

Exact results for KPZ universality in 1+1 dimension (1/2)

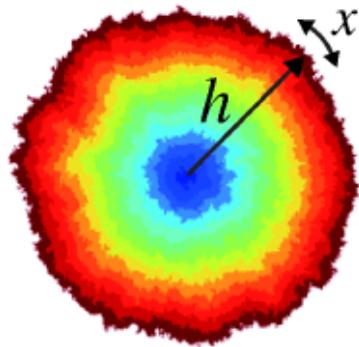


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Université Paul Sabatier, Toulouse



25 July - 4 August 2017
Cargèse summer school
Exact methods in low dimensional statistical physics

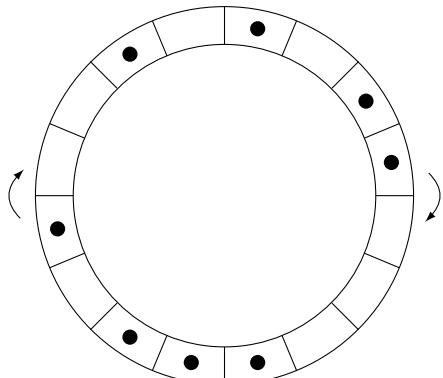
Interface growth



$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi$$

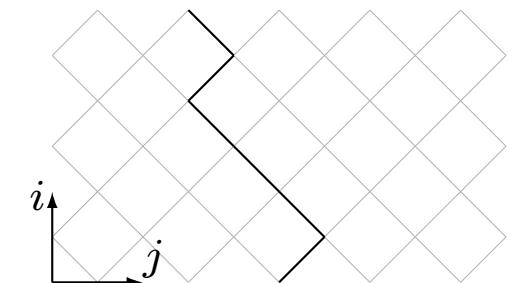
KPZ universality

Driven particles



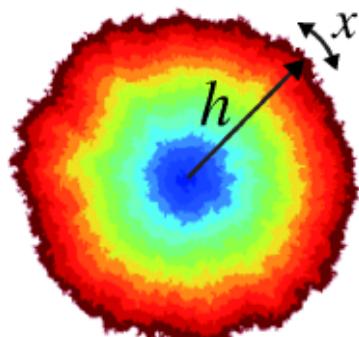
$$\partial_t \rho = \partial_x^2 \rho + \partial_x(\rho^2) + \partial_x \xi$$

Directed polymer in random medium



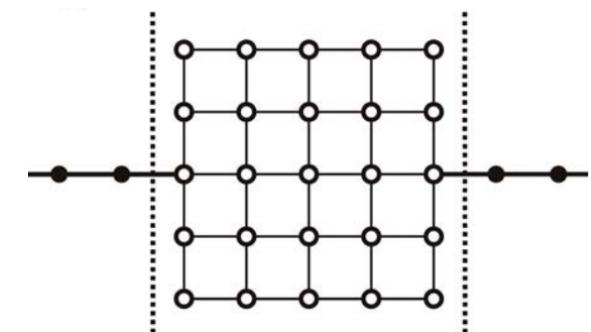
$$E = \sum_{(i,j) \in \text{path}} \varepsilon_{i,j}$$

Interface growth



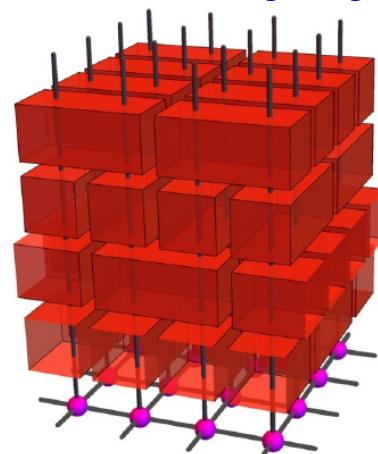
$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi$$

Conductance localized systems

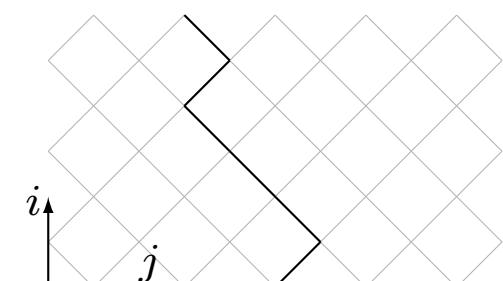


$$\log g \simeq -\frac{2L}{\ell} + \alpha \left(\frac{L}{\ell}\right)^{1/3} \chi$$

Random unitary dynamics

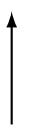


Directed polymer in random medium

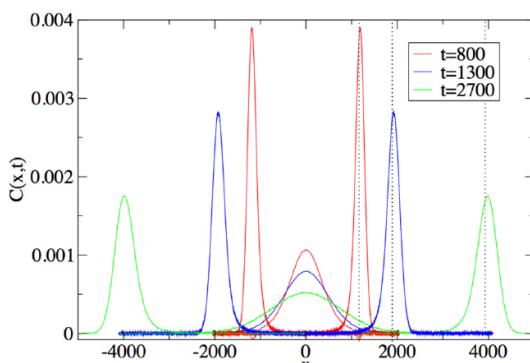


$$E = \sum_{(i,j) \in \text{path}} \varepsilon_{i,j}$$

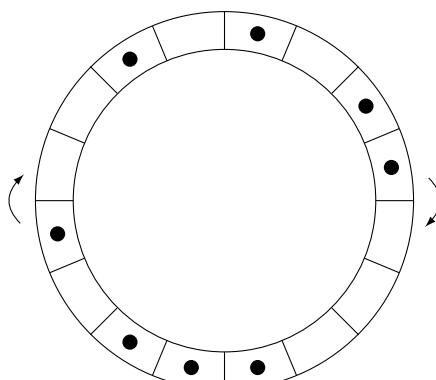
Gross-Pitaevskii



1D fluids



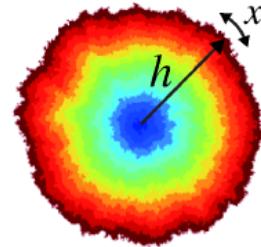
Driven particles



$$\partial_t \rho = \partial_x^2 \rho + \partial_x(\rho^2) + \partial_x \xi$$

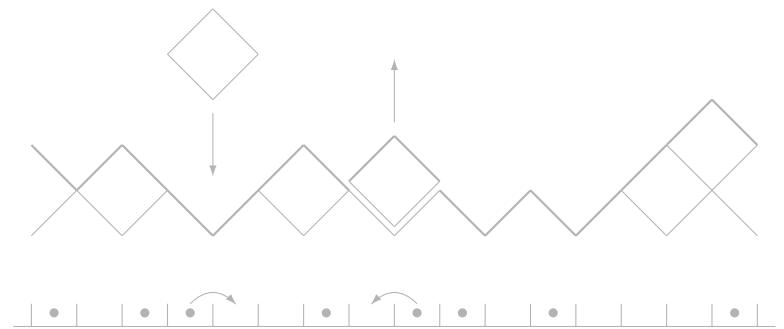
Trapped Fermi gases

I Interface growth



II KPZ universality

III Integrable models



Kardar-Parisi-Zhang equation (1986)

$$\partial_t h(t, x) = v \partial_x^2 h(t, x) + \frac{\lambda}{2} (\partial_x h(t, x))^2 + \xi(t, x)$$

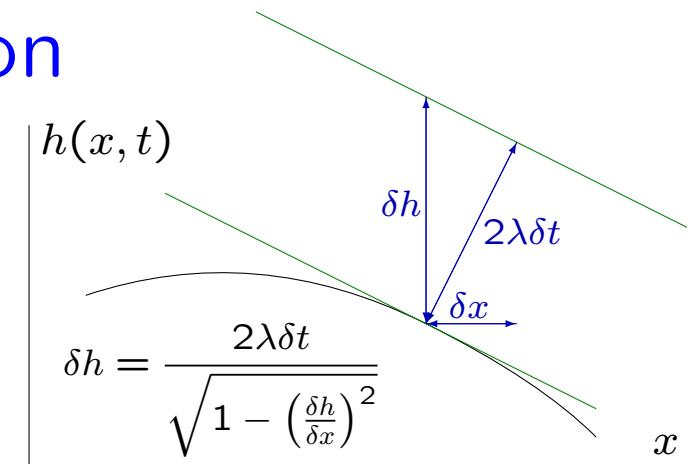


To thirty years of Kardar-Parisi-Zhang

Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h(x, t) = \nu \partial_x^2 h(x, t) + \lambda (\partial_x h(x, t))^2 + \sqrt{D} \xi(x, t)$$

