CFT and 2d chiral topological phases: an introduction

Lecture notes for the 2017 Cargèse Summer School

"Exact methods in low dimensional statistical physics"

J. Dubail

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The goal of these two lectures is to give an introduction to the "Moore-Read construction", namely the approach to 2d chiral topological phases based on those trial wavefunctions that can be constructed from the conformal blocks of a 2d conformal field theory (CFT).

Although this approach has been most popular and fruitful in the context of the Fractional Quantum Hall effect, it may also be applied to other systems, such as chiral spin liquids, or a $p_x + ip_y$-paired superconductor. The latter system is particularly simple because it requires to deal only with free fermions, and it is related to the simplest possible 2d CFT: the one of the critical 2d Ising model.

For this reason, in these two lectures, I focus solely on the $p_x + ip_y$ superconductor.

I am assuming that the reader is familiar with the 2d Ising model and with the CFT that describes its critical point. From there, the goal is to

1. explain how the BCS (Bardeen-Cooper-Schrieffer) wavefunction of the $p_x + ip_y$ superconductor is related to a correlator in the chiral part of the Ising CFT

2. derive the bulk-edge correspondence: the fact that the chiral $1 + 1d$ CFT that describes the low-energy edge excitations must be the same as the 2d CFT that gives the bulk wavefunction

3. relate the non-abelian adiabatic statistics of the vortices to the monodromy of the conformal blocks.

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Lecture 1

CFT trial states and bulk-edge correspondence

1.1 What is a chiral topological phase?

Let $H(\lambda)$ be the hamiltonian of a d-dimensional system in the thermodynamic limit, that depends continuously on some parameter λ (this parameter may be multi-dimensional, $\lambda = (\lambda_1, \lambda_2, \lambda_3 \ldots)$). By 'hamiltonian for a d-dimensional system', I mean a hamiltonian that is a sum of *local* terms acting on a collection of degrees of freedom that live in $\mathbb{R}^d$, or on a d-dimensional lattice. If $H(\lambda)$ has an energy gap $\Delta E > 0$ above a finite set of degenerate ground states, then by continuity there exists an open neighborhood of λ in the space of parameters where the gap stays non-zero and the number of ground states is constant. One then says that the system is in a *topological phase*.

A *topological phase* may be *topologically trivial*, in the sense that it can be continuously deformed into the vacuum (or, more generally, a trivial product state) without closing the energy gap. It may also be *topologically non-trivial*. I will give an example of both situations shortly.

In 2d, there exist *topologically non-trivial phases* that possess the following property: when the system has a boundary, there are gapless excitations that are localized along the one-dimensional boundary. When these excitations propagate in one direction only, one talks about a *2d chiral topological phase*. In such a system, time-reversal and parity symmetry must be broken.

The most prominent examples of *2d chiral topological phases* are the Integer and Fractional Quantum Hall effects (including the Laughlin states at different filling fractions, the hierarchical states, the Moore-Read state,
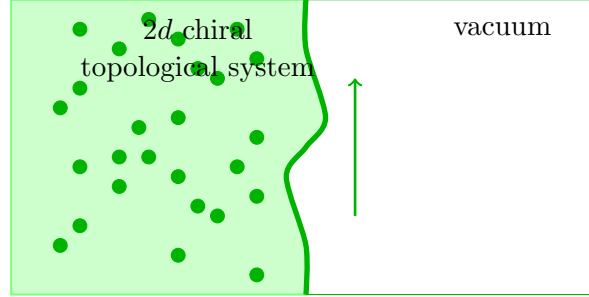
Figure 1.1: Cartoon of an interface between a 2d chiral topological phase (on the left) and the vacuum (on the right) with a small deformation of the fluid at the edge. The energy gap is finite in the bulk, but the edge excitations are gapless and they propagate in one direction only. For many chiral topological phases (including the case of $p_x + ip_y$ superconductors treated in these two lectures), the gapless edge excitations are described by a chiral $1 + 1d$ CFT.

etc.), but there exist also other examples, like chiral spin liquids, Chern insulators, or $p_x + ip_y$-paired superconductors, to which I now turn.

1.2 BCS theory of the $p_x + ip_y$-paired superconductor

The BCS (Bardeen-Cooper-Schrieffer) microscopic theory of superconductivity is based on a hamiltonian of the following type:

$$ H = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{pmatrix} \frac{|k|^2}{2m} - \mu & \Delta(k) \\ \Delta(k)^* & -\frac{|k|^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}. \quad (1.1) $$

Here $c_k^\dagger / c_k$ is the creator/annihilator of a spinless fermion in two dimensions with momentum $k = (k_x, k_y)$, satisfying the canonical anti-commutation relation $\{c_k^\dagger, c_{k'}\} = (2\pi)^2 \delta^{(2)}(k - k')$. $\frac{|k|^2}{2m}$ is the kinetic energy of a fermion and μ is the chemical potential. The pairing terms $\Delta(k)^* c_{-k} c_k$ and $\Delta(k) c_k^\dagger c_{-k}^\dagger$ —where $\Delta(k)$ is a complex amplitude called the *gap function*— break particle number conservation. This is the key point in BCS theory. These terms arise from a treatment of quartic interaction terms at the mean-field level, $c^\dagger c^\dagger c c \to \frac{1}{2}(c^\dagger c^\dagger \langle cc \rangle + \langle c^\dagger c^\dagger \rangle cc)$. In the BCS approach, the *gap function* $\Delta(k)$ is related to $\langle c_{-k} c_k \rangle$, and must typically be calculated self-consistently.

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Here, we forget the origin of the amplitude $\Delta(k)$, and we simply assume that it is some given function with the following behavior at small $|k|$,

$$\Delta(k) \underset{|k|\to 0}{\simeq} (k_x - ik_y) \times \text{const.} \tag{1.2}$$

This is a gap function for a superconductor with Cooper pairs that have orbital momentum $\ell = -1$ (p-wave pairing); the name "$p_x + ip_y$" refers to the complex combination of k_x and k_y in $\Delta(k)$. [In fact, here we are really dealing with a "$p_x - ip_y$" superconductor, rather than "$p_x + ip_y$". This choice of sign is unimportant, but the minus sign leads to nicer formulas below.]

The hamiltonian (1.1) is easily diagonalized, and one finds that it has single-particle eigenvalues $\pm\sqrt{(|k|^2/(2m) - \mu)^2 + |\Delta(k)|^2}$. It has a finite energy gap as long as $\mu \neq 0$. The gap vanishes at $\mu = 0$. This model thus possesses two topological phases, one with $\mu > 0$, the other with $\mu < 0$. One way of seeing that the two phases are topologically distinct is to consider the following continuous map, defined for $\mu \neq 0$,

$$\mathbb{R}^2 \ni k \longmapsto \frac{\begin{pmatrix} \text{Re}\,\Delta(k) \\ \text{Im}\,\Delta(k) \\ \frac{|k|^2}{2m} - \mu \end{pmatrix}}{\sqrt{(|k|^2/(2m) - \mu)^2 + |\Delta(k)|^2}} \in S^2. \tag{1.3}$$

For $\mu < 0$, the index of that map is zero (it never reaches the south pole of the sphere S^2). On the contrary, for $\mu > 0$, the index is one. The two phases are phenomenologically very different:

- the phase with $\mu < 0$ is *topologically trivial*: it is a continuous deformation of the vacuum at $\mu \to -\infty$.

- the phase with $\mu > 0$ is *topologically non-trivial*, and it is in fact a 2d chiral topological phase, with chiral gapless edge modes, as well as interesting bulk physics.

From now on, we assume that we are in the topologically non-trivial phase, with $\mu > 0$.

1.3 The $p_x + ip_y$ trial state

The (unnormalized) ground state of the BCS hamiltonian is a gaussian superposition of pairs of particles with opposite momenta,

$$|\Psi\rangle = \exp\left(\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} g_k c^\dagger_{-k} c^\dagger_k\right) |0\rangle, \tag{1.4}$$

where $|0\rangle$ is the vacuum, and

$$g_k := \frac{\frac{|k|^2}{2m} - \mu - \sqrt{(\frac{|k|^2}{2m} - \mu)^2 + |\Delta(k)|^2}}{\Delta(k)^*}. \qquad (1.5)$$

Because we are assuming $\mu > 0$, one sees that g_k has the following behavior at small $|k|$,

$$g_k \underset{|k| \to 0}{\simeq} \frac{1}{k_x + ik_y} \times \text{const.} \qquad (1.6)$$

[For $\mu < 0$, one would get $g_k \propto k_x - ik_y$ instead.] This small-$|k|$ behavior implies that, at large distances, the Fourier transform of g_k behaves as

$$g(x, y) := \int \frac{d^2k}{(2\pi)^2} e^{ik_x x + ik_y y} g_k \underset{|x+iy| \to \infty}{\simeq} \frac{\kappa}{x + iy}$$

for some non-zero constant $\kappa \in \mathbb{C}$. This observation motivates the introduction of the following *trial state*,

$$|\Psi_{\text{trial}}\rangle := \exp\left(\frac{\kappa}{2} \int d^2 z_1 d^2 z_2 \frac{1}{z_1 - z_2} c^\dagger(z_1, \bar{z}_1) c^\dagger(z_2, \bar{z}_2)\right) |0\rangle. \qquad (1.7)$$

[Here we use complex coordinates $z = x + iy$. $c^\dagger(z, \bar{z})$ is the Fourier transform of $c_k^\dagger$; it creates a fermion at position z.] This trial state differs from the true BCS ground state when particles are close to each other, but it mimics the behavior of the true ground state when particles are well separated. So one expects that it shares some of the key features of the true ground state.

