A crash course on two-dimensional conformal field theory

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Abstract

We provide a brief but self-contained introduction to conformal field theory on the Riemann sphere. We first introduce general axioms such as local conformal invariance, and derive Ward identities and BPZ equations. We then define Liouville theory by specific axioms on its spectrum and degenerate fields. We solve the theory and study its four-point functions.

Contents

Acknowledgements

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This text was written with brevity as the main concern, so as to be the basis for about 100 minutes' worth of lectures. For more details, see the "Minimal lectures on 2d CFT" [1] (20 pages) and the full review article [2] (120 pages), both on arXiv.

The first two sections are about the main ideas of 2d CFT, as developed in the 1980s. Then we will focus on the particular case of Liouville theory, a CFT that was mostly solved in the 1990s, with however some subtleties understood only in 2015. Liouville theory was introduced by Polyakov as a model of 2d gravity, and later applied to string theory. More recently, it has been related to statistical systems such as Conformal Loop Ensembles (Ikhlef–Jacobsen–Saleur) or Random Energy Models (Raoul Santachiara's talks in this school).

1 Conformal field theory

1.1 The Virasoro algebra and its representations

By definition, conformal transformations are transformations that preserve angles. In two dimensions with a complex coordinate z, any holomorphic map preserves angles, and infinitesimal conformal transformations have the generators

$$\ell_n = -z^{n+1}\frac{\partial}{\partial z} \qquad (n \in \mathbb{Z}) \ , \tag{1}$$

including $\ell_{-1} = -\frac{\partial}{\partial z}$ for translations. The commutation relations are

$$[\ell_n, \ell_m] = (n-m)\ell_{m+n} \ , \tag{2}$$

and the corresponding Lie algebra is called the Witt algebra. In a quantum theory, symmetry transformations act projectively on states. Projective representations of an algebra are equivalent to representations of a centrally extended algebra.

Definition 1 (Virasoro algebra)
The central extension of the Witt algebra is called the Virasoro algebra. It has the generators $(L_n)_{n\in\mathbb{Z}}$ and $\mathbf{1}$, and the commutation relations

$$[\mathbf{1}, L_n] = 0 \quad , \quad [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n-1)n(n+1)\delta_{n+m,0}\mathbf{1} \ , \tag{3}$$

where the number c is called the central charge.

The spectrum, i.e. the space of states, must be a representation of the Virasoro algebra. We want to interpret L_0 as the energy operator, so its eigenvalues must be bounded from below. But

$$L_0|v\rangle = \Delta|v\rangle \qquad \Rightarrow \qquad L_0 L_n|v\rangle = (\Delta - n)L_n|v\rangle \ . \tag{4}$$

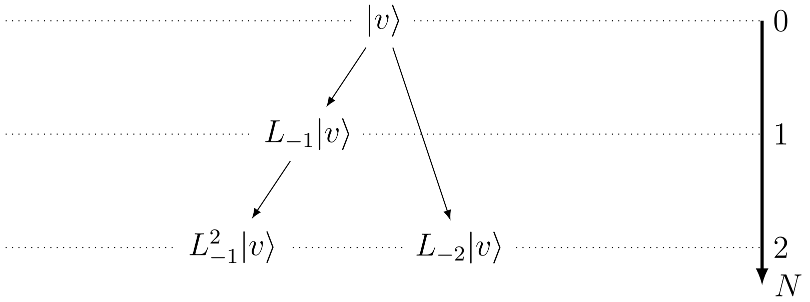
So $L_{n>0}$ decrease conformal dimensions, and must therefore kill the state with the lowest dimension.

Definition 2 (Primary and descendent states, level, Verma module)
A primary state with conformal dimension Δ is a state $|v\rangle$ such that

$$L_0|v\rangle = \Delta|v\rangle \quad , \quad L_{n>0}|v\rangle = 0 \ . \tag{5}$$

The Verma module $\mathcal{V}_\Delta$ is the representation whose basis is $\left\{\prod_{i=1}^k L_{-n_i}|v\rangle\right\}_{0<n_1\leq\cdots\leq n_k}$. The level of the state $\prod_{i=1}^k L_{-n_i}|v\rangle$ is $N = \sum_{i=1}^k n_i \geq 0$. A state of level $N \geq 1$ is called a descendent state.