Stable thermodynamic phase
growing inside metastable phase



Kinetic roughening

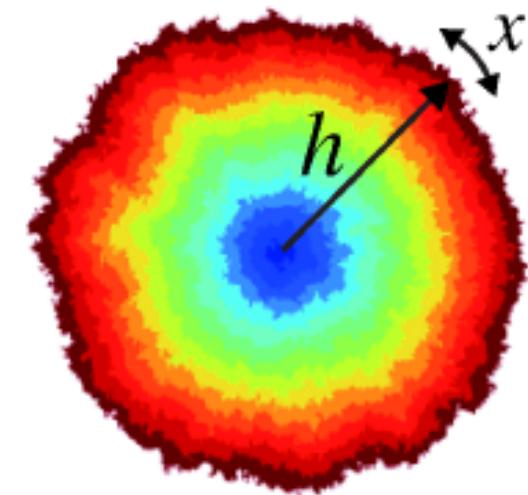
Roughness exponent $\alpha = 1/2$

Dynamical exponent $z = 3/2$

Family-Vicsek scaling $w = \ell^\alpha f_{\text{FV}}(t/\ell^z)$

Roughening $t \ll \ell^z$: $w \sim t^{1/3}$

Saturation $t \gg \ell^z$: $w \sim \ell^\alpha$



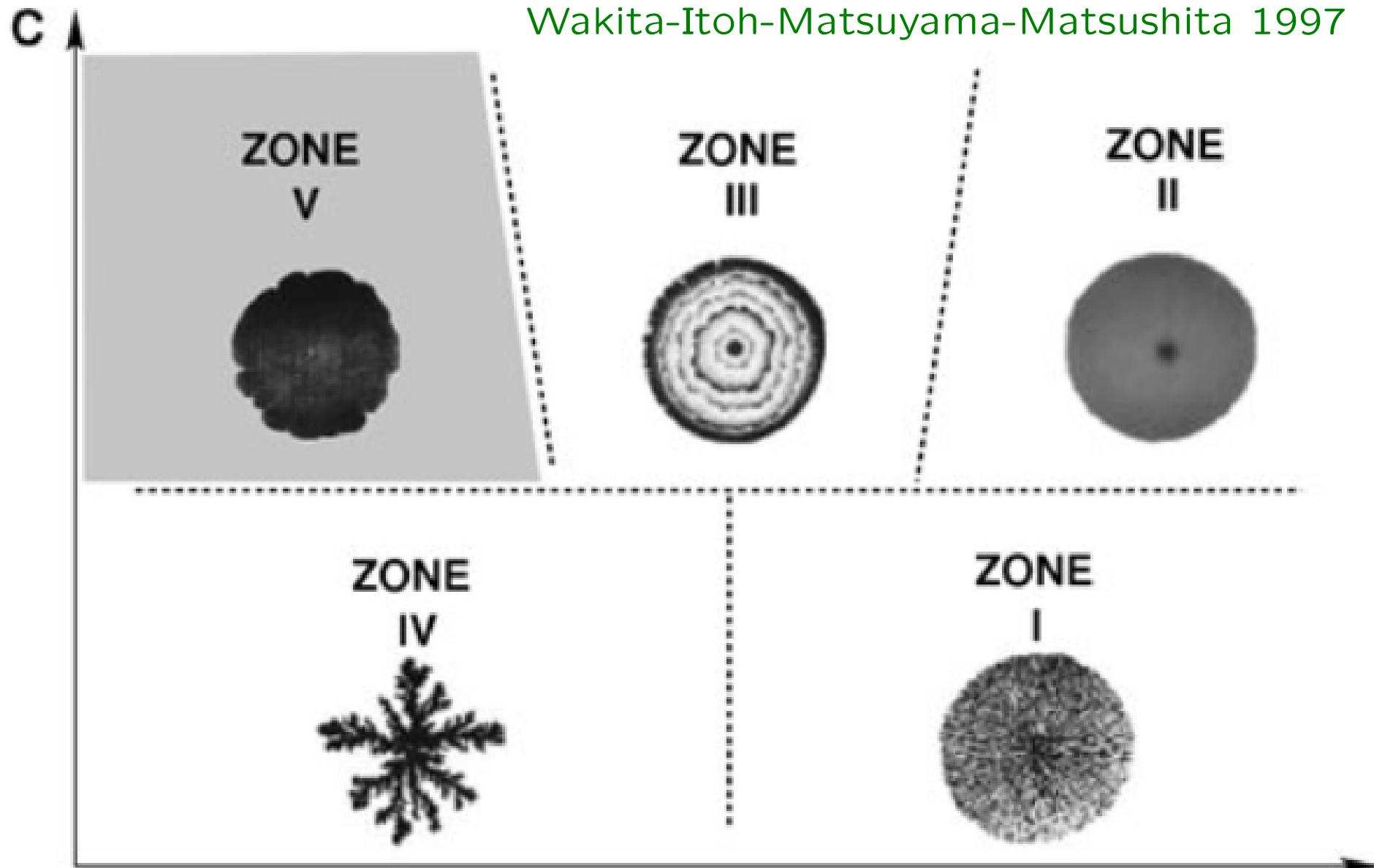
Noise + non-linearity ???

Regularity structures (M. Hairer 2013)
 \Rightarrow proofs of universality

Turbulent liquid crystals
Takeuchi-Sano 2010

Growth of bacterial colonies (*Bacillus subtilis*)

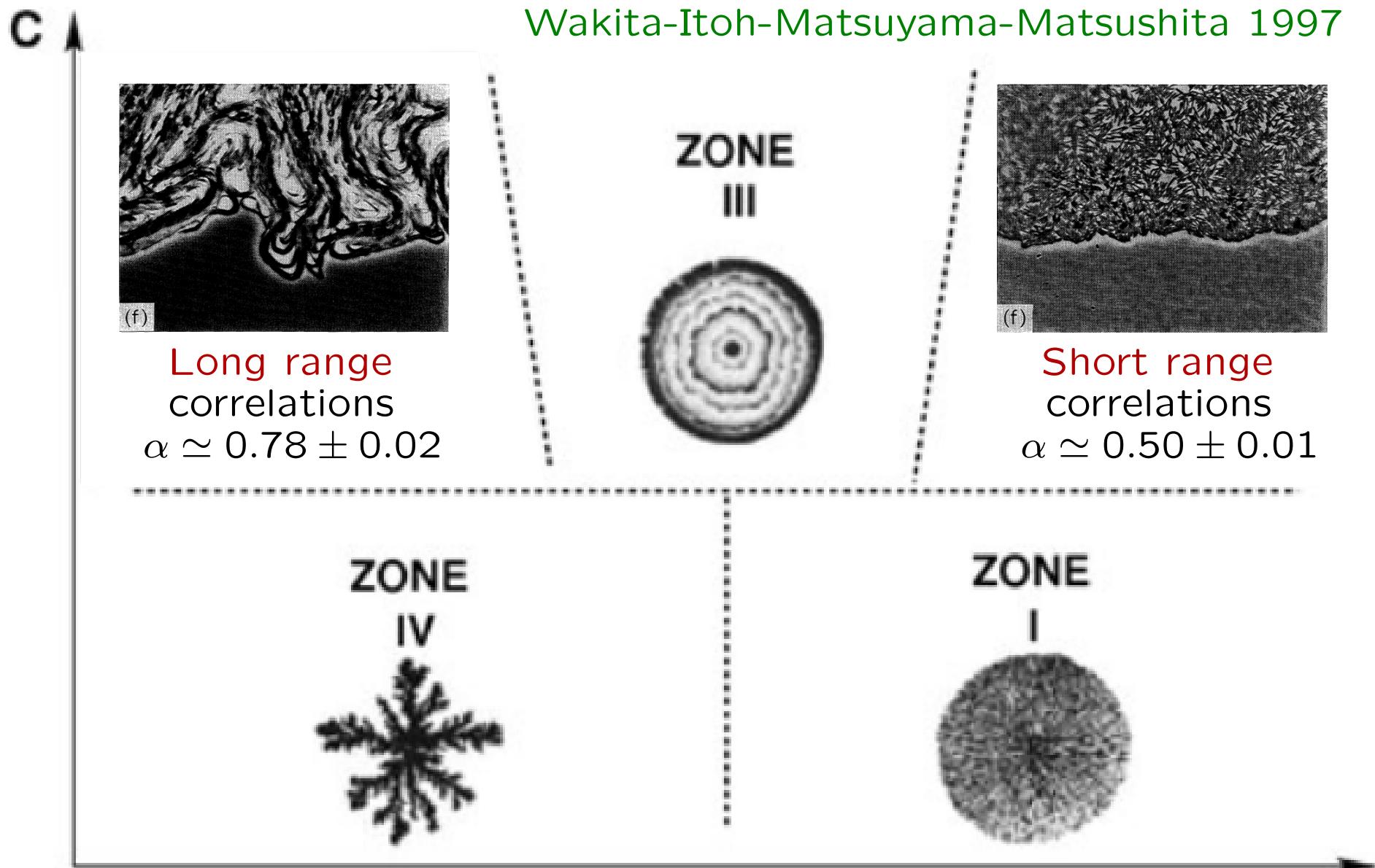
Wakita-Itoh-Matsuyama-Matsuhashita 1997



Agar concentration $C_A \Rightarrow$ cell mobility \nearrow with C_A^{-1}
Nutrient concentration $C \Rightarrow$ cell growth rate \nearrow with C

Growth of bacterial colonies (*Bacillus subtilis*)