Another way to motivate the introduction of this trial state is to look for a hamiltonian H_{trial} that has $|\psi_{\text{trial}}\rangle$ as a unique ground state, and then argue that there is a continuous path from H to H_{trial} such that the gap never closes. Then H_{trial} is viewed as a **particular point inside the topological phase, with a ground state that takes a particularly simple mathematical form**. This simple form allows to perform certain types of calculations that would be harder at more generic points inside the phase. But it is crucial to emphasize that **one expects the key physical features of $|\psi_{\text{trial}}\rangle$ to be properties of the entire topological phase, and not just of a particular point inside it**.

[For the record, such a hamiltonian H_{trial} is easy to construct. For instance, one can take

$$H_{\text{trial}} = \frac{1}{2m} \int \frac{d^2k}{(2\pi)^2} \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{pmatrix} \frac{|k|^2 - |\kappa|^2}{2} & \kappa(k_x - ik_y) \\ \kappa^*(k_x + ik_y) & -\frac{|k|^2 - |\kappa|^2}{2} \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}.$$

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It has a finite gap $\frac{|\kappa|^2}{2m} > 0$. Notice that there is obviously no unique choice for H_{trial}, since one can multiply the 2×2 matrix in the middle of the integrand by any function $f(k) \geq \epsilon > 0$, and this will do the job as well. For instance, if one multiplies it by $f(k) = \frac{1/2}{|k|^2 + |\kappa|^2}$, then the 2×2 matrix has constant eigenvalues ± 1, so one gets a 'spectrally flat' trial hamiltonian, which can be useful for some purposes. However, in the rest of these notes, I will not rely on a such a specific hamiltonian; instead, I will focus on the properties of the trial state $|\Psi_{\text{trial}}\rangle$ itself.]

1.4   Relation to the Ising CFT

The factor $\frac{1}{z_1 - z_2}$ in Eq. (1.7) should look familiar to the reader. It is the two-point function of the chiral part of the energy operator in the Ising CFT, $\varepsilon(z, \bar{z}) = i\psi(z)\bar{\psi}(\bar{z})$,

$$\langle \psi(z_1)\psi(z_2) \rangle = \frac{1}{z_1 - z_2}. \tag{1.8}$$

The field $\psi(z)$ has conformal dimension $\frac{1}{2}$, and it is well-known that it is in fact a free self-adjoint fermion, also called a *Majorana fermion*, or *Majorana field*. The correlation function of $2m$ Majorana fields is given by Wick's theorem,

$$\langle \psi(z_1)\psi(z_2)\ldots\psi(z_{2m}) \rangle = \text{Pf}\left[\frac{1}{z_i - z_j}\right]. \tag{1.9}$$

For odd numbers of fields, the correlator vanishes. Here $\text{Pf}(A_{ij})$ is the Pfaffian of the $2m \times 2m$ antisymmetric matrix A that has entries A_{ij}. Using Wick's theorem, one can rewrite the $p_x + ip_y$ trial state (1.7) in the form of a correlator in the Ising CFT,

$$|\Psi_{\text{trial}}\rangle = \left\langle \exp\left(i\,\kappa^{1/2}\int d^2z\,\psi(z)\,c^\dagger(z, \bar{z})\right)\right\rangle |0\rangle. \tag{1.10}$$

[The factor i comes from the fact that the ψ's anticommute with the $c^\dagger$'s: for instance, focusing on the configuration with two particles as one expands the exponential, one gets the term $\frac{(i\kappa^{1/2})^2}{2!}\left\langle \psi_1 c_1^\dagger \psi_2 c_2^\dagger \right\rangle = \frac{\kappa}{2}\frac{1}{z_1 - z_2}c_1^\dagger c_2^\dagger$ as required by Eq. (1.7).]

Notice that one can use the anti-chiral part of $\varepsilon(z, \bar{z})$, $\bar{\psi}(\bar{z})$, to write the conjugate (or 'bra') of $|\Psi_{\text{trial}}\rangle$,

$$\langle \Psi_{\text{trial}}| = \langle 0|\left\langle \exp\left(\kappa^{*1/2}\int d^2z\,c(z, \bar{z})\,\bar{\psi}(\bar{z})\right)\right\rangle. \tag{1.11}$$

[In the bra, there is no factor i coming from the anti-commutation; for instance, for two particles one gets $\frac{(\kappa^{*1/2})^2}{2!} \left\langle c_1 \overline{\psi}_1 c_2 \overline{\psi}_2 \right\rangle = \frac{\kappa}{2} c_2 c_1 \left\langle \overline{\psi}_1 \overline{\psi}_2 \right\rangle = \frac{\kappa}{2} c_2 c_1 \frac{1}{\overline{z}_1 - \overline{z}_2}$, as needed.]

Thinking of the 'ket' and of the 'bra' as correlators of chiral and anti-chiral operators respectively is a trick that is extremely useful. I will rely on it very heavily to derive the key properties of the $p_x + ip_y$ trial state.

1.5 Short-range correlations in the bulk

The $p_x + ip_y$ superconductor is a gapped phase of matter, so it must have short-range (*i.e.* exponentially decaying) connected correlations only. It cannot have power-law correlations. This may sound somewhat puzzling to some readers, since I have been emphasizing the role of $2d$ CFT in the construction of the trial state, and CFT is usually associated with power-law correlations.

It is important to dissipate any possible confusion about this. The $2d$ CFT that is used to construct the trial state does not have anything to do with a gapless theory that would describe a critical $2 + 1d$ system. It does *not* predict power-law correlation functions in the bulk; this would be utter nonsense in the context of topological phases. Instead, CFT is perfectly consistent with exponentially decaying correlations in the bulk. Let me explain why this is so.

For concreteness, I focus on the two-point correlation function

$$\frac{\left\langle \Psi_{\text{trial}} \right| c(w_1, \bar{w}_1) c(w_2, \bar{w}_2) \left| \Psi_{\text{trial}} \right\rangle}{\left\langle \Psi_{\text{trial}} \middle| \Psi_{\text{trial}} \right\rangle}$$

whose decay rate may be interpreted as the size of a Cooper pair. Of course, since we are dealing with a translation-invariant free fermion model, it is easy to calculate this two-point function exactly. Its Fourier transform $\left\langle \Psi_{\text{trial}} \right| c_k c_{-k} \left| \Psi_{\text{trial}} \right\rangle / \left\langle \Psi_{\text{trial}} \middle| \Psi_{\text{trial}} \right\rangle = g_k/(1+|g_k|^2)$ is obtained directly from the BCS ground state (1.4). It is real-analytic in k_x and k_y, which implies that in real space the correlator (1.5) decays exponentially with distance $|w_1 - w_2|$. But this is not really the way we want to do it.

Instead, the question of interest to us is: how should we understand this exponential decay from the point of view of the Ising CFT? Let me start

with the denominator in Eq. (1.5):

$$\langle \Psi_{\text{trial}} | \Psi_{\text{trial}} \rangle$$
$$= \langle 0 | \left\langle \exp \left(\kappa^{*1/2} \int d^2 z \, c(z, \bar{z}) \, \bar{\psi}(\bar{z}) \right) \exp \left(i \, \kappa^{1/2} \int d^2 z \, \psi(z) \, c^\dagger(z, \bar{z}) \right) \right\rangle | 0 \rangle$$
$$= \left\langle \exp \left(i \, |\kappa| \int d^2 z \, \bar{\psi}(\bar{z}) \psi(z) \right) \right\rangle = \left\langle e^{-|\kappa| \int d^2 z \, \varepsilon(z, \bar{z})} \right\rangle,$$

where $\varepsilon(z, \bar{z}) = i\psi(z)\bar{\psi}(\bar{z})$ is the energy operator in the Ising CFT. A similar calculation for the numerator gives $\langle \Psi_{\text{trial}} | c(w_1, \bar{w}_1) c(w_2, \bar{w}_2) | \Psi_{\text{trial}} \rangle = -\kappa \left\langle \psi(w_1) \psi(w_2) e^{|\kappa| \int d^2 z \, \varepsilon(z, \bar{z})} \right\rangle$. Thus, the two-point function in the trial state is rephrased as a ratio of expectation values in the Ising CFT,

$$\frac{\langle \Psi_{\text{trial}} | c(w_1, \bar{w}_1) c(w_2, \bar{w}_2) | \Psi_{\text{trial}} \rangle}{\langle \Psi_{\text{trial}} | \Psi_{\text{trial}} \rangle} = -\kappa \, \frac{\left\langle \psi(w_1) \psi(w_2) e^{-|\kappa| \int d^2 z \, \varepsilon(z, \bar{z})} \right\rangle}{\left\langle e^{-|\kappa| \int d^2 z \, \varepsilon(z, \bar{z})} \right\rangle}$$

The right-hand side is nothing but the correlation function of $\psi(w_1)$ and $\psi(w_2)$, evaluated in the **Ising CFT perturbed by the energy operator**, where the action of the theory has been replaced by

$$S_{\text{CFT}} \rightarrow S_{\text{CFT}} + |\kappa| \int d^2 z \, \varepsilon(z, \bar{z}).$$

This perturbation is relevant, and it drives the Ising model towards the low-temperature phase. Therefore, the correlation function

$$\langle \psi(w_1) \psi(w_2) \rangle_{\text{low T}} := \frac{\left\langle \psi(w_1) \psi(w_2) e^{-|\kappa| \int d^2 z \, \varepsilon(z, \bar{z})} \right\rangle}{\left\langle e^{-|\kappa| \int d^2 z \, \varepsilon(z, \bar{z})} \right\rangle}$$

decays exponentially, and not as a power-law as one could perhaps have thought naively.

