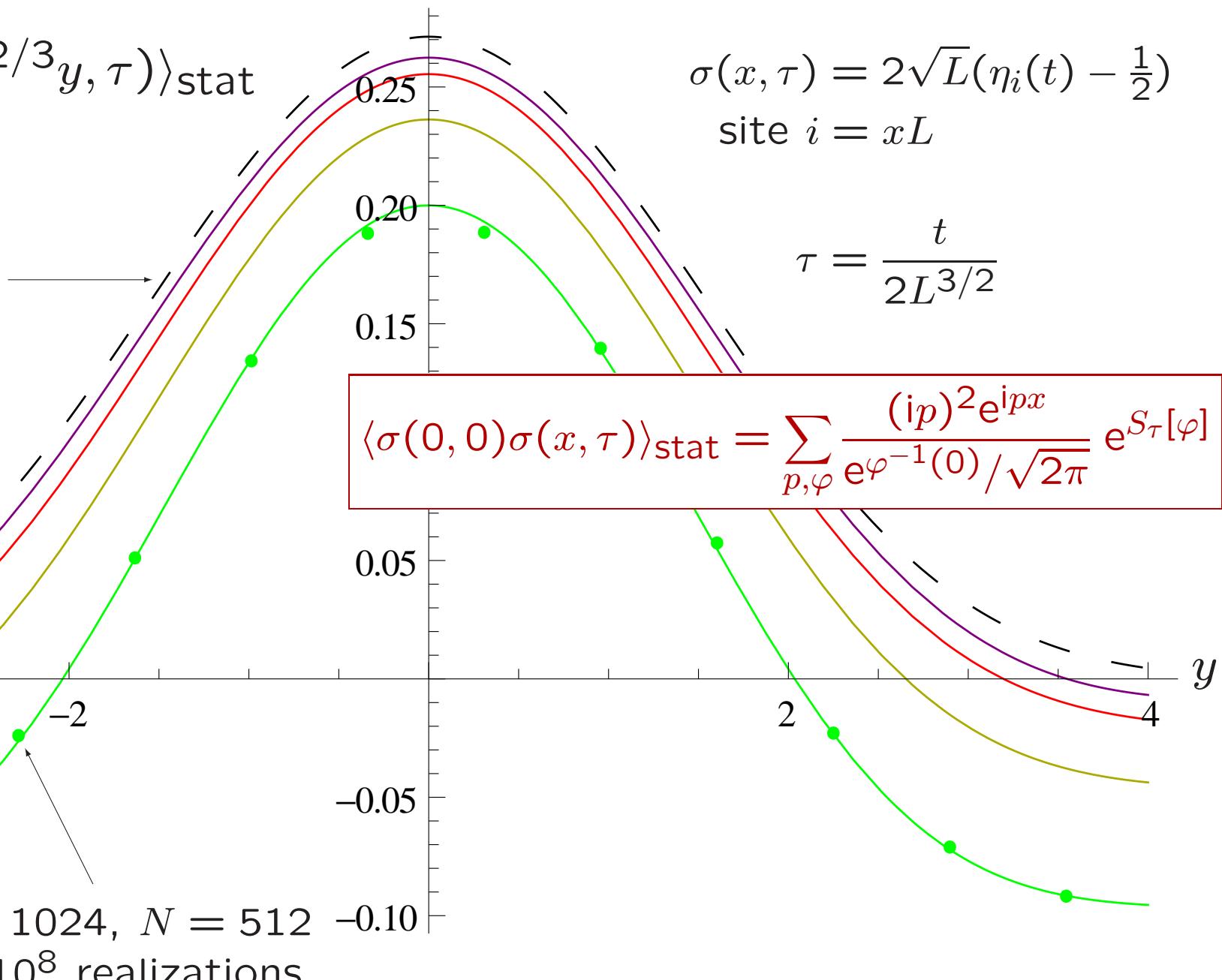
Average density with step initial condition