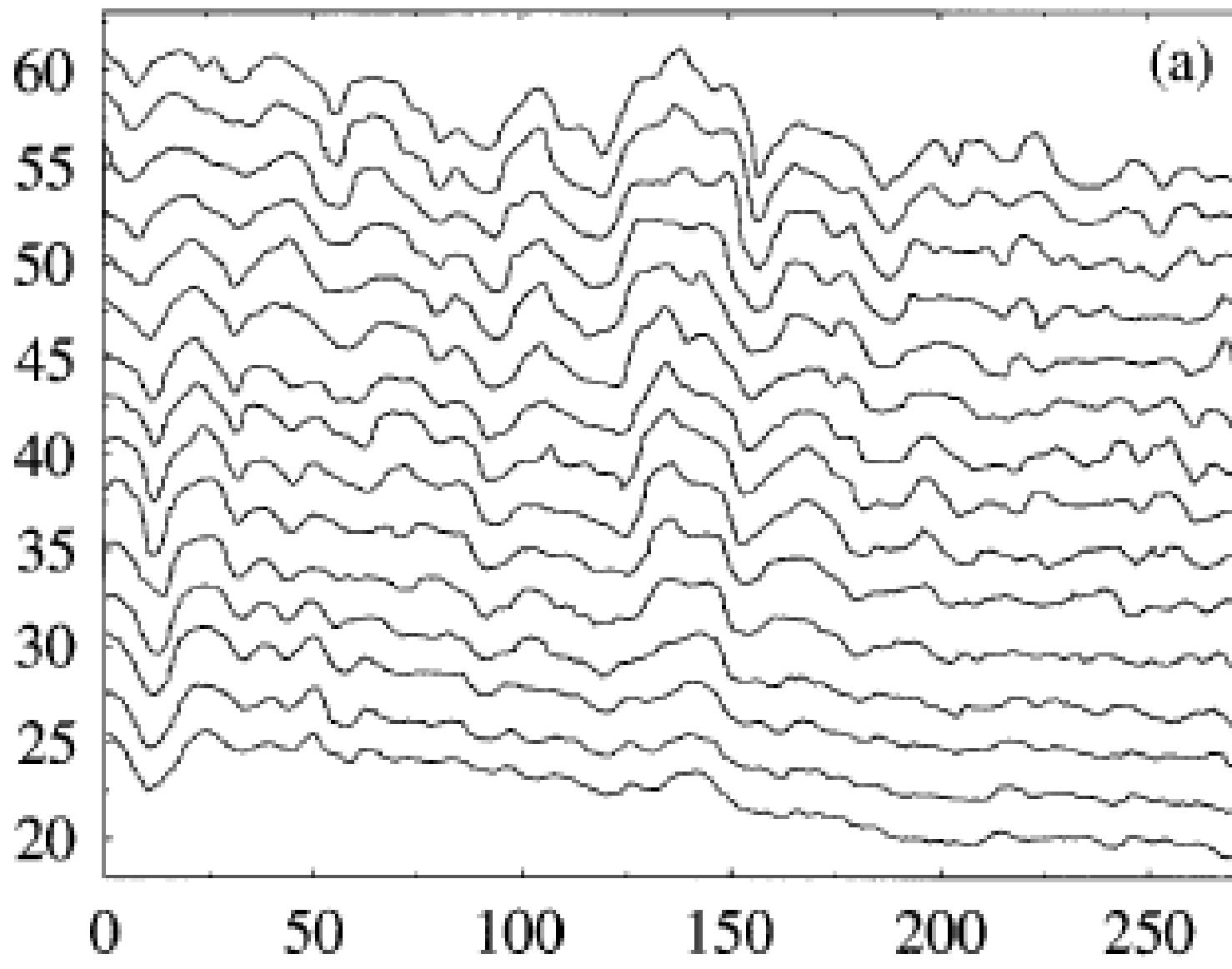
Wakita-Itoh-Matsuyama-Matsuhashita 1997



Agar concentration $C_A \Rightarrow$ cell mobility \nearrow with C_A^{-1}
Nutrient concentration $C \Rightarrow$ cell growth rate \nearrow with C

Slow flameless combustion of paper

Maunuksela-Myllys-Kähkönen-Timonen-Provatas-Alava-Ala-Nissila 1997



Structure of paper



Correlations

15× fibre length

~ cm

Paper + KNO₃



Uniform propagation

Control air flow

⇒ laminar

$$\alpha = 0.48 \pm 0.01$$

$$\frac{\alpha}{z} = 0.32 \pm 0.01$$

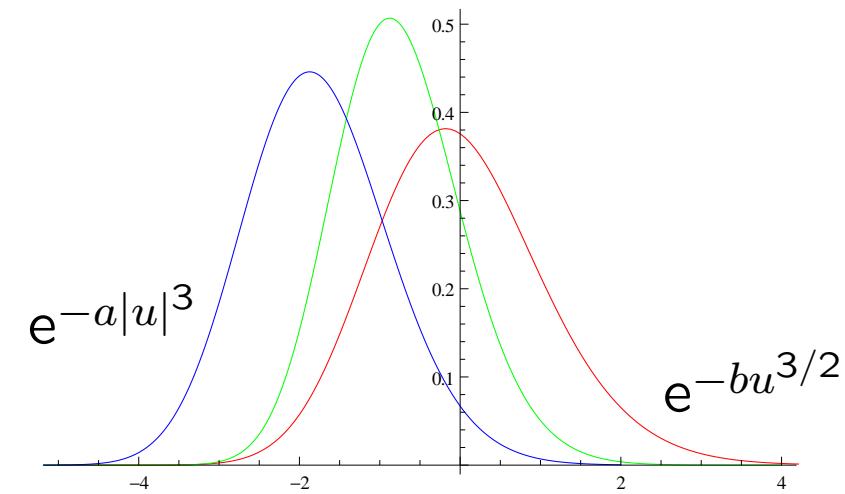
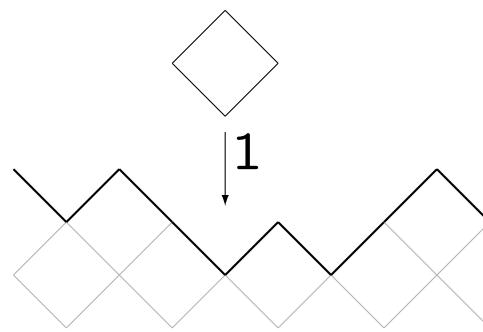
Beyond exponents: universal scaling functions

Single step model on \mathbb{Z}

Interface Height $H_i(t)$

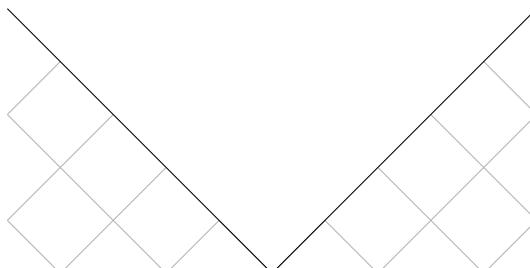
Height fluctuations

$$\chi_t = \frac{H_0(t) - t/4}{(t/16)^{1/3}}$$



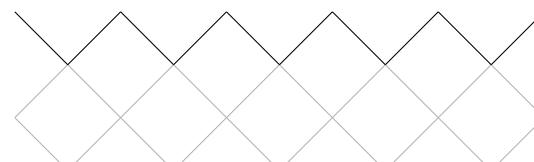
Droplet growth

$\mathbb{P}(\chi_t > -u) \rightarrow F_{\text{GUE}}(u)$
GUE Tracy-Widom
(Johansson 2000)



Flat interface

$\mathbb{P}(\chi_t > -u) \rightarrow F_{\text{GOE}}(2^{2/3}u)$
GOE Tracy-Widom
(Sasamoto 2005)



Stationary interface

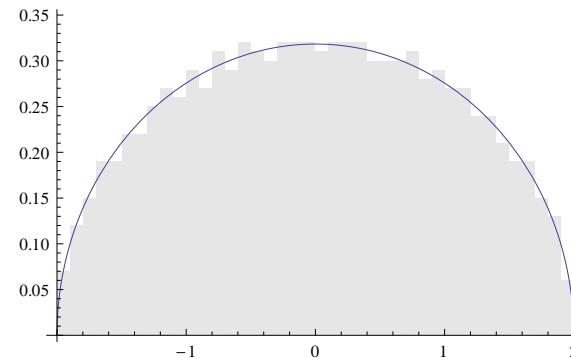
$\mathbb{P}(\chi_t > -u) \rightarrow F_{\text{BR}}(u)$
Baik-Rains distribution
(Ferrari-Spohn 2006)

$P(\swarrow) = 1/2$
 $P(\searrow) = 1/2$
independently

Tracy-Widom distributions & random matrix theory

Gaussian unitary ensemble: $N \times N$ Hermitian matrices A $\mathbb{P}(A) \propto e^{-\frac{1}{2} \text{tr}(A^2)}$

Eigenvalues $\lambda_1(A) < \dots < \lambda_N(A)$
typically in interval $[-2\sqrt{N}, 2\sqrt{N}]$



Bulk density of the eigenvalues:
Wigner's semicircle law

Fluctuations: Tracy-Widom

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\frac{\lambda_N(A) - 2\sqrt{N}}{N^{-1/6}} \leq s \right) = F_{\text{GUE}}(s)$$

Fredholm determinant $F_{\text{GUE}}(s) = \det(\mathbb{1} - P_{[s, \infty)} \mathbb{K})$

$$\text{Airy kernel } K(u, v) = \frac{\text{Ai}(u) \text{Ai}'(v) - \text{Ai}'(u) \text{Ai}(v)}{u - v}$$

Painlevé II equation Hastings-McLeod solution $q''(u) = 2q^3(u) + u q(u)$

$$F_{\text{GUE}}(s) = \exp \left(- \int_s^\infty du (u - s) q^2(u) \right)$$

Gaussian orthogonal ensemble: Hermitian \rightarrow symmetric

Numerical evaluations: Fredholm determinant easier (F. Bornemann 2010)