Let us plot a basis of a Verma module up to the level $N = 2$:



(6)

If a Verma module is not irreducible, i.e. if it has a nontrivial submodule, then the state with the lowest conformal dimension in that submodule must again be a primary state, while also being a descendent state.

Definition 3 (Null vectors)
A descendent state that is also primary is called a null vector or singular vector.

In the Verma module $\mathcal{V}_\Delta$, let us find out whether $L_{-1}|v\rangle$ is a null vector:

$$L_n L_{-1}|v\rangle \underset{n\geq 1}{=} [L_n, L_{-1}]|v\rangle = (n+1)L_{n-1}|v\rangle = \begin{cases} 0 & \text{if } n \geq 2 \\ 2\Delta|v\rangle & \text{if } n = 1 \end{cases} \tag{7}$$

So $L_{-1}|v\rangle$ is a null vector if and only if $\Delta = 0$, and the Verma module $\mathcal{V}_0$ is reducible. Similarly, there exists a level two null vector if and only if

$$\Delta = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right) . \tag{8}$$

In order to simplify this formula, let us introduce other notations for c and Δ. We define

$$\text{the background charge } Q\ , \quad c = 1 + 6Q^2\ , \tag{9}$$

$$\text{the coupling constant } b\ , \quad Q = b + \frac{1}{b}\ , \tag{10}$$

$$\text{the momentum } \alpha\ , \quad \Delta = \alpha(Q - \alpha)\ . \tag{11}$$

The condition (8) for the existence of a level two singular vector becomes

$$\alpha = -\frac{1}{2}b^{\pm 1}\ . \tag{12}$$

To summarize, the null vectors $L|v\rangle$ at levels $N \leq 2$ and the conditions on α for their existence are

N	$\alpha_{\langle r,s\rangle}$	$L_{\langle r,s\rangle}$	$\langle r,s\rangle$
1	0	L_{-1}	$\langle 1,1\rangle$
2	$-\frac{b}{2}$	$L_{-1}^2 + b^2 L_{-2}$	$\langle 2,1\rangle$
	$-\frac{1}{2b}$	$L_{-1}^2 + b^{-2} L_{-2}$	$\langle 1,2\rangle$

(13)

The general pattern is that null vectors at a level N are parametrized by pairs r, s of strictly positive integers such that $rs = N$. For a null vector to exist, the momentum must be of the type

$$\alpha_{\langle r,s\rangle} = \frac{Q}{2} - \frac{1}{2}(rb + sb^{-1}) \qquad \text{with} \qquad N = rs\ . \tag{14}$$

For generic c, the submodule generated by a null vector is the Verma module $\mathcal{V}_{\Delta_{\langle r,s\rangle}+rs} \subset \mathcal{V}_{\Delta_{\langle r,s\rangle}}$. Then the coset $\frac{\mathcal{V}_{\Delta_{\langle r,s\rangle}}}{\mathcal{V}_{\Delta_{\langle r,s\rangle}+rs}}$ is an irreducible representation, where the null vector vanishes.

1.2 Fields and correlation functions

Let us define fields on the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Axiom 4 (State-field correspondence)
For any state $|w\rangle$ in the spectrum, there is an associated field $V_{|w\rangle}(z)$. The map $|w\rangle \mapsto V_{|w\rangle}(z)$ is linear. We define the action of the Virasoro algebra on such fields as

$$L_n V_{|w\rangle}(z) = L_n^{(z)} V_{|w\rangle}(z) = V_{L_n|w\rangle}(z) \ . \tag{15}$$

Definition 5 (Primary field, descendent field, degenerate field)
Let $|v\rangle$ be the primary state of the Verma module $\mathcal{V}_\Delta$. We define the primary field $V_\Delta(z) = V_{|v\rangle}(z)$. This field obeys

$$L_{n\geq 0} V_\Delta(z) = 0 \quad , \quad L_0 V_\Delta(z) = \Delta V_\Delta(z) \ . \tag{16}$$