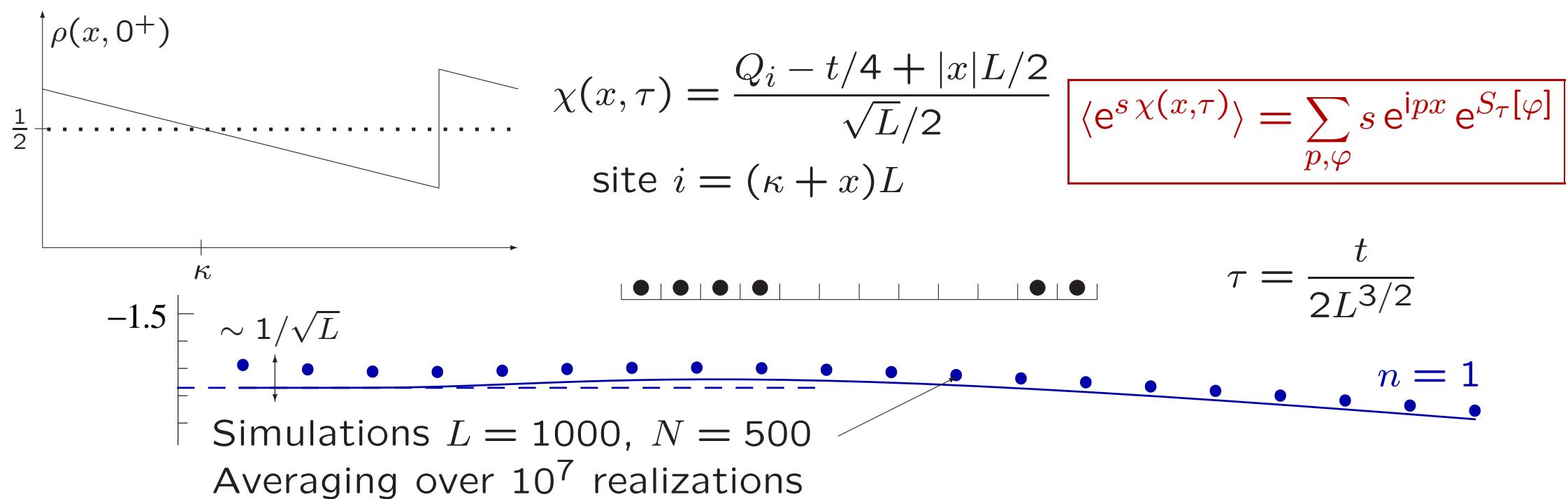
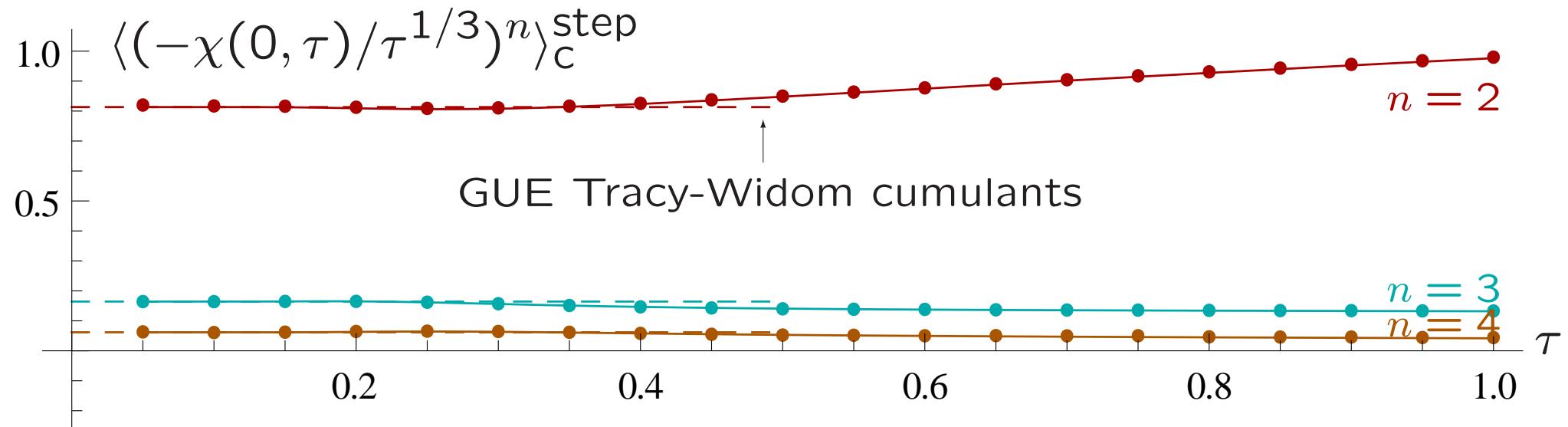
Stationary two-point function of the density