Domain growth in turbulent liquid crystals

K.A. Takeuchi and M. Sano 2010 Phys. Rev. Lett. 104:230601

Thin layer of **liquid crystal** with nematic order

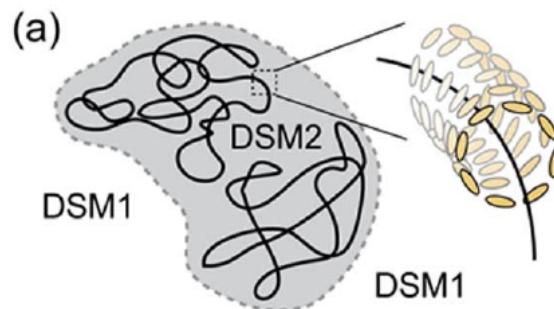


Rod-like molecules \perp to surface

External voltage \Rightarrow **topological defects**

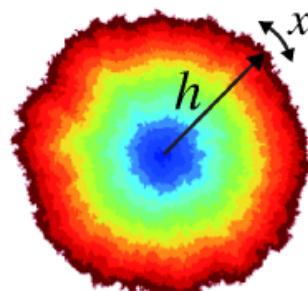


Large voltage \Rightarrow **turbulence** \Rightarrow **short range** correlations \Rightarrow KPZ

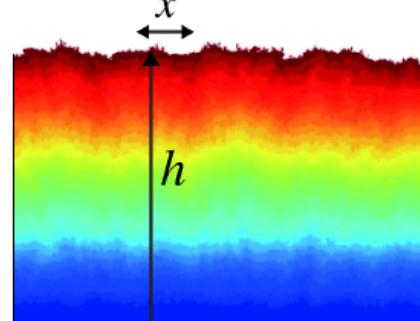


Dynamic scattering modes
DSM1: few defects, of small size
DSM2: many elongated defects
Different light transmittance

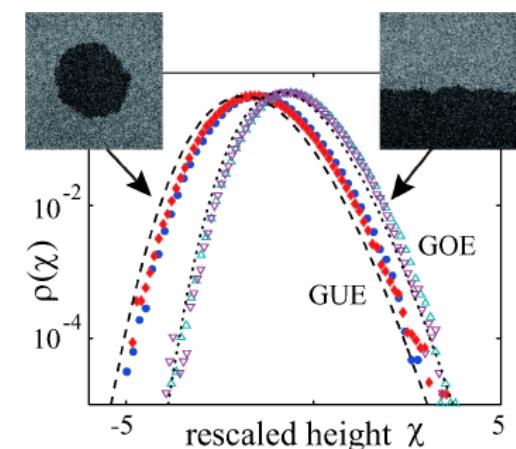
Droplet growth



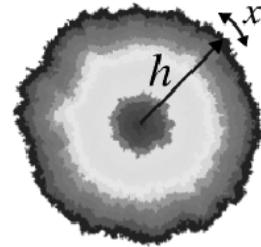
Growth of a **flat line**



$$h(x, t) \underset{t \rightarrow \infty}{\approx} ct + \alpha t^{1/3} \chi$$

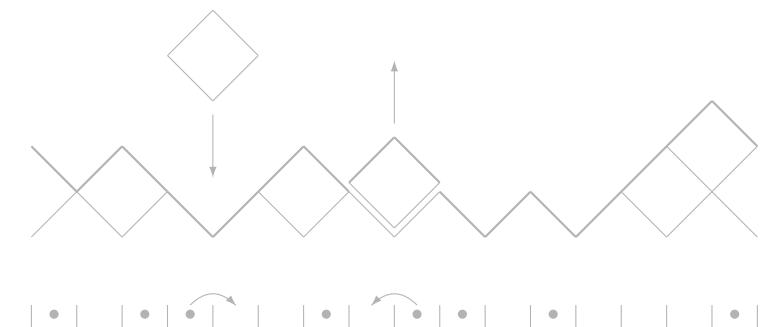


I Interface growth

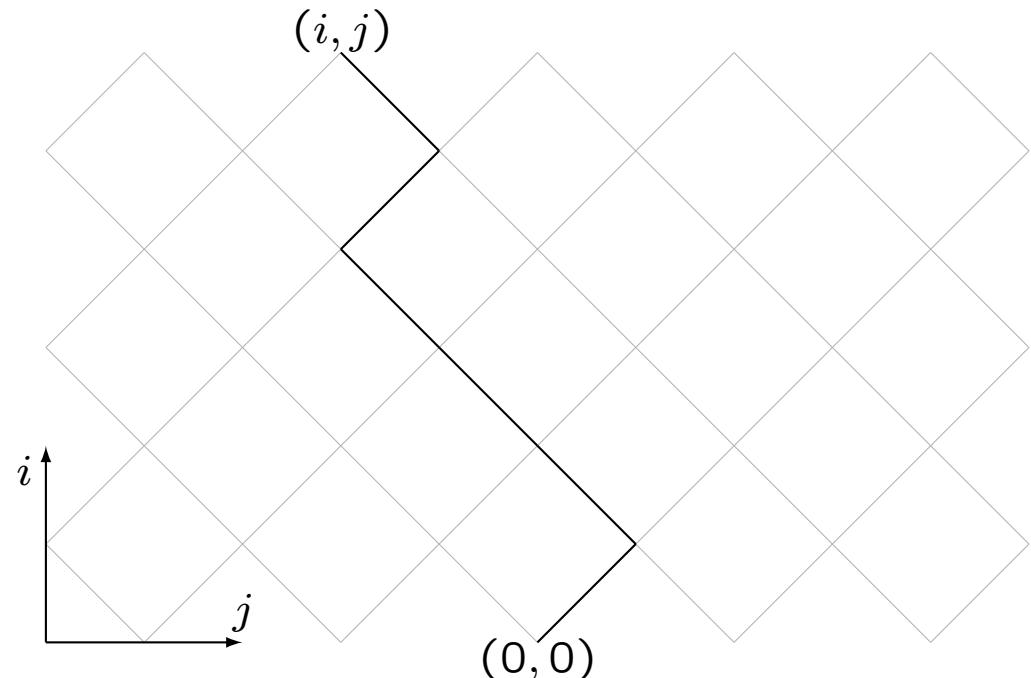


II KPZ universality

III Integrable models



Directed polymer in a random medium



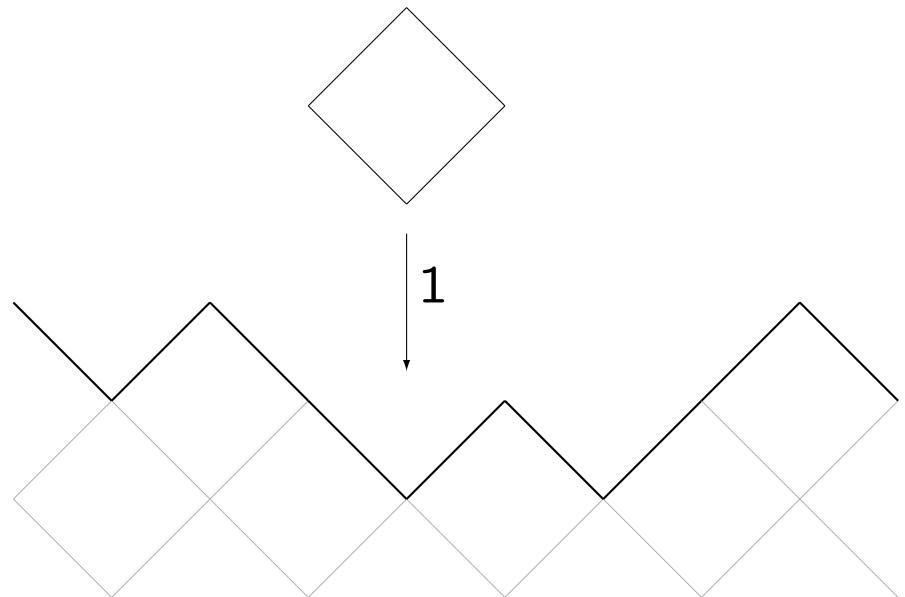
Directed polymer in a random medium

$$E = \sum_{(i,j) \in \text{path}} \varepsilon_{i,j}$$

Zero temperature

$$E_{i,j} = \min(E_{i-1,j}, E_{i-1,j+1}) + \varepsilon_{i,j}$$

Finite temperature
Free energy fluctuations



Interface growth

Single step model

$$t_{i,j} = \max(t_{i-1,j}, t_{i-1,j+1}) + \tau_{i,j}$$



KPZ equation finite time
Height fluctuations

Conductance in two-dimensional localized systems

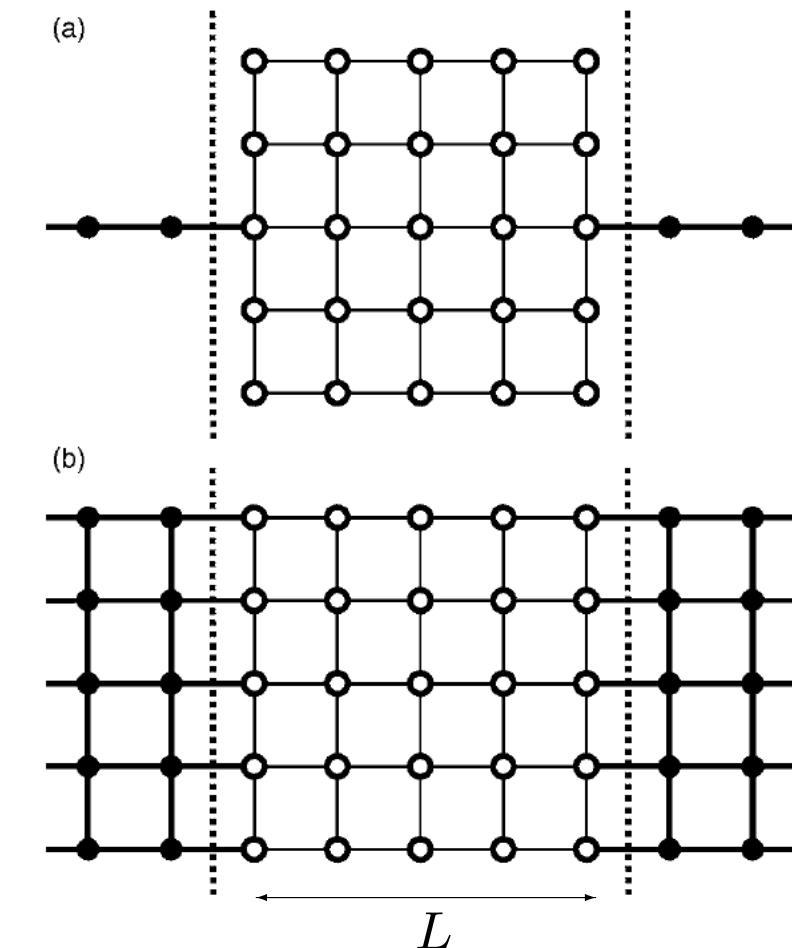
Somoza-Ortuño-Prior 2007 Phys. Rev. Lett. 99:116602