Similarly, descendent fields correspond to descendent states. And the degenerate field $V_{\langle r,s\rangle}(z)$ corresponds to the primary state of the coset representation $\frac{\mathcal{V}_{\Delta_{\langle r,s\rangle}}}{\mathcal{V}_{\Delta_{\langle r,s\rangle}+rs}}$, and therefore obeys

$$L_{\langle r,s\rangle} V_{\langle r,s\rangle}(z) = 0 \qquad \text{– for example,} \ \ L_{-1} V_{\langle 1,1\rangle}(z) = 0 \ . \tag{17}$$

Axiom 6 (Dependence of fields on z)
For any field $V(z)$, we have

$$\frac{\partial}{\partial z} V(z) = L_{-1} V(z) \ . \tag{18}$$

Using this axiom for both $V(z)$ and $L_n^{(z)} V(z)$, we find

$$\frac{\partial}{\partial z} L_n^{(z)} = [L_{-1}^{(z)}, L_n^{(z)}] = -(n+1) L_{n-1}^{(z)} \ , \qquad (\forall n \in \mathbb{Z}) \ . \tag{19}$$

These infinitely many equations can be encoded into one functional equation,

$$\frac{\partial}{\partial z} \sum_{n\in\mathbb{Z}} \frac{L_n^{(z)}}{(y-z)^{n+2}} = 0 \ . \tag{20}$$

Definition 7 (Energy-momentum tensor)
The energy-momentum tensor is a field, that we define by the formal Laurent series

$$T(y) = \sum_{n\in\mathbb{Z}} \frac{L_n^{(z)}}{(y-z)^{n+2}} \ . \tag{21}$$

In other words, for any field $V(z)$, we have

$$T(y) V(z) = \sum_{n\in\mathbb{Z}} \frac{L_n V(z)}{(y-z)^{n+2}} \ . \tag{22}$$

In particular, for a primary field $V_\Delta(z)$, we find

$$T(y) V_\Delta(z) = \frac{\Delta}{(y-z)^2} V_\Delta(z) + \frac{1}{y-z} \frac{\partial}{\partial z} V_\Delta(z) + O(1) \ . \tag{23}$$

This is our first example of an operator product expansion.

The energy-momentum tensor $T(y)$ is locally holomorphic as a function of y, and acquires poles in the presence of other fields. Since we are on the Riemann sphere, it must also be holomorphic at $y = \infty$.

Axiom 8 (Behaviour of $T(y)$ at infinity)

$$T(y) \underset{y\to\infty}{=} O\left(\frac{1}{y^4}\right) . \tag{24}$$

Definition 9 (Correlation function)
To N fields $V_1(z_1),\dots,V_N(z_N)$, we associate a number called their correlation function or N-point function, and denoted as

$$\Big\langle V_1(z_1)\cdots V_N(z_N)\Big\rangle . \tag{25}$$

Correlation functions depend linearly on fields.

Axiom 10 (Commutativity of fields)
Correlation functions do not depend on the order of the fields,

$$V_1(z_1)V_2(z_2) = V_2(z_2)V_1(z_1) , \qquad (z_1 \neq z_2) . \tag{26}$$

1.3 Ward identities and BPZ equations

Let us work out the consequences of conformal symmetry for correlation functions. In order to study an N-point function Z of primary fields, we introduce an auxiliary $(N+1)$-point function $Z(y)$ where we insert the energy-momentum tensor,

$$Z = \left\langle \prod_{i=1}^N V_{\Delta_i}(z_i) \right\rangle \quad , \quad Z(y) = \left\langle T(y)\prod_{i=1}^N V_{\Delta_i}(z_i) \right\rangle . \tag{27}$$

$Z(y)$ is a meromorphic function of y, with poles at $y = z_i$, whose residues are given by eq. (23) (using the commutativity of fields). This implies

$$Z(y) = \sum_{i=1}^N \left(\frac{\Delta_i}{(y-z_i)^2} + \frac{1}{y-z_i}\frac{\partial}{\partial z_i}\right) Z . \tag{28}$$