The key point, which is absolutely crucial when one looks at trial states constructed from 2d CFT —not only the $p_x + ip_y$ trial state, but also the Laughlin state, the Moore-Read state and others—, is that the correlation functions of physical observables in the bulk are not given directly by CFT correlators; this would be in contradiction with the fact that bulk correlations must decay exponentially. Instead, they are obtained from the **CFT with a perturbation that makes it flow towards a massive RG fixed point, such that all correlations are short-range**.

1.6 The edge CFT is the same as the bulk CFT

Bulk correlations are exponentially decaying. But, when the system has a boundary, what about correlations along the edge? In section 1.1, I advertised the fact that the $p_x + ip_y$-superconductor is an example of a chiral topological phase, with a gapless chiral edge described by a chiral $1+1d$ CFT. Is there a relation between this "physical" $1+1d$ CFT, and the (chiral part of the) 2d CFT that is used to construct the trial state?

1.6.1 Long-range correlations along the edge

First, let me illustrate why correlation functions along the edge should be long-range (i.e. power-law decaying). For simplicity, I consider an interface between the vacuum in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C}; \mathrm{Im}\, z > 0\}$, and the $p_x + ip_y$ superconductor in the lower half-plane $\mathbb{C} \setminus \mathbb{H}$. I write the following trial state for that system,

$$\left| \Psi_{\text{trial}}^{\mathbb{C}\setminus\mathbb{H}} \right\rangle := \left\langle \exp\left(i\,\kappa^{1/2} \int_{\mathbb{C}\setminus\mathbb{H}} d^2z\ \psi(z)\, c^\dagger(z, \bar{z}) \right) \right\rangle |0\rangle , \qquad (1.12)$$

where the integration is restricted to the lower half-plane $\mathbb{C} \setminus \mathbb{H}$. It is the simplest generalization of the state (1.10) that has no particles in $\mathbb{H}$. Moreover, since the bulk correlation length ξ is finite, it obviously has exactly the same bulk properties as the state (1.10), as long as one is at a distance from the edge that is larger than ξ. For these two reasons, it is a reasonable choice for a trial state in the presence of a boundary.

[There is another —perhaps better— justification of the fact that it is a good trial state: it is possible to exhibit a hamiltonian for which $\left| \Psi_{\text{trial}}^{\mathbb{C}\setminus\mathbb{H}} \right\rangle$ is the exact ground state. It turns out that one possibility is the restriction to $\mathbb{C} \setminus \mathbb{H}$ of the the spectral flattening of the hamiltonian H_{trial} briefly discussed below Eq. (1.7).]

Now, for concreteness, imagine that we are interested in the correlation function of two points located along the boundary, for instance:

$$\frac{\left\langle \Psi_{\text{trial}}^{\mathbb{C}\setminus\mathbb{H}} \right| c^\dagger(w_1, \overline{w}_1) c(w_2, \overline{w}_2) \left| \Psi_{\text{trial}}^{\mathbb{C}\setminus\mathbb{H}} \right\rangle}{\left\langle \Psi_{\text{trial}}^{\mathbb{C}\setminus\mathbb{H}} \middle| \Psi_{\text{trial}}^{\mathbb{C}\setminus\mathbb{H}} \right\rangle}, \qquad \text{with} \quad w_1, w_2 \in \mathbb{R}.$$

Acting with $c_1^\dagger$ on $\left\langle \Psi_{\text{trial}}^{\mathbb{C}\setminus\mathbb{H}} \right|$, and with c_2 on $\left| \Psi_{\text{trial}}^{\mathbb{C}\setminus\mathbb{H}} \right\rangle$, the numerator becomes $\langle 0| \left\langle e^{\kappa^{*1/2} \int c\bar{\psi}} \bar\psi(\bar{w}_1)\psi(w_2) e^{i\kappa^{1/2}\int \psi c^\dagger} \right\rangle |0\rangle$, up to some constant factor. Then,
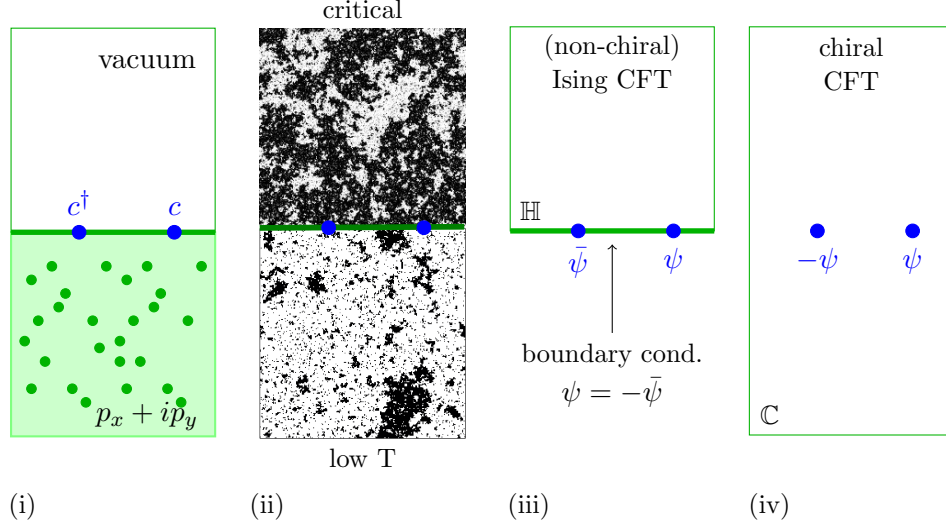
Figure 1.2: Sketch of the argument which shows that correlation functions along the edge are given by the bulk chiral CFT. (i) The calculation of the two-point function $\langle c^\dagger c \rangle$ of operators along the edge maps to the one of (ii) a correlation function $\langle \bar{\psi}\psi \rangle$ in the Ising model, where the upper half is critical, and the lower half is at low temperature. (iii) At the RG fixed point, this is equivalent to the critical Ising CFT in the upper half-plane $\mathbb{H}$, with the conformal boundary condition $\psi = -\bar{\psi}$ along the real axis. (iv) The correlation function is then given by the two-point function $-\langle \psi\psi \rangle$ in the *chiral* CFT in the plane $\mathbb{C}$.

using the same trick as in the previous section, the correlation function may be rewritten as

$$\frac{\left\langle \bar{\psi}(\overline{w}_1)\psi(w_2)\, e^{-|\kappa|\int_{\mathbb{C}\backslash\mathbb{H}} d^2z\, \varepsilon(z,\bar{z})} \right\rangle}{\left\langle e^{-|\kappa|\int_{\mathbb{C}\backslash\mathbb{H}} d^2z\, \varepsilon(z,\bar{z})} \right\rangle},$$

up to multiplication by a constant. This is nothing but a correlation function in an Ising model where the lower half-plane is at low temperature, and the upper half-plane is critical. Under the RG flow, the lower half is sent to a trivial system with zero correlation length, which, from the point of view of the upper half, is just a boundary, where the fields $\psi(w)$ and $\bar{\psi}(\overline{w})$ are constrained by a certain *conformal boundary condition*. This boundary condition can be fixed as follows: (i) it must be a constraint that relates the chiral and anti-chiral fields ψ and $\bar{\psi}$ at the boundary (ii) it must be

local, so it has to be of the form $\psi(w) = e^{i\alpha}\bar{\psi}(\overline{w})$, with some phase $e^{i\alpha}$, (iii) the expectation value of the energy operator $\langle \varepsilon(z, \bar{z}) \rangle_{\mathbb{H}} = \langle i\psi(z)\bar{\psi}(\bar{z}) \rangle_{\mathbb{H}}$ in the upper half-plane should be real and negative (because the local energy should decrease as one approaches the region at low temperature). This implies

$$\psi(w) = -\bar{\psi}(\overline{w}) \qquad \text{when} \quad w \in \mathbb{R}. \tag{1.13}$$

The conclusion of that argument is that, as long as the points w_1 and w_2 are well separated, one finds

$$\frac{\left\langle \Psi_{\text{trial}}^{\mathbb{C}\backslash\mathbb{H}} \middle| c^\dagger(w_1, \overline{w}_1)c(w_2, \overline{w}_2) \middle| \Psi_{\text{trial}}^{\mathbb{C}\backslash\mathbb{H}} \right\rangle}{\left\langle \Psi_{\text{trial}}^{\mathbb{C}\backslash\mathbb{H}} \middle| \Psi_{\text{trial}}^{\mathbb{C}\backslash\mathbb{H}} \right\rangle} \propto \left\langle \bar{\psi}(\overline{w}_1)\psi(w_2) \right\rangle_{\mathbb{H}} = -\left\langle \psi(w_1)\psi(w_2) \right\rangle.$$

This shows that the physical correlation function of operators located at the edge of the $p_x + ip_y$ trial state is automatically equal to a correlation function in the chiral Ising CFT, up to constant factors that do not dependent on the positions.

The argument and its conclusion can easily be generalized to correlation functions with more points.