$$\tau^{2/3} \langle \sigma(0,0) \sigma(\tau^{2/3}y, \tau) \rangle_{\text{stat}}$$

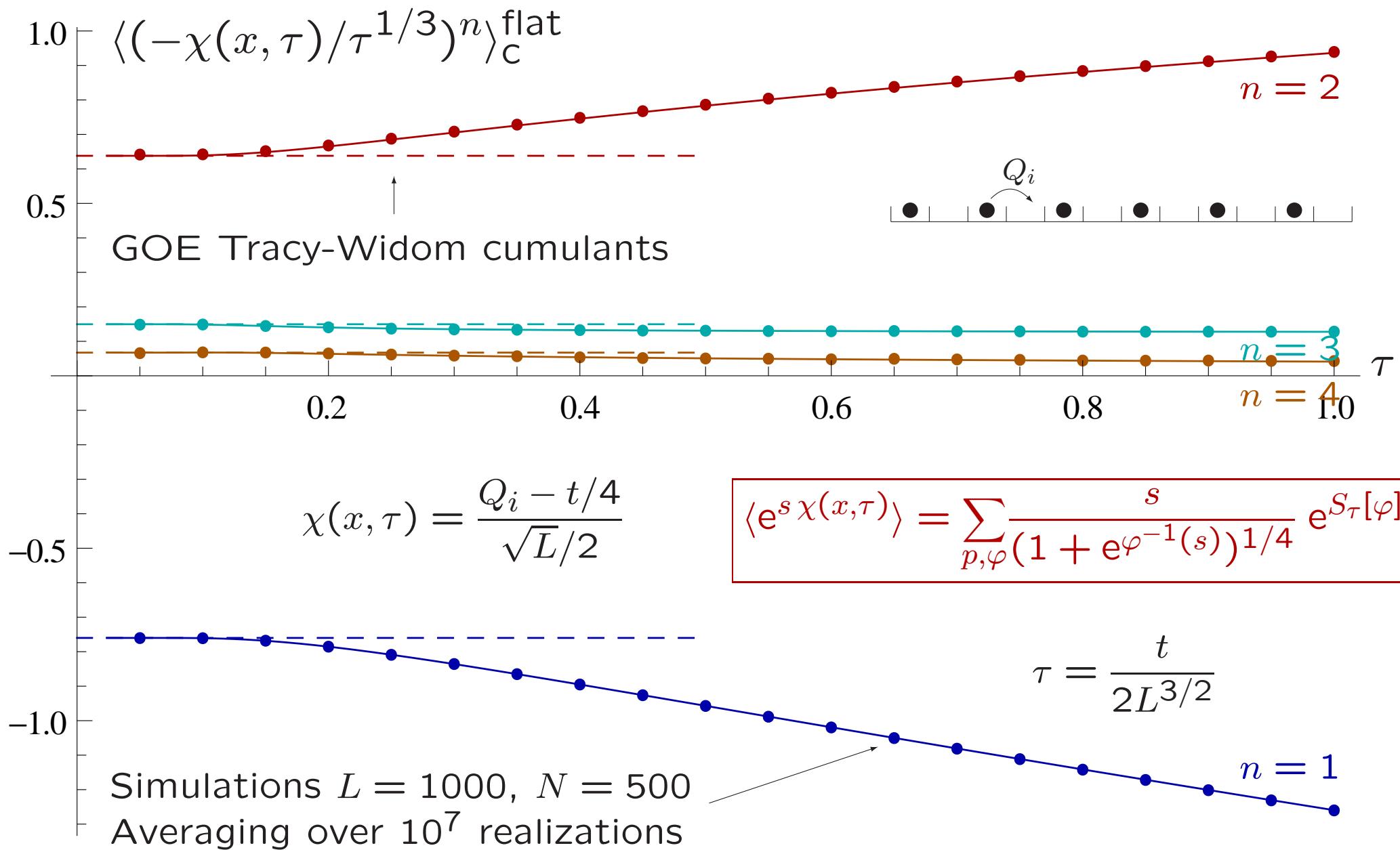
Prähofer-Spohn scaling function



Current fluctuations with step initial condition

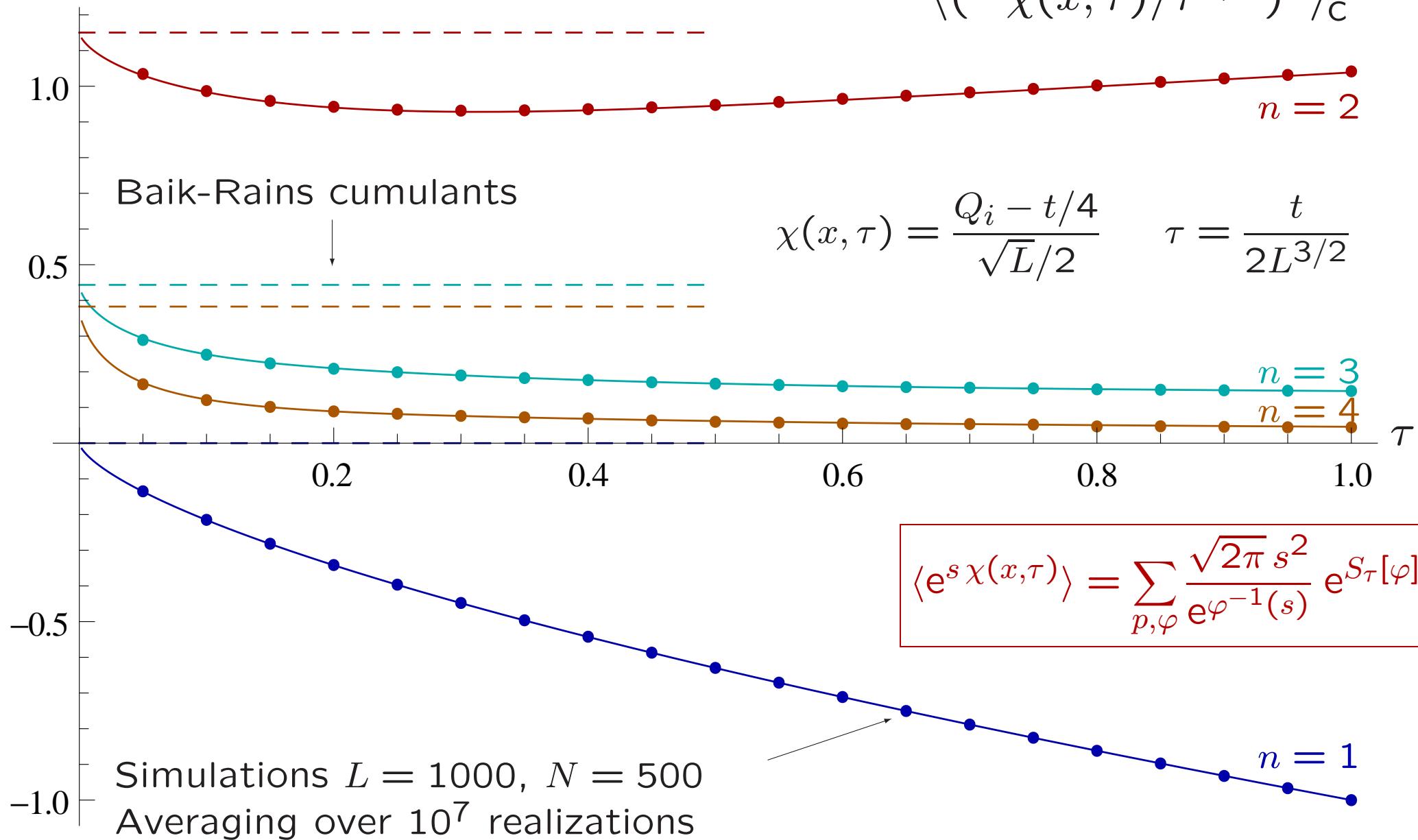