Moreover, the behaviour (24) of $T(y)$ implies that the coefficients of y^{-1}, y^{-2}, y^{-3} in the large y expansion of $Z(y)$ must vanish,

$$\sum_{i=1}^N \partial_{z_i} Z = \sum_{i=1}^N \left(z_i\partial_{z_i} + \Delta_i\right) Z = \sum_{i=1}^N \left(z_i^2\partial_{z_i} + 2\Delta_i z_i\right) Z = 0 . \tag{29}$$

These three equations are called global Ward identities. Let us solve them. For a one-point function, we have

$$\partial_z \Big\langle V_\Delta(z)\Big\rangle = 0 , \quad \Delta\Big\langle V_\Delta(z)\Big\rangle = 0 , \quad \Rightarrow \quad \Big\langle V_\Delta(z)\Big\rangle \propto \delta_{\Delta,0} . \tag{30}$$

In the case of two-point functions, we find

$$\Big\langle V_{\Delta_1}(z_1)V_{\Delta_2}(z_2)\Big\rangle \propto \delta_{\Delta_1,\Delta_2}(z_1-z_2)^{-2\Delta_1} . \tag{31}$$

For three-point functions, there are as many equations (29) as unknowns z_1, z_2, z_3, and therefore a unique solution with no constraints on Δ_i,

$$\left\langle \prod_{i=1}^3 V_{\Delta_i}(z_i)\right\rangle \propto (z_1-z_2)^{\Delta_3-\Delta_1-\Delta_2} \times 2 \text{ permutations} . \tag{32}$$

For four-point functions, the general solution is

$$\left\langle \prod_{i=1}^4 V_{\Delta_i}(z_i) \right\rangle = \left(\prod_{i<j} (z_i - z_j)^{\delta_{ij}} \right) G\left(\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} \right) , \tag{33}$$

where $\sum_{j\neq i} \delta_{ij} = -2\Delta_i$, and G is an arbitrary function. Equivalently, the three global Ward identities effectively reduces the four-point function to a function of one variable,

$$G(z) = \left\langle V_{\Delta_1}(z) V_{\Delta_2}(0) V_{\Delta_3}(\infty) V_{\Delta_4}(1) \right\rangle . \tag{34}$$

Correlation functions of descendents can be deduced from correlation functions of primaries. For example,

$$\left\langle L_{-2} V_{\Delta_1}(z_1) V_{\Delta_2}(z_2) \cdots \right\rangle = \frac{1}{2\pi i} \oint_{z_1} \frac{dy}{y-z_1} Z(y) = \sum_{i=2}^N \left(\frac{1}{z_1 - z_i} \frac{\partial}{\partial z_i} + \frac{\Delta_i}{(z_i - z_1)^2} \right) Z , \tag{35}$$

where we used first eq. (22) for $L_{-2}V_{\Delta_1}(z_1)$, and then eq. (28) for $Z(y)$. This can be generalized to any correlation function of descendent fields. The resulting equations are called local Ward identities.

Correlation functions that involve degenerate fields obey additional equations. In particular,

$$\left(L_{-1}^2 + b^2 L_{-2}\right) V_{\langle 2,1\rangle}(z_1) = 0 \qquad \text{so that} \qquad L_{-2} V_{\langle 2,1\rangle}(z_1) = -\frac{1}{b^2} \frac{\partial^2}{\partial z_1^2} V_{\langle 2,1\rangle}(z_1) . \tag{36}$$

Together with the local Ward identity (35), this leads to the second-order Belavin–Polyakov–Zamolodchikov partial differential equation

$$\left(\frac{1}{b^2} \frac{\partial^2}{\partial z_1^2} + \sum_{i=2}^N \left(\frac{1}{z_1 - z_i} \frac{\partial}{\partial z_i} + \frac{\Delta_i}{(z_1 - z_i)^2} \right) \right) \left\langle V_{\langle 2,1\rangle}(z_1) \prod_{i=2}^N V_{\Delta_i}(z_i) \right\rangle = 0 . \tag{37}$$

In the case of a four-point function $N = 4$, the BPZ equation amounts to a differential equation for a function of one variable.

2 Conformal bootstrap

Conformal symmetry leads to linear equations for correlation functions: Ward identities and BPZ equations. In order to fully determine correlation functions, we need additional, nonlinear equations, and therefore additional axioms: single-valuedness of correlation functions, and existence of operator product expansions. Using these axioms for studying conformal field theories is called the conformal bootstrap method.