1.6.2   Trial states for the edge excitations

The connection with boundary CFT can be pushed further, and this is what I do in the next two subsections. These two subsections are slightly more technical than the rest of these notes. In particular I am assuming that the reader knows radial quantization and is familiar with Ishibashi states in boundary CFT.

For our purposes, it is more convenient to work in a disc rather than in the half-plane. So, from now on, I work with a $p_x + ip_y$ superconductor that is filling the disc $D := \{z \in \mathbb{C},\ |z| \leq R\}$; the exterior of the disc $\mathbb{C} \setminus D$ is occupied by the vacuum. Also, without loss of generality, one can decide that one measures all lengths in units of the radius R (e.g. $z \to z/R$), such that $R = 1$ in all the formulas.

I am going to use radial quantization,

$$\psi(z) = \sum_{n \in \mathbb{Z}+\frac{1}{2}} z^{-n-\frac{1}{2}} \psi_n \qquad \bar{\psi}(\bar{z}) = \sum_{n \in \mathbb{Z}+\frac{1}{2}} \bar{z}^{-n-\frac{1}{2}} \bar{\psi}_n,$$

with $\{\psi_n, \psi_m\} = \{\bar{\psi}_n, \bar{\psi}_m\} = \delta_{n+m,0}$. The modes ψ_n of the chiral Majorana field generate an algebra, usually called the *chiral algebra*. Let $|1\rangle$ be the

vacuum of the CFT, that satisfies

$$\psi_n \left|1\right\rangle \;=\; 0 \qquad \text{for all } \; n > 0.$$

Other states are obtained by acting on $\left|1\right\rangle$ with the negative modes,

$$\psi_{-\frac{1}{2}}\left|1\right\rangle, \; \psi_{-\frac{3}{2}}\left|1\right\rangle, \; \psi_{-\frac{5}{2}}\left|1\right\rangle, \; \psi_{-\frac{1}{2}}\psi_{-\frac{3}{2}}\left|1\right\rangle, \; etc.$$

The CFT Hilbert space is the infinite-dimensional space spanned by these states, equipped with the inner product that is compatible with the hermitic structure $\psi_n^\dagger = \psi_{-n}$. It is an irreducible representation of the chiral algebra.

In radial quantization, correlation functions $\langle \psi(z_1) \ldots \psi(z_n) \rangle$ are constructed as matrix elements of the form $\langle 1 | \, \psi(z_1) \ldots \psi(z_n) \, |1\rangle$. Below, I will need to consider matrix elements where the 'bra' $\langle 1|$ is replaced by an arbitrary state $\langle v|$ in the CFT Hilbert space: $\langle v| \, \psi(z_1) \ldots \psi(z_n) \, |1\rangle$. I will write this object as '$\langle v \, | \psi(z_1) \ldots \psi(z_n) \rangle\rangle$'. One way of constructing it explicitly is to use contour integrals,

$$\langle v \, | \psi(z_1) \ldots \psi(z_n) \rangle \; := $$
$$\oint_{\mathcal{C}_p} d\zeta_p \, \zeta_p^{n_p - \frac{1}{2}} \ldots \oint_{\mathcal{C}_1} d\zeta_1 \, \zeta_1^{n_1 - \frac{1}{2}} \, \langle \psi(\zeta_p) \ldots \psi(\zeta_1)\psi(z_1) \ldots \psi(z_n) \rangle,$$
$$\text{with} \qquad |v\rangle = \psi_{-n_1} \ldots \psi_{-n_p} \left|1\right\rangle.$$

Here the contour $\mathcal{C}_1$ encircles all the z_i's. Then $\mathcal{C}_2$ encircles the contour $\mathcal{C}_1$, $\mathcal{C}_3$ encircles $\mathcal{C}_2$, and so on.

Now let me use this formalism to construct new trial states. For any state $|v\rangle$ in the CFT Hilbert space, I define

$$\left| \Psi_{\text{trial}, \langle v|}^D \right\rangle \; := \; \frac{1}{\sqrt{\mathcal{Z}}} \left\langle v \left| (-i)^{\frac{1 - (-1)^F}{2}} \exp\left(i \, \kappa^{1/2} \int_D d^2 z \, \psi(z) \, c^\dagger(z, \bar{z}) \right) \right\rangle \right. |0\rangle.$$
$$(1.14)$$

Here $(-1)^F$ is the operator that measures fermion parity in the CFT Hilbert space. The factor $(-i)^{\frac{1 - (-1)^F}{2}}$ is equal to 1 for configurations with even numbers of ψ's, and $-i$ for odd numbers. In the latter case, the $-i$ is needed in order to cancel the i coming from the exponential, such that the

expansion of the latter reads

$$
\begin{aligned}
\sqrt{\mathcal{Z}}\left|\Psi^{D}_{\mathrm{trial},\,\langle v|}\right\rangle \;&=\; \langle v\,|1\rangle\ |0\rangle \\[4pt]
&+\; \kappa^{1/2}\int_{D} d^2 z_1\, \langle v\,|\psi(z_1)\rangle\ c^{\dagger}(z_1,\bar z_1)\,|0\rangle \\[4pt]
&+\; \frac{\kappa}{2!}\int_{D} d^2 z_1 d^2 z_2\, \langle v\,|\psi(z_1)\psi(z_2)\rangle\ c_1^{\dagger}c_2^{\dagger}\,|0\rangle \\[4pt]
&+\; \frac{\kappa^{3/2}}{3!}\int_{D} d^2 z_1 d^2 z_2 d^2 z_3\, \langle v\,|\psi(z_1)\psi(z_2)\psi(z_3)\rangle\ c_1^{\dagger}c_2^{\dagger}c_3^{\dagger}\,|0\rangle \\[4pt]
&+\;\ldots
\end{aligned}
$$

Below, I will also need to manipulate the corresponding 'bra'. It will be convenient to write it as

$$
\begin{aligned}
\sqrt{\mathcal{Z}}\left\langle\Psi^{D}_{\mathrm{trial},\,\langle\bar v|}\right| \;&=\; \langle 0|\left\langle v\left|(-1)^{F}\exp\left(\kappa^{*1/2}\int_{D} d^2 z\, c(z,\bar z)\,\bar\psi(\bar z)\right)\right\rangle\right|0\rangle . \\[6pt]
&=\; \langle 0|\ \overline{\langle v\,|1\rangle}\ +\ \kappa^{*1/2}\int_{D} d^2 z_1\, \overline{\langle v\,|\psi(z_1)\rangle}\ \langle 0|\, c(z_1,\bar z_1) \\[4pt]
&+\; \frac{\kappa^{*}}{2!}\int_{D} d^2 z_1 d^2 z_2\, \overline{\langle v\,|\psi(z_1)\psi(z_2)\rangle}\ \langle 0|\, c_2 c_1 \\[4pt]
&+\; \frac{\kappa^{*3/2}}{3!}\int_{D} d^2 z_1 d^2 z_2 d^2 z_3\, \overline{\langle v\,|\psi(z_1)\psi(z_2)\psi(z_3)\rangle}\ \langle 0|\, c_3 c_2 c_1 \\[4pt]
&+\;\ldots
\end{aligned}
$$

The normalization factor $\mathcal{Z}$ in (1.14) is introduced for later convenience, and is defined as

$$
\mathcal{Z} \;:=\; \left\langle e^{-|\kappa|\int_{D} d^2 z\,\varepsilon(z,\bar z)}\right\rangle .
$$

By construction, all these new trial states have the same bulk properties as the trial state (1.10), at distances from the boundary that are larger than the bulk correlation length ξ. But they may differ close to the boundary.

The definition (1.14) provides an (anti-)linear map $|v\rangle \mapsto \left|\Psi^{D}_{\mathrm{trial},\,\langle v|}\right\rangle$ from the CFT Hilbert to a certain subspace of the physical Hilbert space, spanned by all the trial states for edge excitations. One can refer to this subspace of the physical Hilbert space as the *Hilbert space of edge excitations.*

1.6.3   Edge state inner products

It is possible to say much more about the map $|v\rangle \mapsto \left|\Psi^{D}_{\mathrm{trial},\,\langle v|}\right\rangle$ that sends the CFT Hilbert space onto the Hilbert space of edge excitations. Indeed, the goal of this subsection is to show that, in the thermodynamic limit, **it**

is an isometry:

$$\left\langle \Psi^D_{\text{trial}, \langle v_1|} \middle| \Psi^D_{\text{trial}, \langle v_2|} \right\rangle = \langle v_2 | v_1 \rangle . \tag{1.15}$$

Hence the two Hilbert spaces are the same. **This gives a precise meaning to the statement that "the edge theory is the same as the bulk CFT"**.