Anderson's model

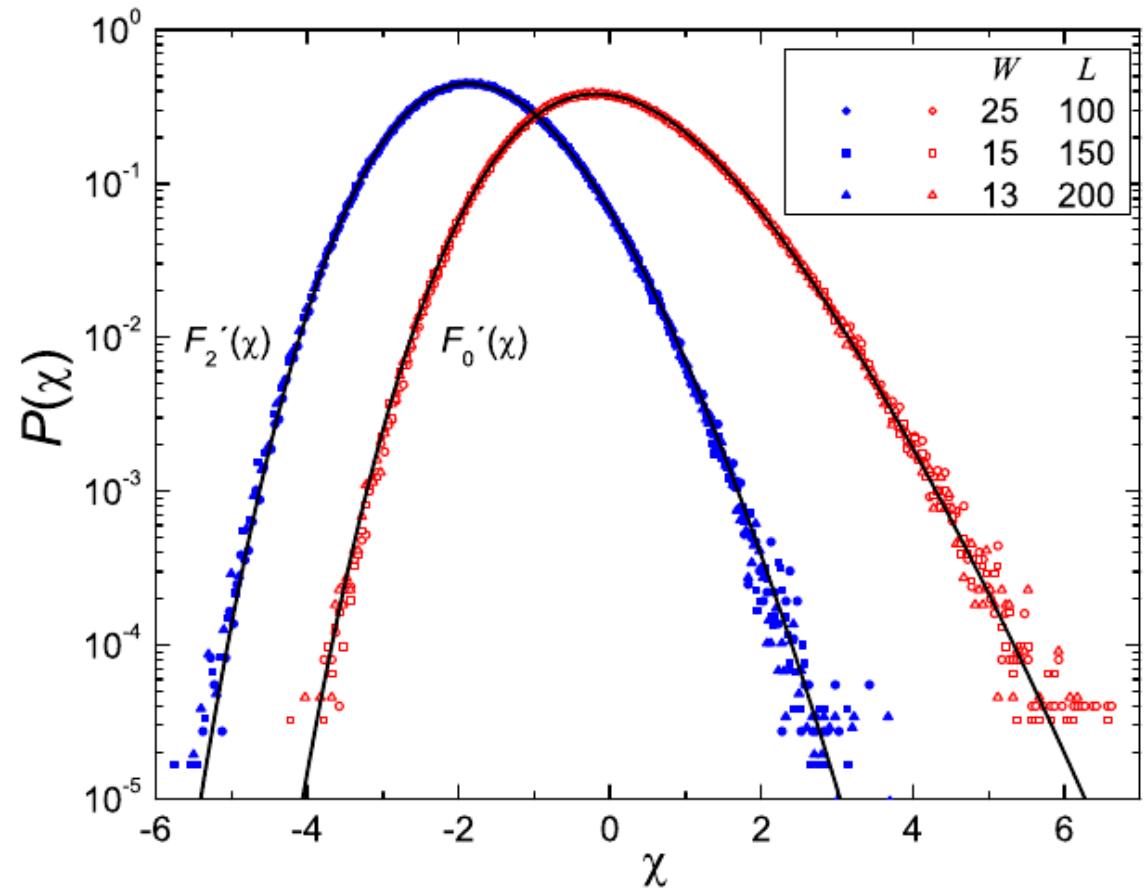
$$H = \sum_i \epsilon_i a_i^\dagger a_i + \sum_{\langle i,j \rangle} a_j^\dagger a_i + \text{h.c.} \Rightarrow \langle a | G | b \rangle = \sum_{\text{paths}} \prod_{i \in \text{path}} \frac{1}{E - \epsilon_i}$$

Conductance g : $\log g \simeq -\frac{2L}{\ell} + \alpha \left(\frac{L}{\ell}\right)^{1/3} \chi$

Green's function



Localization length $\ell \ll L$



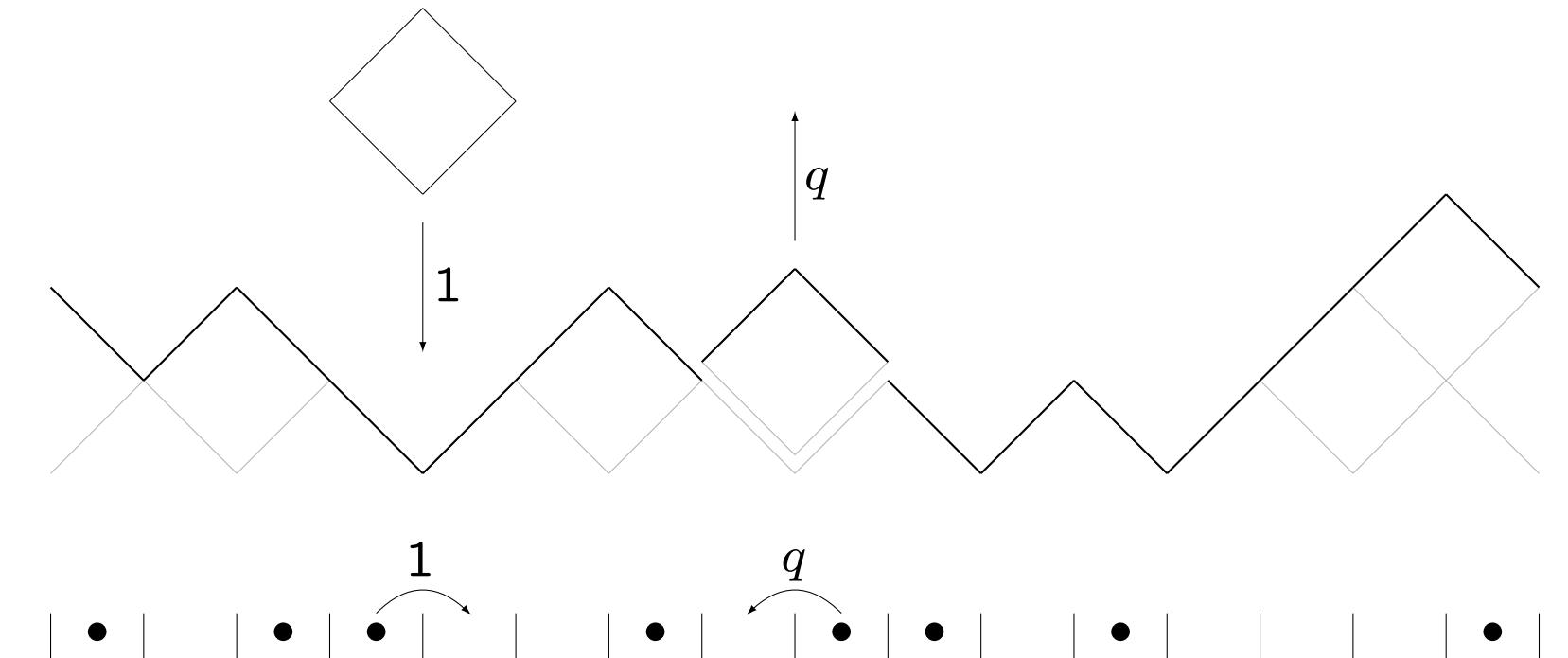
Burgers' equation and interacting particle systems

KPZ equation $\partial_t h = \nu \partial_x^2 h + \lambda (\partial_x h)^2 + \sqrt{D} \xi$

Burgers' equation $\partial_t \rho = \nu \partial_x^2 \rho + \lambda \partial_x (\rho^2) + \sqrt{D} \partial_x \xi$ $\rho(x, t) = \partial_x h(x, t)$
random forcing

Locally conserved density field $\rho(x, t)$

KPZ universality: current fluctuations for irreversible particle systems



Anomalous fluctuations in one-dimensional fluids

Van Beijeren 2012, Spohn 2014

1D fluids with several conservation laws (mass, energy, momentum)

Microscopic model: deterministic Hamiltonian dynamics
anharmonic chains

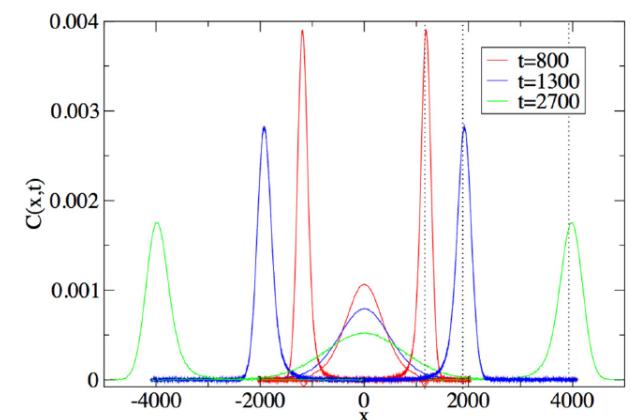
Large scale hydrodynamics $\partial_t \vec{\rho}(x, t) + \partial_x \vec{j}(\vec{\rho}(x, t)) = 0$