Current fluctuations with flat initial condition



Current fluctuations with stationary initial condition

$$\langle (-\chi(x, \tau)/\tau^{1/3})^n \rangle_c^{\text{stat}}$$



Probability distribution of current fluctuations

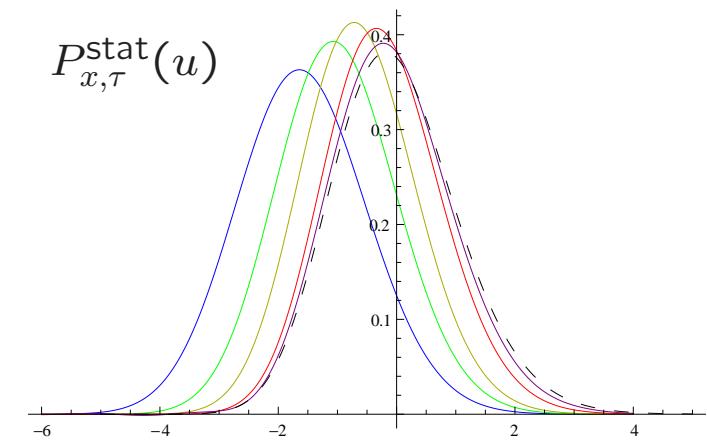
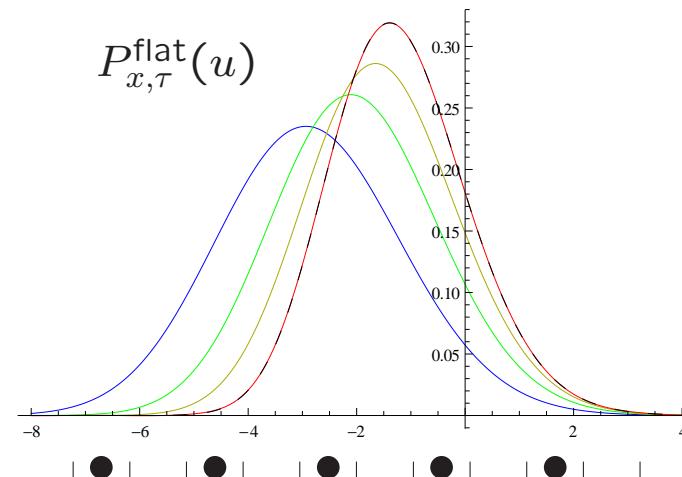
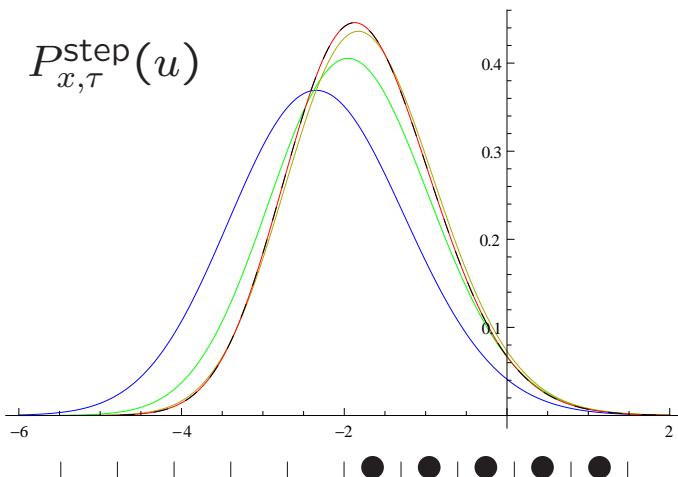
Generating function $G_{x,\tau}(s) = \langle e^{s\chi(x,\tau)} \rangle = \sum_{p,\varphi} \mathcal{A}_{x,s}[\varphi] e^{\int_{-\infty}^{\varphi^{-1}(s)} dv (\varphi'(v)^2 + \tau\varphi(v))}$