Let me explain where formula (1.15) comes from. I start by clarifying the meaning of 'thermodynamic limit' here. It would, in principle, mean that the radius R is sent to infinity, but since we are measuring all lengths in units of R, what it really means is that one takes the limit $|\kappa| \to \infty$, keeping $R = 1$. Then the main idea is that the following state (where $|1, \bar{1}\rangle := |1\rangle \overline{|1\rangle}$ is the vacuum of the non-chiral CFT and $\varepsilon(z, \bar{z}) = i\psi(z)\bar{\psi}(\bar{z})$ is the energy operator),

$$\frac{e^{-|\kappa| \int_D d^2 z\, \varepsilon(z, \bar{z})} |1, \bar{1}\rangle}{\left\langle e^{-|\kappa| \int_D d^2 z\, \varepsilon(z, \bar{z})} \right\rangle},$$

must become a *conformal boundary state*, or *Ishibashi state*, when $|\kappa| \to 1$. An Ishibashi state is a state that automatically encodes the conformal boundary condition (1.13). In the radially quantized CFT, this condition reads

$$(\psi_n + i\,\bar{\psi}_{-n}) |\text{Ishibashi}\rangle = 0.$$

[This is obtained from Eq. (1.13) by conformally mapping $\mathbb{H}$ to $\mathbb{C} \setminus D$, which gives the condition $z^{\frac{1}{2}}\psi(z) = -i\,\bar{z}^{\frac{1}{2}}\bar{\psi}(\bar{z})$ if $|z| = 1$.] Moreover, it is normalized such that $\langle 1, \bar{1} | \text{Ishibashi}\rangle = \langle 1 | 1\rangle = 1$. It follows that the Ishibashi state possesses the following property for all $|v_1\rangle, |v_2\rangle$:

$$\langle \overline{v_1}, v_2 | i^{\frac{1-(-1)^F}{2}} |\text{Ishibashi}\rangle = (\overline{\langle v_1 |}\, \langle v_2 | i^{\frac{1-(-1)^F}{2}}) |\text{Ishibashi}\rangle = \langle v_2 | v_1\rangle . \tag{1.16}$$

This is all we need to derive Eq. (1.15). Using the same trick as above, one gets

$$\left\langle \Psi^D_{\text{trial}, \langle v_1|} \middle| \Psi^D_{\text{trial}, \langle v_2|} \right\rangle$$

$$= \frac{1}{\mathcal{Z}} \langle 0 | \left\langle \overline{v_1}, v_2 \middle| (-1)^F (-i)^{\frac{1-(-1)^F}{2}} e^{\kappa^{*1/2} \int_D d^2 z\, c(z, \bar{z})\, \bar{\psi}(\bar{z})} e^{i\,\kappa^{1/2} \int_D d^2 z\, \psi(z)\, c^\dagger(z, \bar{z})} \right\rangle |0\rangle$$

$$= \frac{1}{\mathcal{Z}} \left\langle \overline{v_1}, v_2 \middle| i^{\frac{1-(-1)^F}{2}} e^{-|\kappa| \int_D d^2\, \varepsilon(z, \bar{z})} \right\rangle \xrightarrow[|\kappa| \to \infty]{} \langle \overline{v_1}, v_2 | i^{\frac{1-(-1)^F}{2}} |\text{Ishibashi}\rangle ,$$

and one concludes using property (1.16).

Lecture 2

Non-abelian statistics

2.1 Preliminary: a refresher on conformal blocks in the Ising CFT

In the first lecture, I used correlators of the Majorana field $\langle \psi(z_1) \ldots \psi(z_n) \rangle$ as trial wavefunctions. Those correlators are all meromorphic functions of the z_i's.

In general, it is not true that conformal blocks are single-valued [a conformal block is a correlator of chiral primary fields in a chiral CFT]. The chiral Ising CFT contains three primary fields: the identity 1 (with conformal dimension 0), the Majorana field ψ (dimension $\frac{1}{2}$) and another field noted σ (dimension $\frac{1}{16}$), which is the chiral part of the operator that measures the local magnetization in the critical 2d Ising model. It is clear from the dimension of σ that its correlators cannot be single-valued: for instance, the two-point function is (with $\eta_{12} := \eta_1 - \eta_2$)

$$\langle \sigma(\eta_1)\sigma(\eta_2) \rangle = \frac{1}{(\eta_{12})^{\frac{1}{8}}}, \tag{2.1}$$

so it has a branch-cut. If one exchanges η_1 and η_2 clockwise (*i.e.* such that $\eta_{12} \to e^{i\theta}\eta_{21}$ for θ from 0 to π), then $\langle \sigma_1\sigma_2 \rangle$ picks up a phase $\frac{\pi}{8}$.

2.1.1 Action of the braid group on the space of conformal blocks

With four chiral operators σ, the situation is even worse: if one continuously exchanges two of the σ's clockwise, one arrives at a result that is not just the same function multiplied by a phase. Instead, there are two linearly

independent conformal blocks, and one should think of $\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle$ as an arbitrary linear combination

$$\langle \sigma(\eta_1)\sigma(\eta_2)\sigma(\eta_3)\sigma(\eta_4) \rangle = v_1 \mathcal{F}_1(\eta) + v_2 \mathcal{F}_2(\eta) \tag{2.2}$$

with complex coefficients v_1 and v_2, and

$$\begin{aligned}
\mathcal{F}_1(\eta) &= \frac{1}{\sqrt{2}} \left(\frac{\eta_{13}\eta_{24}}{\eta_{12}\eta_{23}\eta_{34}\eta_{14}} \right)^{\frac{1}{8}} \sqrt{1 + \sqrt{1-x}} \\
\mathcal{F}_2(\eta) &= \frac{1}{\sqrt{2}} \left(\frac{\eta_{13}\eta_{24}}{\eta_{12}\eta_{23}\eta_{34}\eta_{14}} \right)^{\frac{1}{8}} \sqrt{1 - \sqrt{1-x}}
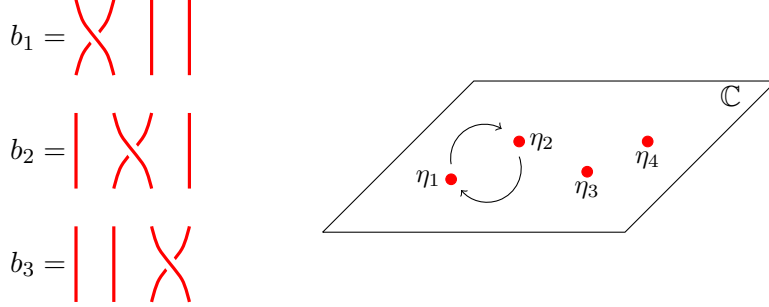\end{aligned} \qquad x := \frac{\eta_{12}\eta_{34}}{\eta_{13}\eta_{24}} .$$

Figure 2.1: Left: the three generators of the braid group B_4. Right: the generator b_1 acts by exchanging the coordinates η_1 and η_2 clockwise.

Then, as one exchanges η_1 and η_2 clockwise, the coefficients of the linear combination change according to the rule

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longrightarrow b_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{3\pi}{8}} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} . \tag{2.3a}$$

Similarly, there are two other 2×2 matrices that encode the effect of exchanging z_2 and z_3 clockwise, and z_3 and z_4 clockwise:

$$b_2 = \frac{e^{i\frac{\pi}{8}}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \qquad b_3 = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{3\pi}{8}} \end{pmatrix} . \tag{2.3b}$$

The matrix b_1, b_2 and b_3 give a 2-dimensional representation of the braid group B_4. This representation is irreducible: it is not possible to find a basis where the three matrices b_1, b_2 and b_3 are simultaneously diagonal, because

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b_2 does not commute with b_1 and b_3. Thus, the space of conformal blocks $\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle$ provides a *non-abelian* representation of the braid group B_4.

It is also important to stress that this representation of the braid group B_4 is *unitary*: it preserves the euclidean norm $v_1^2 + v_2^2$ of the vector of coefficients $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

The generalization to higher number of operators is as nice as one would naively expect: the space of conformal blocks $\langle \sigma_1 \dots \sigma_{2m} \rangle$ provides a unitary, irreducible, 2^{m-1}-dimensional representation of the braid group B_{2m} (while, for odd number of operators, the correlators $\langle \sigma_1 \dots \sigma_{2m+1} \rangle$ are zero, because they are odd under $\mathbb{Z}_2$ symmetry $\sigma \to -\sigma$).

2.1.2 How to combine chiral and anti-chiral blocks to get correlation functions in the critical Ising model

Correlation functions of mutually local operators in the critical 2d Ising model must be single-valued. One such local operator in the Ising model is the local magnetization at a point z. It is a non-chiral field $\sigma(\eta, \bar{\eta})$, which can be roughly viewed as the chiral operator $\sigma(\eta)$ times its anti-chiral copy $\bar{\sigma}(\bar{\eta})$.

The four-point correlation function of the local magnetization must be invariant under the exchange of the points. There exists a unique combination of the blocks $\mathcal{F}_1(\eta)$, $\mathcal{F}_2(\eta)$, $\overline{\mathcal{F}}_1(\bar{\eta})$, $\overline{\mathcal{F}}_2(\bar{\eta})$ that is invariant under the action of B_4: the so-called 'diagonal' combination. Because it is the only possibility, one can safely identify it with the four-point function

$$\langle \sigma(\eta_1, \bar{\eta}_1) \sigma(\eta_2, \bar{\eta}_2) \sigma(\eta_3, \bar{\eta}_3) \sigma(\eta_4, \bar{\eta}_4) \rangle$$
$$= \begin{pmatrix} \mathcal{F}_1(\eta) & \mathcal{F}_2(\eta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathcal{F}}_1(\bar{\eta}) \\ \overline{\mathcal{F}}_2(\bar{\eta}) \end{pmatrix}. \qquad (2.4a)$$