Small fluctuations around equilibrium: **non-linear fluctuating hydrodynamics**

- expand to second order
- add dissipation and noise
- **normal modes**

Burgers' equation for sound modes
 \Rightarrow KPZ **super-diffusive** spreading $\ell \sim t^{2/3}$

Non-KPZ, Lévy 5/3 heat mode



Fermi-Pasta-Ulam chain Das-Dhar-Saito-Mendl-Spohn 2014

Gross-Pitaevskii / non-linear Schrödinger equation

Kulkarni-Huse-Spohn 2015, He-Sieberer-Altman-Diehl 2015

$$i\partial_t \psi = -\frac{1}{2m} \partial_x^2 \psi + g|\psi|^2 \psi$$

Large scale behaviour of

- 1D Bose-Einstein condensates (ultracold atoms, exciton-polariton)
- laser beams in non-linear optical waveguides

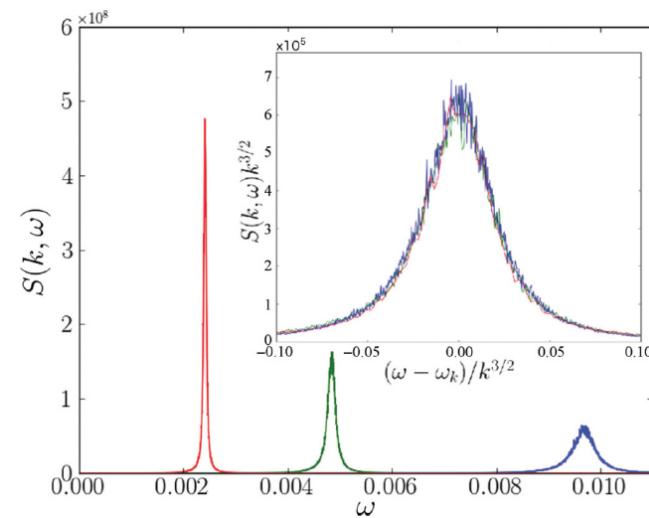
Complex field $\psi(x, t) = \sqrt{\rho(x, t)} e^{i\theta(x, t)}$ \Rightarrow density ρ and velocity $v = \partial_x \theta/m$

Universal fluctuations: **non-linear fluctuating hydrodynamics**

- expand to second order
- add dissipation and noise
- **normal modes** ϕ_{\pm}

$$\langle \phi_{\pm}(x, t) \phi_{\pm}(0, 0) \rangle \underset{t \rightarrow \infty}{\simeq} (\lambda t)^{-2/3} f_{\text{KPZ}}((\lambda t)^{-2/3}(x \pm ct))$$

Counter propagating **super-diffusive KPZ modes**



Quantum entanglement random unitary dynamics

Nahum-Ruhman-Vijay-Haah 2016

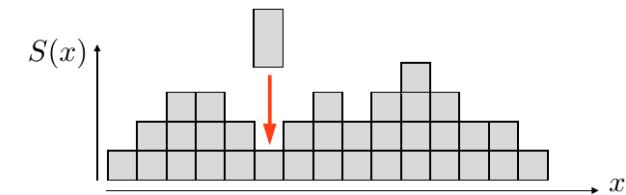
Spin chain in 1D segment, L sites, local Hilbert space of dimension M

Discrete time dynamics $|\psi_t\rangle = U_t |\psi_{t-1}\rangle$

U_t independent **random unitary** matrices, act on random pair of adjacent sites

$\rho(x, t) = \text{tr}_{\text{left of } x} |\psi_t\rangle\langle\psi_t|$
bipartition splitting system at x

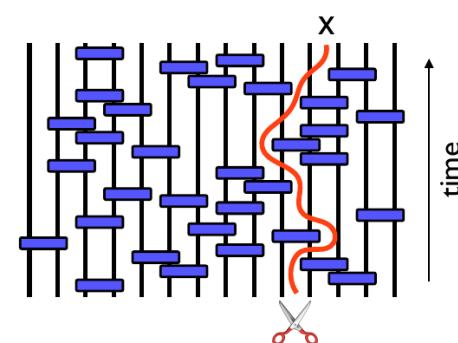
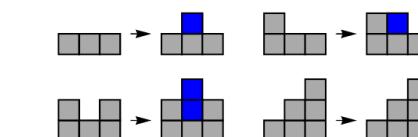
Renyi entropy $S_n(x, t) = \frac{1}{n-1} \log(\text{tr } \rho_x^n)$



$M \rightarrow \infty \Rightarrow$ simple **interface growth** for $S_n(x, t)$
 \Rightarrow KPZ universality

Directed polymer picture: $S_n(x, t) \leq S_{\text{cut}}$
Von Neumann entropy $S_1(x, t) \xrightarrow[M \rightarrow \infty]{} \min_{\text{cut}} S_{\text{cut}}$

KPZ picture for generic, non-integrable deterministic quantum dynamics ?

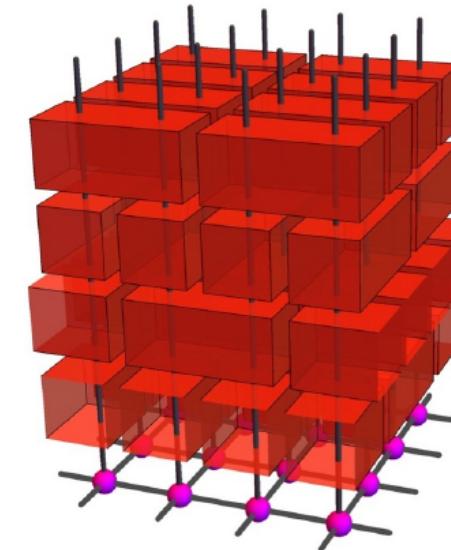


Correlators with parallel random unitary dynamics

Nahum-Vijay-Haah 2017

Parallel update

$$|\psi_t\rangle = U_t |\psi_{t-1}\rangle \text{ with } U_t = \bigotimes_{\text{bonds } b} U_{t,b}$$



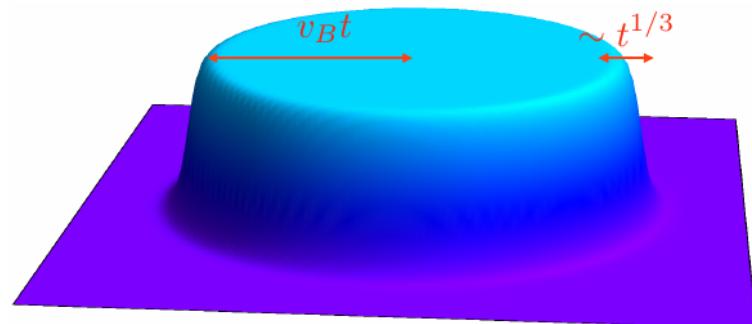
Independent random unitary operators $U_{t,b}$

Local Hermitian operators X_0 acts on site 0
 Y_x acts on site x

ρ_∞ infinite temperature Gibbs state

Out of time order correlator

$$C(x, t) = -\frac{1}{2} \text{tr}(\rho_\infty [U_t^{-1} X_0 U_t, Y_x]^2)$$



2+1D quantum \Rightarrow 1+1D KPZ droplet growth
 $\langle C(v_B t + axt^{1/3}, t) \rangle \xrightarrow{t \rightarrow \infty} 1 - F_{\text{GUE}}(x)$

Trapped Fermi gases

Dean-Le Doussal-Majumdar-Schehr 2015 Phys. Rev. Lett. 114:110402

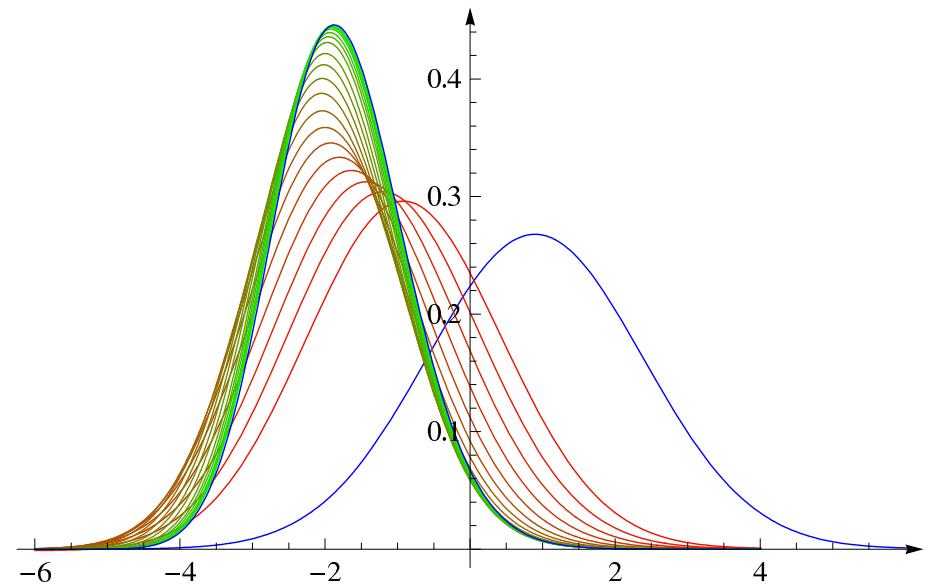
N free spinless Fermions, 1D trap $V(x) = \frac{m\omega^2 x^2}{2}$, equilibrium temperature T

$$\alpha = \sqrt{m\omega/\hbar} \quad \frac{\hbar\omega}{k_B T} = \frac{\beta}{N^{1/3}}$$

$$\mathbb{P}\left(x_{\max} \leq \frac{\sqrt{2N}}{\alpha} + \frac{N^{-1/6}}{\alpha\sqrt{2}} u\right) \xrightarrow[N \rightarrow \infty]{} F_\beta(u)$$