↓ Fourier transform

Probability density function of $\chi(x,\tau)$: $P_{x,\tau}(u) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{-isu} G_{x,\tau}(is)$

↓ Change of variables $s \rightarrow \nu = \varphi^{-1}(s)$
Explicit summation over eigenstates

Cumulative distribution function $P(\chi(x,\tau) > u)$: Fredholm determinant
(P. 2016, Baik-Liu 2016)



Bethe vectors of ASEP: determinants

Normalization of Bethe eigenstates: Gaudin-McCoy-Wu 1981, Korepin 1982

$$\langle \psi | \psi \rangle \propto \left(\prod_{j=1}^N \prod_{k=j+1}^N \frac{(y_j - qy_k)(qy_j - y_k)}{(y_j - y_k)^2} \right) \det \left[\partial_{y_i} \log \left(\left(\frac{1 - y_j}{1 - qy_j} \right)^L \prod_{k=1}^N \frac{qy_j - y_k}{y_j - qy_k} \right) \right]_{i,j}$$

Flat  and step  initial configurations:

Pozsgay 2014

Mossel-Caux 2010

$$\langle \psi | \mathcal{F} \rangle \propto \left(\prod_{j=1}^N \prod_{k=j+1}^N \frac{1}{(y_j - y_k)(1 - qy_j y_k)} \right) \det \left[\left(\frac{1 + y_k}{1 - y_k} \right)^{2j} - \left(\frac{1 + qy_k}{1 - qy_k} \right)^{2j} \right]_{j,k}$$

$$\langle \psi | \mathcal{S} \rangle \propto \left(\prod_{j=1}^N \prod_{k=j+1}^N \frac{1}{y_j - y_k} \right) \det \left[\frac{1}{(1 - qy_k)^j} - \frac{1}{(1 - y_k)^j} \right]_{j,k}$$

Stationary state and sum over final states: Slavnov 1989 + P. 2016

$$\sum_{\mathcal{C}} \langle \psi | e^{\frac{\lambda}{L} X} | \mathcal{C} \rangle \propto \sum_{\mathcal{C}} \langle \mathcal{C} | e^{-\frac{\lambda}{L} X} | \psi \rangle \propto \prod_{j=0}^{N-1} (1 - e^{-\lambda} q^j) \quad X | \mathcal{C} \rangle = \sum_{j=1}^N x_j | \mathcal{C} \rangle$$

States with no particle at site 1: Slavnov 1989 + P. 2016

$$\sum_{\mathcal{C}, \eta_1(\mathcal{C})=0} \langle \psi | e^{\frac{\lambda}{L} X} | \mathcal{C} \rangle \propto \sum_{\mathcal{C}, \eta_1(\mathcal{C})=0} \langle \mathcal{C} | e^{-\frac{\lambda}{L} X} | \psi \rangle \propto (e^{i p/L - \rho \lambda} - e^{-\lambda}) \left(\prod_{j=1}^{N-1} (1 - e^{-\lambda} q^j) \right)$$