In the 2d Ising model (critical or not), there exists another operator, the *disorder operator* $\mu(\eta, \bar{\eta})$, which creates a topological defect such that the order parameter $\sigma(\eta', \bar{\eta}')$ picks up a factor -1 when it turns around $\mu(\eta, \bar{\eta})$. Thus, $\mu(\eta, \bar{\eta})$ and $\sigma(\eta', \bar{\eta}')$ are not mutually local. In CFT, the operator $\mu(\eta, \bar{\eta})$ is constructed out of the chiral and anti-chiral fields $\sigma(\eta)$ and $\bar{\sigma}(\bar{\eta})$, just like the local magnetization. However, the 'mixed' correlation functions that involve both the disorder operator and the local magnetization cannot be given by the diagonal combination. Instead, they are obtained from the other three hermitian combinations (hermiticity is needed because the

correlators of σ and μ are real-valued):

$$\langle\mu(\eta_1,\bar\eta_1)\sigma(\eta_2,\bar\eta_2)\sigma(\eta_3,\bar\eta_3)\mu(\eta_4,\bar\eta_4)\rangle$$
$$= \left(\begin{array}{cc} \mathcal{F}_1(\eta) & \mathcal{F}_2(\eta) \end{array}\right) \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) \left(\begin{array}{c} \overline{\mathcal{F}}_1(\bar\eta) \\ \overline{\mathcal{F}}_2(\bar\eta) \end{array}\right) \qquad (2.4\text{b})$$

$$\langle\sigma(\eta_1,\bar\eta_1)\mu(\eta_2,\bar\eta_2)\sigma(\eta_3,\bar\eta_3)\mu(\eta_4,\bar\eta_4)\rangle$$
$$= \left(\begin{array}{cc} \mathcal{F}_1(\eta) & \mathcal{F}_2(\eta) \end{array}\right) \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right) \left(\begin{array}{c} \overline{\mathcal{F}}_1(\bar\eta) \\ \overline{\mathcal{F}}_2(\bar\eta) \end{array}\right) \qquad (2.4\text{c})$$

$$\langle\sigma(\eta_1,\bar\eta_1)\sigma(\eta_2,\bar\eta_2)\mu(\eta_3,\bar\eta_3)\mu(\eta_4,\bar\eta_4)\rangle$$
$$= \left(\begin{array}{cc} \mathcal{F}_1(\eta) & \mathcal{F}_2(\eta) \end{array}\right) \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) \left(\begin{array}{c} \overline{\mathcal{F}}_1(\bar\eta) \\ \overline{\mathcal{F}}_2(\bar\eta) \end{array}\right). \qquad (2.4\text{d})$$

The fact that those non-diagonal combinations of the conformal blocks correspond to correlation functions involving the disorder operator μ will be crucial in section 2.3.

2.2   Back to the $p_x + ip_y$ superconductor: vortices

We are now ready to talk about non-abelian statistics in the $p_x + ip_y$-paired superfluid.
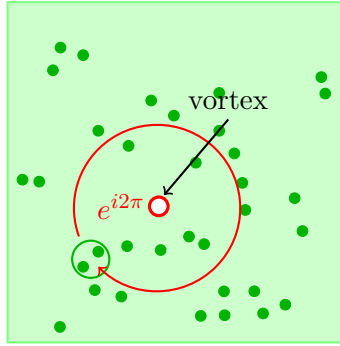

Figure 2.2: A vortex is a topological defect around which the phase of a Cooper pair (or, equivalently, of the gap function) has non-zero winding. Eq. (2.5) defines trial wavefunctions for $p_x + ip_y$ superconductor with vortices that all have winding number $+1$ (the phase jump around the vortex is 2π).

Superconductors in $3d$ can have vortex lines, around which the phase of a Cooper pair (or, equivalently, of the gap function) is winding: $\langle cc \rangle \rightarrow$

$e^{i2\pi} \langle cc \rangle$ when the pair is dragged around the line. In $2d$, the vortex "lines" are just points around which the gap function $\Delta(k)$ has non-zero winding.

It is of course possible to deal with vortices directly in BCS theory: one would write a BCS hamiltonian with a gap function of the form $e^{i\theta}\Delta(k)$ that depends on the angle θ around the vortex, then solve the corresponding Bogoliubov-de Gennes equations. It is possible to do that for the $p_x + ip_y$-superconductor, but this is not what I am going to do here.

Since the purpose of these two lectures is to give an introduction to trial states for 2d chiral topological phases (not an introduction to BCS theory), I am going to focus directly on a trial state that has vortices inserted at positions η_j, and I will simply show that the phase acquired by a Cooper pair around each of these vortices is 2π (the same for all vortices) for that trial state. Thus, the trial state will possess the basic property expected from the true BCS ground state. Then, even if we are not dealing explicitly with a BCS hamiltonian, one can study the trial state for itself. The physical features that are characteristic of the entire topological phase should be apparent in that trial state.

The $p_x + ip_y$ trial state with vortices at positions $\eta = (\eta_1, \eta_2, \ldots, \eta_{2m})$ is

$$|\Psi_{\text{trial}}(\eta)\rangle = \left\langle \sigma(\eta_1)\sigma(\eta_2)\ldots\sigma(\eta_{2m}) \exp\left(i\,\kappa^{1/2}\int d^2z\,\psi(z)\,c^\dagger(z,\bar{z})\right)\right\rangle |0\rangle. \tag{2.5}$$

It is a straighforward generalization of the state $|\Psi_{\text{trial}}\rangle$ without vortices, see Eq. (1.10). The two states (with and without vortices) have the property that, away from the vortices, they look exactly the same.

For simplicity, let me start by considering the case of two vortices. Expanding the exponential, one gets

$$\begin{aligned}
|\Psi_{\text{trial}}(\eta)\rangle &= \langle \sigma(\eta_1)\sigma(\eta_2)\rangle |0\rangle \\
&\quad + \frac{1}{2}\int d^2z_1 d^2z_2\,\langle \sigma(\eta_1)\sigma(\eta_2)\psi(z_1)\psi(z_2)\rangle\,c_1^\dagger c_2^\dagger |0\rangle \\
&\quad + \ldots
\end{aligned}$$

and one sees that the behavior of the wavefunction $\langle \sigma(\eta_1)\sigma(\eta_2)\psi(z_1)\psi(z_2)\rangle$ is determined by the operator product expansion of ψ with σ,

$$\psi(z_i)\sigma(\eta_j) \sim \frac{1}{(z_i - \eta_j)^{\frac{1}{2}}}\sigma(\eta_j) + \ldots$$

Thus, the wavefunction $\langle \sigma(\eta_1)\sigma(\eta_2)\psi(z_1)\psi(z_2)\rangle$ picks up a phase π when z_i is rotated around η_j. The same is obviously true for higher-order terms in

the expansion of the exponential, corresponding to more particles. When a Cooper pair (made of two particles) rotates around one of the vortices, it picks up a phase 2π. So the trial state $|\Psi_{\text{trial}}(\eta)\rangle$ works as advertised: it correctly encodes the phase acquired by the Cooper pairs as they go around the vortices.

There is, however, an important problem with the definition (2.5): for $2m \geq 4$, the correlator $\langle \sigma(\eta_1) \ldots \sigma(\eta_{2m})\psi(z_1)\ldots\psi(z_n)\rangle$ is not well-defined. As briefly reviewed in section 2.1, there are in fact 2^{m-1} linearly independent conformal blocks, and what we write formally as $\langle \sigma(\eta_1)\ldots\sigma(\eta_{2m})\rangle$ can be any linear combination of those 2^{m-1} terms. This still holds when Majorana fields are inserted in the correlator: $\langle \sigma(\eta_1)\ldots\sigma(\eta_{2m})\psi(z_1)\ldots\psi(z_n)\rangle$ is always a superposition of 2^{m-1} blocks, no matter how large n is. Thus, Eq. (2.5) does not define one state; instead, it defines 2^{m-1} states!

For simplicity, from now on I focus only on the case of $2m = 4$ vortices. Then there are two linearly independent states, labeled by the two conformal blocks $\mathcal{F}_1(\eta)$ and $\mathcal{F}_2(\eta)$. We write these two states as $|\Psi_{\text{trial}, \mathcal{F}_1}(\eta)\rangle$ and $|\Psi_{\text{trial}, \mathcal{F}_2}(\eta)\rangle$. To be concrete, these two states are defined as follows ($a = 1, 2$):

$$|\Psi_{\text{trial}, \mathcal{F}_a}(\eta)\rangle =$$
$$\sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d^2 z_i \left\langle \sigma(\eta_1)\sigma(\eta_2)\sigma(\eta_3)\sigma(\eta_4) \prod_{p=1}^{n} \psi(z_p) \right\rangle_a \prod_{q=1}^{n} c^\dagger(z_q, \bar{z}_q) |0\rangle \ ,$$

where $\langle \sigma_1\sigma_2\sigma_3\sigma_4\psi_1\ldots\psi_n\rangle_a$ is the conformal block that satisfies

$$\langle \sigma_1\sigma_2\sigma_3\sigma_4\psi_1\ldots\psi_n\rangle_a \sim \mathcal{F}_a(\eta) \times \langle \psi(z_1)\ldots\psi(z_n)\rangle$$

when $|z_1|, |z_2|, \ldots, |z_n| \gg |\eta_1|, \ldots, |\eta_4|$.

2.3 Overlaps between trial states with vortices

I have argued that, in the presence of four vortices, there are two different trial states $|\Psi_{\text{trial}, \mathcal{F}_{1,2}}(\eta)\rangle$ labeled by the two conformal blocks $\mathcal{F}_1$ and $\mathcal{F}_2$. In fact, there is a caveat in the argument: even though the states look linearly independent —because, when expanding the exponential, the amplitudes of the n-particle configurations $c_1^\dagger \ldots c_n^\dagger |0\rangle$ are given by two linearly independent conformal blocks—, one should check their overlaps in the thermodynamic limit. It could be that their overlap actually goes to one; if so, then we have just found two different ways of writing the same physical state.