$\beta \rightarrow \infty$: $F_\beta(u) \rightarrow F_{\text{GUE}}(u)$

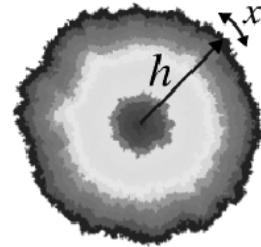
$\beta \rightarrow 0$: $F_\beta(u) \rightarrow$ moving Gaussian



F_β also describes KPZ height fluctuations at finite time β^3 for droplet growth

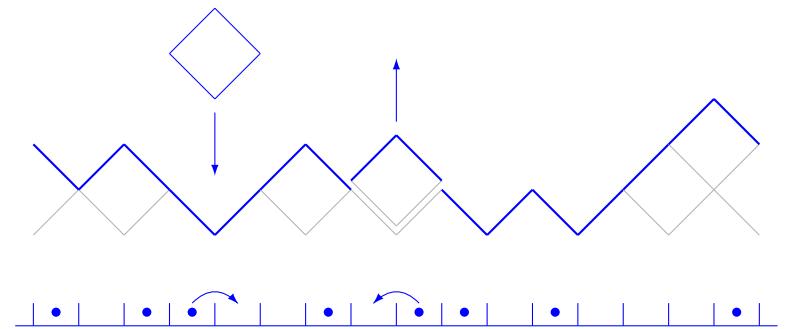
Mapping to KPZ ?

I Interface growth



II KPZ universality

III Integrable models



Quantum & stochastic integrability

Why **integrability** ? Complete failure of perturbative methods
mean field approximations

Same algebraic structures shared by quantum and **stochastic** integrability:

Eigenstates given exactly by **Bethe ansatz**: $|\psi\rangle = B(e^{iq_1}) \dots B(e^{iq_N})|0\rangle$
Bethe equations $e^{iLq_j} = f(q_j; q_1, \dots, q_N)$

Transfer matrix $T(\lambda) = P + \lambda H + \lambda^2 Q_2 + \lambda^3 Q_3 + \dots$

Yang-Baxter equation $\Rightarrow [T(\lambda), T(\mu)] = 0$ conserved quantities

Different expectation values of observables

Quantum: $\langle\psi(t)|\mathcal{O}|\psi(t)\rangle \Rightarrow$ generalized Gibbs ensemble $\beta_k \leftrightarrow \mathcal{Q}_k$
 \Rightarrow **integrable \neq generic**

Stochastic: $\sum_C \langle \mathcal{C} | \mathcal{O} | P(t) \rangle \Rightarrow$ all \mathcal{Q}_k reduce to the same operator
 \Rightarrow **integrable is generic**

Single step model \sim ASEP

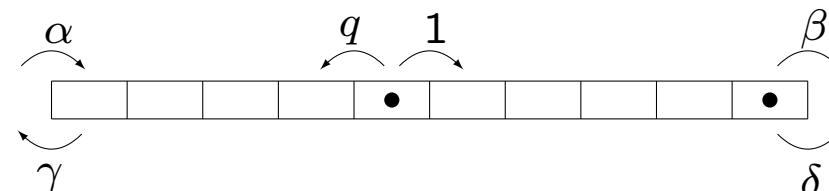
Asymmetric simple exclusion process

Generator \sim twisted XXZ spin chain

$$H = \sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z)$$

$$\Delta = \frac{q^{1/2} + q^{-1/2}}{2} > 1 \quad S_{L+1}^\pm = q^{\mp L/2} S_1^\pm$$

Open boundaries



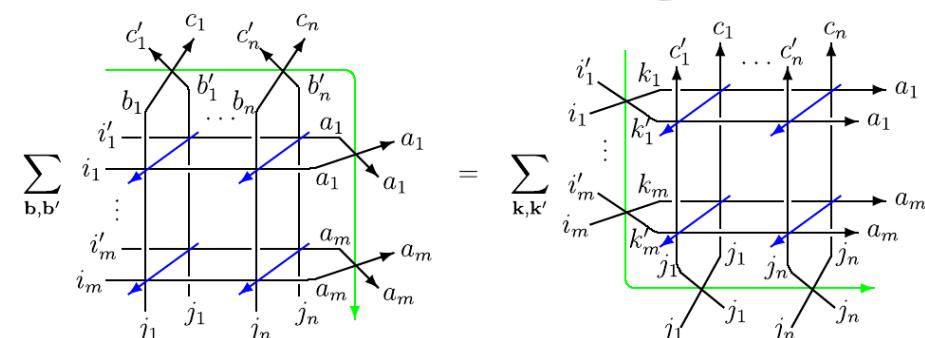
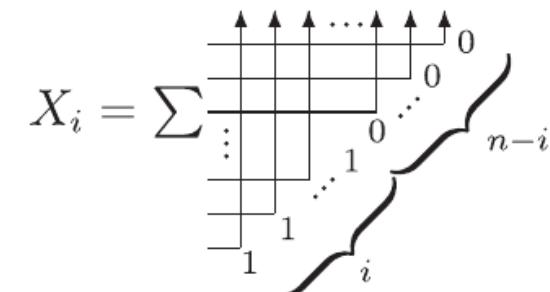
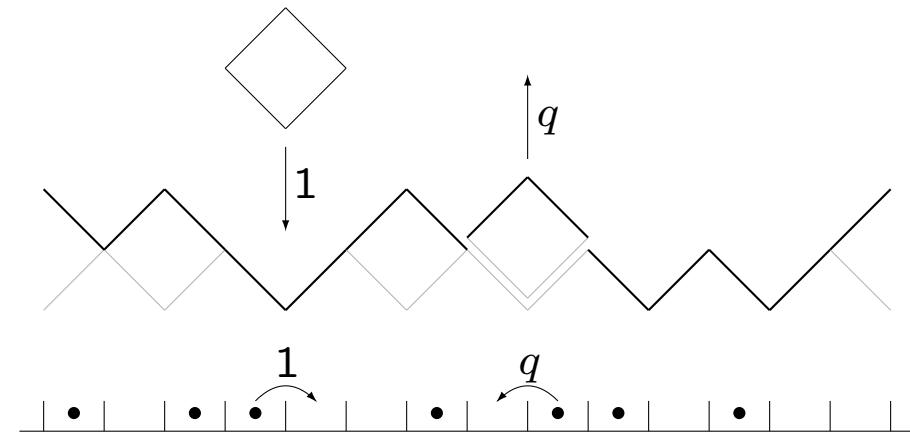
Multi-species with hierarchy $jk \xrightleftharpoons[q]{\alpha} kj$ ($0 \leq k < j \leq n$)

Twisted $SU(n+1)$ chain

Stationary state $P_{\text{st}}(\mathcal{C}) = \text{tr}_\infty[X_{\eta_1} \dots X_{\eta_L}]$