Bethe vectors of TASEP ($q = 0$): product formulas

Normalization of Bethe eigenstates: the Gaudin determinant simplifies
Motegi-Sakai-Sato 2012

$$\langle \psi | \psi \rangle \propto \left(\prod_{j=1}^N \prod_{k=j+1}^N \frac{1}{(y_j - y_k)^2} \right) \left(\prod_{j=1}^N \left(L - N + \frac{N}{y_j} \right) \right) \sum_{j=1}^N \frac{y_j}{N + (L - N)y_j}$$

Flat initial configuration



$$\langle \psi | \mathcal{F} \rangle \propto \left(\prod_{j=1}^N \prod_{k=j+1}^N (1 - y_j y_k) \right)$$

Step initial configuration



$$\langle \psi | \mathcal{S} \rangle \propto 1$$

Stationary state and sum over final states: Bogoliubov 2009 + P. 2016

$$\sum_{\mathcal{C}} \langle \psi | e^{\frac{\lambda}{L} X} | \mathcal{C} \rangle \propto \sum_{\mathcal{C}} \langle \mathcal{C} | e^{-\frac{\lambda}{L} X} | \psi \rangle \propto 1 \quad X | \mathcal{C} \rangle = \sum_{j=1}^N x_j | \mathcal{C} \rangle$$

States with no particle at site 1: Bogoliubov 2009 + P. 2016

$$\sum_{\mathcal{C}, \eta_1(\mathcal{C})=0} \langle \psi | e^{\frac{\lambda}{L} X} | \mathcal{C} \rangle \propto \sum_{\mathcal{C}, \eta_1(\mathcal{C})=0} \langle \mathcal{C} | e^{-\frac{\lambda}{L} X} | \psi \rangle \propto e^{i p/L - \rho \lambda} - e^{-\lambda}$$

Bethe vectors of ASEP: large L asymptotics

Depend on q only through global normalization $\mathcal{N}(q) = \prod_{j=1}^{\infty} (1 - q^j)$

Action $S_0 = \lim_{\Lambda \rightarrow \infty} -2m^2 \log \Lambda + \int_{-\Lambda}^{\varphi^{-1}(s)} du \varphi'(u)^2$ $m = |\mathbb{P}| = |\mathbb{H}|$

Vandermonde determinant of wave numbers $V(A) = \prod_{\substack{k, k' \in 2\pi A \\ k > k'}} \frac{i k - i k'}{4}$
 Total momentum $p = \sum_{k \in 2\pi\mathbb{P}} k - \sum_{k \in 2\pi\mathbb{H}} k$

Normalization of
Bethe eigenstates $\frac{\mathcal{N}(q)^2}{\langle \psi | \psi \rangle} \simeq \frac{\sqrt{L}/2}{e^{s\sqrt{L}/2}} \frac{V(\mathbb{P})^2 V(\mathbb{H})^2 e^{S_0}}{(-1)^m 16^m \varphi'(\varphi^{-1}(s))}$

Flat initial configuration $\frac{\langle \psi | \mathcal{F} \rangle}{\mathcal{N}(q)} \simeq e^{s\sqrt{L}/4} \mathbf{1}_{\{\mathbb{P}=\mathbb{H}\}} \frac{4^m (1 + e^{\varphi^{-1}(s)})^{-1/4}}{V(\mathbb{P}) V(\mathbb{H}) e^{S_0/2}}$

Step initial configuration $\frac{\langle \psi | \mathcal{S} \rangle}{\mathcal{N}(q)} \simeq 1$

Conclusions

Exact results **KPZ universality** 1+1 dimension by **stochastic integrability**

Infinite system $\dots \bullet \dots$

- Bethe ansatz propagator as multiple integrals \rightarrow **Fredholm determinants**
- universal scaling functions from random matrix theory

Finite volume

- Bethe eigenstates: singular **Euler-Maclaurin** asymptotics
Richardson extrapolation
- effective field theory $S_\tau[\varphi] = \int_{-\infty}^{\varphi^{-1}(s)} dv (\varphi'(v)^2 + \tau\varphi(v))$