To rule out this possibility, we need to focus on the overlap matrix $M(\eta, \bar{\eta})$, with entries

$$M_{ab}(\eta, \bar{\eta}) := \frac{\langle \Psi_{\text{trial}, \mathcal{F}_a}(\eta) \, | \, \Psi_{\text{trial}, \mathcal{F}_b}(\eta) \rangle}{\sqrt{\langle \Psi_{\text{trial}, \mathcal{F}_a}(\eta) \, | \, \Psi_{\text{trial}, \mathcal{F}_a}(\eta) \rangle} \sqrt{\langle \Psi_{\text{trial}, \mathcal{F}_b}(\eta) \, | \, \Psi_{\text{trial}, \mathcal{F}_b}(\eta) \rangle}}.$$
(2.6)

To calculate the matrix M, it is convenient to introduce another 2×2 hermitian matrix $\tilde{M}(\eta, \bar{\eta})$ defined as

$$\tilde{M}_{ab}(\eta, \bar{\eta}) := \frac{\langle \Psi_{\text{trial}, \mathcal{F}_a}(\eta) \, | \, \Psi_{\text{trial}, \mathcal{F}_b}(\eta) \rangle}{\langle \Psi_{\text{trial}} \, | \, \Psi_{\text{trial}} \rangle}, \tag{2.7}$$

with a denominator that is the norm of the trial state without vortices, see Eq. (1.10). The goal is to use the same trick as in the first lecture: the fact that, when one calculates overlaps between trial states, one gets expectation values of two integrals of the form $\langle 0| e^{\kappa^{*1/2} \int c \bar{\psi}} e^{i\kappa^{1/2} \int \psi c^\dagger} |0\rangle$, which give the exponential of the energy operator in the Ising CFT, $e^{-|\kappa| \int \varepsilon}$. The only difference with the previous lecture is that, now, we need to be careful about the insertions of the chiral and anti-chiral operators $\sigma(\eta)$, $\bar{\sigma}(\bar{\eta})$.

To do this properly, it is convenient to consider the following four quantities:

$$\text{tr}[I \cdot \tilde{M}] \qquad \text{tr}[\sigma^x \cdot \tilde{M}] \qquad \text{tr}[\sigma^y \cdot \tilde{M}] \qquad \text{tr}[\sigma^z \cdot \tilde{M}], \tag{2.8}$$

where I is the identity and σ^x, σ^y, σ^z are the Pauli matrices. Indeed, looking back at Eq. (2.4.a), we see that $\text{tr}[I \cdot \tilde{M}]$ is nothing but the four-point correlation of the order parameter $\sigma(\eta, \bar{\eta})$, but this time it includes the perturbation by the energy operator,

$$\text{tr}[I \cdot \tilde{M}(\eta, \bar{\eta})]$$
$$= \frac{\langle 0| \left\langle \left(\prod_{j=1}^4 \sigma(\eta_j, \bar{\eta}_j) \right) e^{\kappa^{*1/2} \int d^2z \, c(z, \bar{z}) \, \bar{\psi}(\bar{z})} e^{i \, \kappa^{1/2} \int d^2z \, \psi(z) \, c^\dagger(z, \bar{z})} \right\rangle |0\rangle}{\langle \Psi_{\text{trial}} \, | \, \Psi_{\text{trial}} \rangle}$$
$$= \frac{\left\langle \left(\prod_{j=1}^4 \sigma(\eta_j, \bar{\eta}_j) \right) e^{-|\kappa| \int d^2z \, \varepsilon(z, \bar{z})} \right\rangle}{\left\langle e^{-|\kappa| \int d^2z \, \varepsilon(z, \bar{z})} \right\rangle}$$
$$= \langle \sigma(\eta_1, \bar{\eta}_1) \sigma(\eta_2, \bar{\eta}_2) \sigma(\eta_3, \bar{\eta}_3) \sigma(\eta_4, \bar{\eta}_4) \rangle_{\text{low T}}.$$

Similarly, looking at Eqs. (2.4.b,c,d), we see that:

$$\begin{aligned}
\text{tr}[\sigma^x \cdot \tilde{M}(\eta, \bar{\eta})] &= \langle \mu(\eta_1, \bar{\eta}_1) \sigma(\eta_2, \bar{\eta}_2) \sigma(\eta_3, \bar{\eta}_3) \mu(\eta_4, \bar{\eta}_4) \rangle_{\text{low T}} \\
\text{tr}[\sigma^y \cdot \tilde{M}(\eta, \bar{\eta})] &= \langle \sigma(\eta_1, \bar{\eta}_1) \mu(\eta_2, \bar{\eta}_2) \sigma(\eta_3, \bar{\eta}_3) \mu(\eta_4, \bar{\eta}_4) \rangle_{\text{low T}} \\
\text{tr}[\sigma^z \cdot \tilde{M}(\eta, \bar{\eta})] &= \langle \sigma(\eta_1, \bar{\eta}_1) \sigma(\eta_2, \bar{\eta}_2) \mu(\eta_3, \bar{\eta}_3) \mu(\eta_4, \bar{\eta}_4) \rangle_{\text{low T}}.
\end{aligned}$$

In the low temperature phase, the local magnetization σ acquires a non-zero expectation value. The disorder operator μ, on the other hand, has expectation value zero. [In contrast, in the high-temperature phase, the expectation value of the disorder operator would be non-zero, while the local magnetization would be zero. The two are exchanged by Kramers-Wannier duality.] Moreover, since the correlation length ξ is finite in the low-temperature phase, the connected parts of these correlation functions decay exponentially with the distance between the points. This implies

$$
\begin{aligned}
\mathrm{tr}[I \cdot \tilde{M}(\eta, \bar{\eta})] &= C + O(e^{-|\eta_i - \eta_j|/\xi}) \\
\mathrm{tr}[\sigma^x \cdot \tilde{M}(\eta, \bar{\eta})] &= O(e^{-|\eta_i - \eta_j|/\xi}) \\
\mathrm{tr}[\sigma^y \cdot \tilde{M}(\eta, \bar{\eta})] &= O(e^{-|\eta_i - \eta_j|/\xi}) \\
\mathrm{tr}[\sigma^z \cdot \tilde{M}(\eta, \bar{\eta})] &= O(e^{-|\eta_i - \eta_j|/\xi}),
\end{aligned}
$$

where C is a non-zero constant that does not depend on the positions $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$. This, together with the fact that $\tilde{M}$ is hermitian, shows that $\tilde{M}$ is proportional to the identity matrix,

$$
\tilde{M}(\eta, \bar{\eta}) = C/2 \times I, \tag{2.9}
$$

up to corrections that are exponentially suppressed. Thus, up to these exponentially small corrections, the overlap matrix (2.6) is the identity:

$$
M_{ab}(\eta, \bar{\eta}) = \delta_{ab}. \tag{2.10}
$$

This remarkably simple result is extremely important. Not only does it show that the two states $|\Psi_{\mathrm{trial}, \mathcal{F}_1}(\eta)\rangle$, $|\Psi_{\mathrm{trial}, \mathcal{F}_2}(\eta)\rangle$ are linearly independent: it shows that they provide an **orthonormal basis** of the subspace they span in the physical Hilbert space.

It is possible to generalize the argument and its conclusion to higher numbers of vortices $2m$. The linear map that turns a conformal block $\mathcal{F}_a(\eta)$, $a = 1, \ldots, 2^{m-1}$, into a trial state $|\Psi_{\mathrm{trial}, \mathcal{F}_a}(\eta)\rangle$ is, in fact, an *isometry* between two Hilbert spaces: the space of conformal blocks (with the inner product compatible with the unitary action of B_{2m}) and the 2^{m-1}-dimensional subspace of the physical Hilbert space spanned by the trial states.

2.4 Adiabatic transport of the vortices

Finally, let me turn to non-abelian statistics. Imagine that one is able to move the vortices around, and that one can do it adiabatically. 'Adiabatically' means 'sufficiently slowly', such that system stays in the *adiabatic*

subspace of degenerate lowest-energy states at all times. Here, this *adiabatic subspace* is the one spanned by $|\Psi_{\mathrm{trial},\,\mathcal{F}_1}(\eta)\rangle$ and $|\Psi_{\mathrm{trial},\,\mathcal{F}_2}(\eta)\rangle$. For convenience, in this last section I switch to a lighter notation:

$$|\Psi_a(\eta)\rangle \;:=\; \frac{|\Psi_{\mathrm{trial},\,\mathcal{F}_a}(\eta)\rangle}{\sqrt{\langle\Psi_{\mathrm{trial},\,\mathcal{F}_a}(\eta)\,|\Psi_{\mathrm{trial},\,\mathcal{F}_a}(\eta)\rangle}}\,, \qquad a = 1, 2.$$

According to (2.10), $\{|\Psi_1(\eta)\rangle, |\Psi_2(\eta)\rangle\}$ is an orthonormal basis of the adiabatic subspace. Now, imagine that we start from the state $v_1\,|\Psi_1(\eta)\rangle + v_2\,|\Psi_2(\eta)\rangle$ —which one can also write as $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in that basis—, and that we adiabatically exchange η_1 and η_2 clockwise. What is the final state $\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}$ of the system?
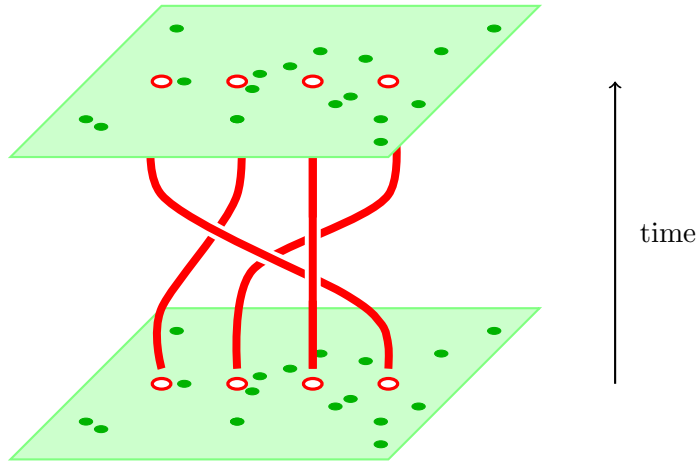

Figure 2.3: The adiabatic exchange of vortices gives a physical realization of the braid group B_4. The question is: is it the same representation of B_4 as the one that is realized in the space of conformal blocks (see Eqs. (2.3a,b))? To answer affirmatively, one must check that there is no unwanted multiplication by a Berry phase (which, here, is a 2×2 "Berry matrix").