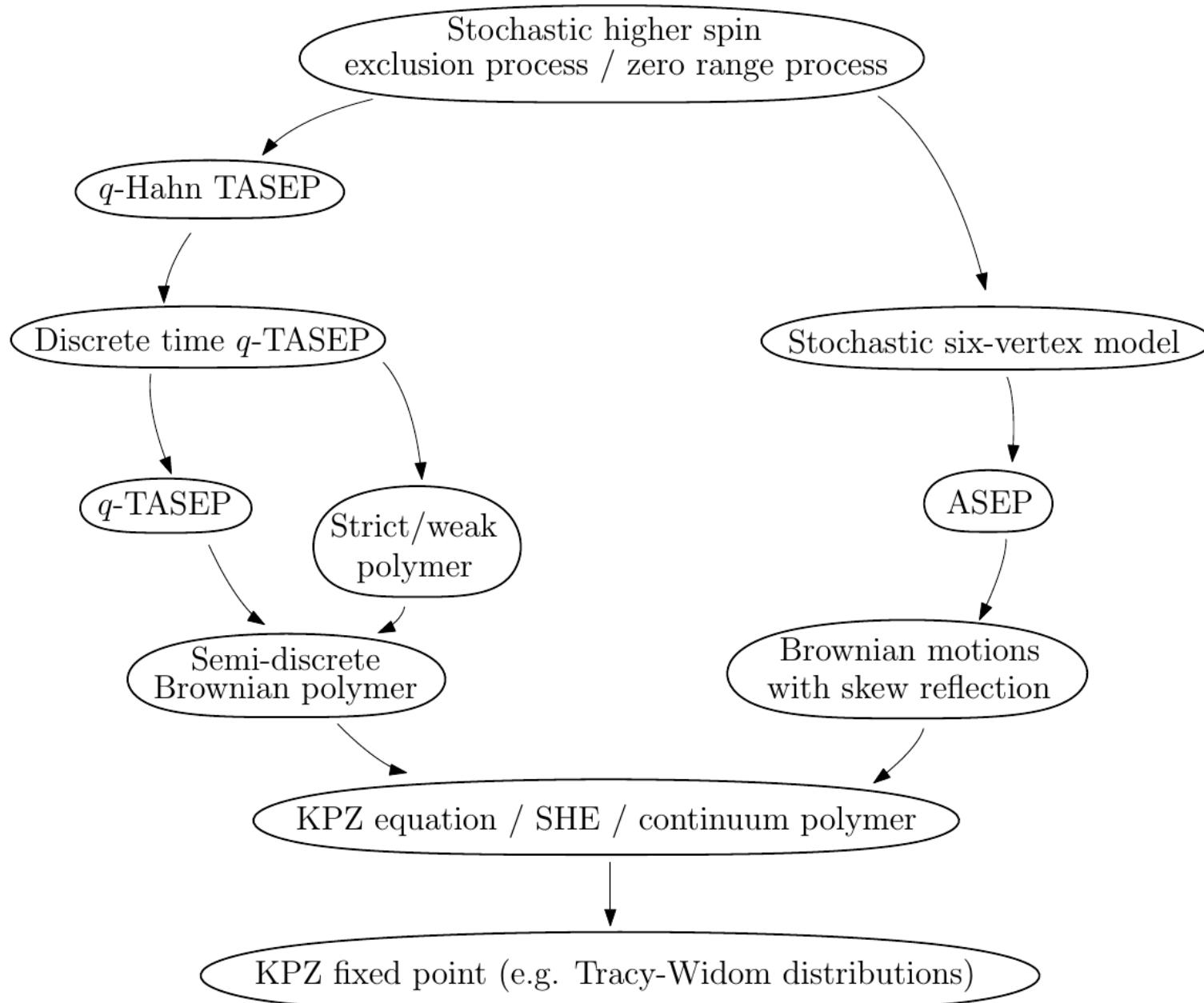
Tetrahedron equation

Kuniba-Maruyama-Okado 2016



Hierarchy stochastic integrable models

Borodin-Corwin 2011, . . . , Corwin-Petrov 2016, . . .



Replica solution of the KPZ equation

Kardar 1987, Dotsenko 2010, Calabrese-Le Doussal-Rosso 2010

KPZ equation

$$\partial_t h(x, t) = \nu \partial_x^2 h(x, t) + \lambda (\partial_x h(x, t))^2 + \sqrt{D} \xi(x, t)$$

Cole-Hopf transformation $Z(x, t) = e^{\frac{\lambda}{\nu} h(x, t)}$ \Rightarrow stochastic heat equation
 $\partial_t Z(x, t) = \nu \partial_x^2 Z(x, t) + \frac{\lambda \sqrt{D}}{\nu} \xi(x, t) Z(x, t)$

Feynman-Kac formula \Rightarrow path integral & directed polymer

$$Z(x, t) = \int^{x(t)=x} \mathcal{D}[x(\tau)] \exp \left[\int_0^t d\tau \left(-\frac{(\partial_\tau x(\tau))^2}{4\nu} + \frac{\lambda \sqrt{D}}{\nu} \xi(x(\tau), \tau) \right) \right]$$

Averaging over noise with n replica \Rightarrow Lieb-Liniger attractive δ -Bose gas
 $-\partial_t \langle Z(x_1, t) \dots Z(x_n, t) \rangle = H_n \langle Z(x_1, t) \dots Z(x_n, t) \rangle$

$$H_n = -\frac{1}{2} \sum_{j=1}^n \frac{d^2}{dx_j^2} - \frac{1}{2} \sum_{j \neq k}^n \delta(x_j - x_k)$$

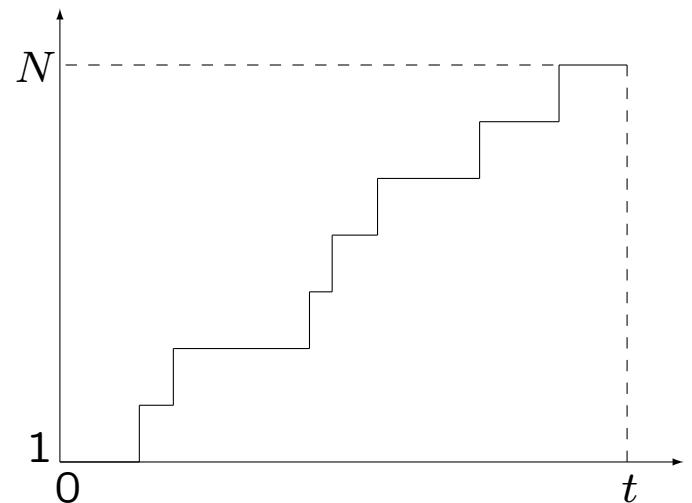
Semi-discrete directed polymer

N. O'Connell and M. Yor 2001

Gaussian Random environment

$$\eta_1(s), \dots, \eta_N(s), s \geq 0$$

$$\langle \eta_j(s) \eta_k(s') \rangle = \delta_{j,k} \delta(s - s')$$



Partition function of the directed polymer

$$Z_{N,t}(\beta) = \int_{0 < t_1 < \dots < t_{N-1} < t} dt_1 \dots dt_{N-1} e^{\beta \left(\int_0^{t_1} ds \eta_1(s) + \int_{t_1}^{t_2} ds \eta_2(s) + \dots + \int_{t_{N-1}}^t ds \eta_N(s) \right)}$$

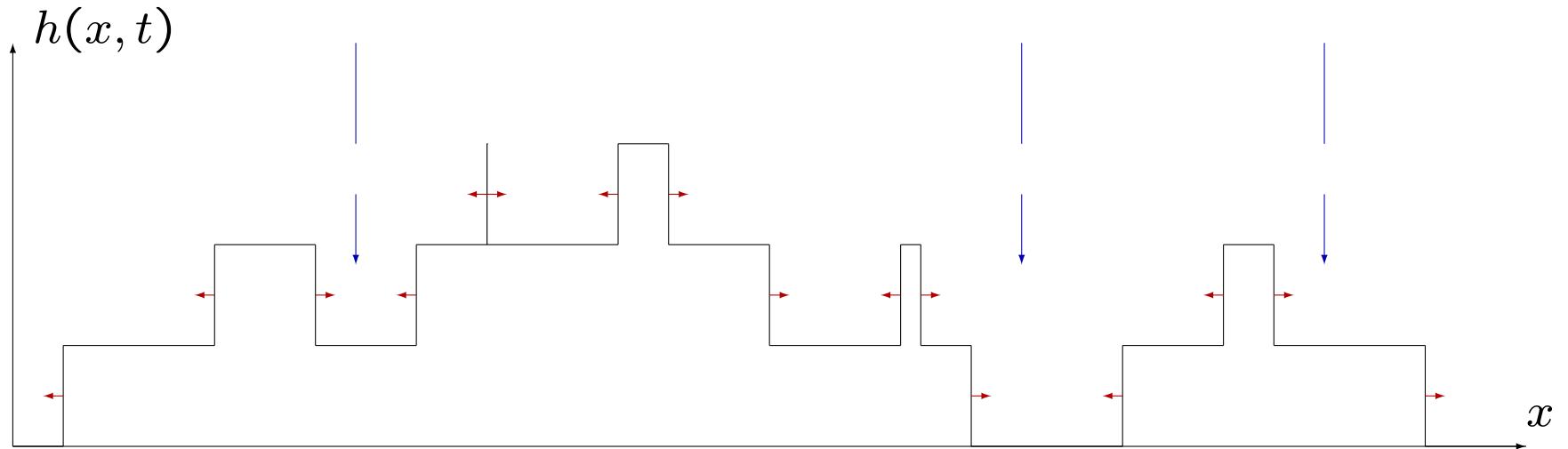
Relation to quantum Toda lattice (N. O'Connell 2012)

$$H = \sum_{j=1}^N \frac{d^2}{dx_j^2} - 2 \sum_{j=1}^{N-1} e^{x_{j+1} - x_j} \quad \text{groundstate } \psi_0(x_1, \dots, x_N)$$

$\log Z_{N,t}(1) \sim$ first coordinate diffusion process $\frac{1}{2}\Delta + \nabla \log \psi_0 \cdot \nabla$ in \mathbb{R}^N

Polynuclear growth model (PNG)

M. Prähofer and H. Spohn 2000



x and t continuous, $h(x, t)$ discrete

- deterministic lateral growth at constant speed c
- random nucleation events (vertical deposition)
- colliding interfaces merge

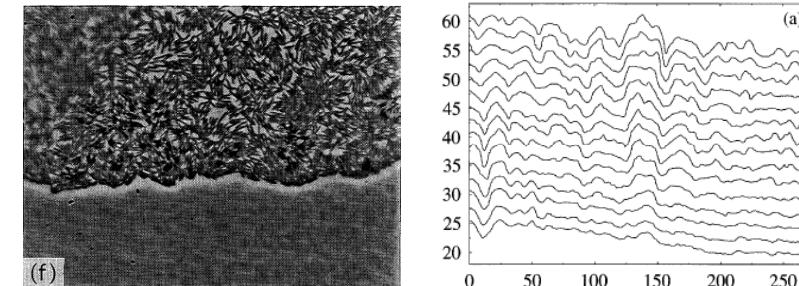
Height fluctuations (indirectly) related to free fermions

Conclusions KPZ universality

Various phenomena

- Interface growth
- Driven particles
- Directed polymer in random medium
- Conductance in 2D localized systems
- Anomalous fluctuations of 1D fluids
- Gross-Pitaevskii dynamics
- Random unitary dynamics

Some experimental realizations



Stochastic integrability / integrable probability

Next lecture: exact KPZ results 1+1 D universal scaling functions

Asymmetric simple exclusion process (ASEP)

- infinite system
- finite volume