The answer is given by a combination of two effects.

- First, the system accumulates a Berry phase, or more precisely a "Berry matrix" (here a 2×2 unitary matrix). Indeed, when one adiabatically transports the state from η to $\eta + d\eta$, it gets multiplied by the matrix

$$I \;+\; i(A_\eta \cdot d\eta + A_{\bar\eta} \cdot d\bar\eta) \tag{2.11}$$

where A_η and $A_{\bar\eta}$ are the components of the Berry connection. In the present case, the latter are 2×2 matrices with entries

$$\left[A_{\eta_j}\right]_{ab} := i \langle \Psi_a(\eta)| \partial_{\eta_j} |\Psi_b(\eta)\rangle$$

and $A_{\bar\eta} = A_\eta^\dagger$. The unitary Berry matrix that is accumulated along the path takes the form of a path-ordered exponential $\mathcal{P}\, e^{i \int A_\eta \cdot d\eta + A_{\bar\eta} \cdot d\bar\eta}$.

- Second, since, when exchanging η_1 and η_2, we are continuously deforming the conformal block $\mathcal{F}_a$ in the definition of $|\Psi_a(\eta)\rangle$, the state $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ should get multiplied by the matrix b_1, according to Eq. (2.3a).

So, the adiabatic exchange of the vortices results in a mixture of two different factors. One is particularly simple and nice: it is the one coming from the monodromy of the conformal blocks, namely the matrix b_1. The other is the Berry matrix, which could possibly be complicated, and could perhaps spoil the nice properties that are inherited from the monodromy.

The miracle is that there is actually no contribution from the Berry matrix, because the Berry connection happens to be zero,

$$A_\eta = A_{\bar\eta} = 0.$$

[This is explained below.] Thus, adiabatically exchanging η_1 and η_2 clockwise, one finds that the state is transformed *exactly* as in (2.3a),

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow b_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Of course, the same conclusion holds for b_2 and b_3 (see Eqs. (2.3a,b)). So the adiabatic exchange of the vortices gives *exactly the same representation* of B_4 as above.

To see why the Berry connection vanishes, notice that the unnormalized trial state $|\Psi_{\mathrm{trial},\mathcal{F}_a}(\eta)\rangle$ is analytic in η_j (as long as η_j is away from the positions of the other vortices η_p). This implies

$$\partial_{\eta_j} \langle \Psi_{\mathrm{trial},\mathcal{F}_a}(\eta) | \Psi_{\mathrm{trial},\mathcal{F}_b}(\eta)\rangle = \langle \Psi_{\mathrm{trial},\mathcal{F}_a}(\eta)| \partial_{\eta_j} |\Psi_{\mathrm{trial},\mathcal{F}_b}(\eta)\rangle,$$

which permits the following manipulation. Using $\tilde{M}$ defined in Eq. (2.7),

as well as the result (2.9), one finds:

$$
\begin{aligned}
[A_{\eta_j}]_{ab} &= i\partial_{\eta_j}\left(\frac{\langle\Psi_{\text{trial},\,\mathcal{F}_a}(\eta)\,|\Psi_{\text{trial},\,\mathcal{F}_b}(\eta)\rangle}{\sqrt{\langle\Psi_{\text{trial},\,\mathcal{F}_a}(\eta)\,|\Psi_{\text{trial},\,\mathcal{F}_a}(\eta)\rangle}\sqrt{\langle\Psi_{\text{trial},\,\mathcal{F}_b}(\eta)\,|\Psi_{\text{trial},\,\mathcal{F}_b}(\eta)\rangle}}\right) \\
&\quad -i\partial_{\eta_j}\left(\frac{1}{\sqrt{\langle\Psi_{\text{trial},\,\mathcal{F}_a}(\eta)\,|\Psi_{\text{trial},\,\mathcal{F}_a}(\eta)\rangle}}\right)\frac{\langle\Psi_{\text{trial},\,\mathcal{F}_a}(\eta)\,|\Psi_{\text{trial},\,\mathcal{F}_b}(\eta)\rangle}{\sqrt{\langle\Psi_{\text{trial},\,\mathcal{F}_b}(\eta)\,|\Psi_{\text{trial},\,\mathcal{F}_b}(\eta)\rangle}} \\
&= i\partial_{\eta_j}\left(\frac{\tilde{M}_{ab}(\eta,\bar{\eta})}{\sqrt{\tilde{M}_{aa}(\eta,\bar{\eta})}\sqrt{\tilde{M}_{bb}(\eta,\bar{\eta})}}\right) - i\partial_{\eta_j}\left(\frac{1}{\tilde{M}_{aa}(\eta,\bar{\eta})}\right)\frac{\tilde{M}_{ab}(\eta,\bar{\eta})}{\sqrt{\tilde{M}_{bb}(\eta,\bar{\eta})}} \\
&= 0
\end{aligned}
$$

as claimed.

In summary, by adiabatically exchanging the four vortices, one naturally gets an action of the braid group B_4 on the space of the degenerate lowest energy states, which is the space spanned by $|\Psi_1(\eta)\rangle$ and $|\Psi_2(\eta)\rangle$. This gives a 'physical' representation of B_4. Another representation of B_4 is the one given by the conformal blocks, see Eqs. (2.3a,b). [This can be generalized to the case of $2m$ vortices, where one has to compare two representations of the braid group B_{2m}, each of dimension 2^{m-1}.] The miracle is that **the two representations of the braid group are exactly the same!** Thus, the adiabatic statistics of the vortices in the $p_x + ip_y$ superconductor can be read directly in the monodromy of the conformal blocks. In particular, they obey **non-abelian adiabatic statistics**.

Further reading

The construction of trial wavefunctions for the Fractional Quantum Hall effect (FQH) from conformal blocks was pioneered in the classic paper:

- *G. Moore and N. Read, "Nonabelions in the fractional quantum hall effect", Nucl. Phys. B 360, 362-396 (1991).*

A good introduction to the FQH physics that predated Moore-Read is:

- *S. Girvin "The Quantum Hall Effect: Novel Excitations and Broken Symmetries", Lecture Notes from Les Houches (1998); arXiv:cond-mat/9907002.*

For more recent lecture notes on the FQH that include also some post-Moore-Read aspects, see for instance

- *D. Tong "Lectures on the Quantum Hall Effect", arXiv:1606.06687.*

The classic reference on $p_x + ip_y$ superconductors is:

- *N. Read and D. Green "Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect", Phys. Rev. B 61, 10267 (2000); arXiv:cond-mat/9906453.*

For a review on non-abelian statistics and its possible use for topological quantum computation, see

- *C. Nayak, S. Simon, A. Stern, M. Freedman, S. Das Sarma, "Non-Abelian anyons and topological quantum computation", Rev. Mod. Phys. 80, 1083 (2008); arXiv:0707.1889.*

The present notes are more specifically based on the following papers. The $p_x + ip_y$ trial state and the treatment of the bulk-edge correspondence have appeared in

- *J. Dubail and N. Read "Entanglement Spectra of Complex Paired Superfluids", Phys. Rev. Lett. 107, 157001 (2011); arXiv:1105.4808*

- *J. Dubail, E. Rezayi and N. Read "Edge-state inner products and real-space entanglement spectrum of trial quantum Hall states", Phys. Rev. B 86, 245310 (2012); arXiv:1207.7119,*

while the part on adiabatic transport and non-abelian statistics is adapted from

- *D. Arovas, J. Schrieffer and F. Wilczek, "Fractional Statistics and the Quantum Hall Effect", Phys. Rev. Lett. 53, 722 (1984)*

- *C. Nayak and F. Wilczek, "2n-quasihole states realize 2n-1-dimensional spinor braiding statistics in paired quantum Hall states", Nucl. Phys. B 479, 529-553 (1996); arXiv:cond-mat/9605145*

- *N. Read, "Non-Abelian adiabatic statistics and Hall viscosity in quantum Hall states and px+ipy paired superfluids", Phys. Rev. B 79, 045308 (2009); arXiv:0805.2507*

- *P. Bonderson, V. Gurarie and C. Nayak, "Plasma analogy and non-Abelian statistics for Ising-type quantum Hall states", Phys. Rev. B 83, 075303 (2011); arXiv:1008.5194.*