2.1 Single-valuedness

Axiom 11 (Single-valuedness)
Correlation functions are single-valued functions of the positions, i.e. they have trivial monodromies.

Our two-point function (31) however has nontrivial monodromy unless $\Delta_1 \in \frac{1}{2}\mathbb{Z}$, as a result of solving holomorphic Ward identities. We would rather have a single-valued function of the type $|z_1 - z_2|^{-4\Delta_1} = (z_1 - z_2)^{-2\Delta_1}(\bar{z}_1 - \bar{z}_2)^{-2\Delta_1}$. This suggests that we need anti-holomorphic Ward identities as well, and therefore a second copy of the Virasoro algebra.

Axiom 12 (Left and right Virasoro algebras)
We have two mutually commuting Virasoro symmetry algebras with the same central charge, called left-moving or holomorphic, and right-moving or anti-holomorphic. Their generators are written $L_n, \bar{L}_n$, with in particular

$$\frac{\partial}{\partial z} V(z) = L_{-1} V(z) \quad , \quad \frac{\partial}{\partial \bar{z}} V(z) = \bar{L}_{-1} V(z) \ . \tag{38}$$

Let us consider left- and right-primary fields $V_{\Delta_i, \bar{\Delta}_i}(z_i)$, with the two-point functions

$$\left\langle \prod_{i=1}^{2} V_{\Delta_i, \bar{\Delta}_i}(z_i) \right\rangle \propto \delta_{\Delta_1, \Delta_2} \delta_{\bar{\Delta}_1, \bar{\Delta}_2} (z_1 - z_2)^{-2\Delta_1} (\bar{z}_1 - \bar{z}_2)^{-2\bar{\Delta}_1} \ . \tag{39}$$

This is single-valued if and only if our two fields have half-integer spins,

$$\Delta - \bar{\Delta} \in \frac{1}{2}\mathbb{Z} \ . \tag{40}$$

The simplest case is $\Delta = \bar{\Delta}$, which leads to the definition

Definition 13 (Diagonal states, diagonal fields and diagonal spectrums)
A primary state or field is called diagonal if it has the same left and right conformal dimensions. A spectrum is called diagonal if all primary states are diagonal.

From now on we assume that all fields are diagonal, and write $V_\Delta(z) = V_{\Delta,\Delta}(z)$.

2.2 Operator product expansion and crossing symmetry

Axiom 14 (Operator product expansion)
Let $V_1(z_1)$ and $V_2(z_2)$ be two fields, and $|w_i\rangle$ be a basis of the spectrum. There exist coefficients $C^i_{12}(z_1, z_2)$ such that we have the operator product expansion (OPE)

$$V_1(z_1) V_2(z_2) = \sum_i C^i_{12}(z_1, z_2) V_{|w_i\rangle}(z_2) \ . \tag{41}$$

If the spectrum is made of diagonal primary states and their descendent states, the OPE of two primary fields must be of the type

$$V_{\Delta_1}(z_1) V_{\Delta_2}(z_2) = \sum_{\Delta \in S} C_{\Delta_1, \Delta_2, \Delta} |z_1 - z_2|^{2(\Delta - \Delta_1 - \Delta_2)} \Big(V_\Delta(z_2) + O(z_1 - z_2) \Big) \ , \tag{42}$$

where $C_{\Delta_1, \Delta_2, \Delta}$ is the z_i-independent OPE coefficient. $O(z_1 - z_2)$ denotes the contributions of descendents fields, which are determined by conformal Ward identities. Using the OPE, we can reduce a three-point function to a combination of two-point functions, and we find

$$\left\langle \prod_{i=1}^{3} V_{\Delta_i}(z_i) \right\rangle = C_{\Delta_1, \Delta_2, \Delta_3} |z_1 - z_2|^{2(\Delta_3 - \Delta_1 - \Delta_2)} |z_1 - z_3|^{2(\Delta_2 - \Delta_1 - \Delta_3)} |z_2 - z_3|^{2(\Delta_1 - \Delta_2 - \Delta_3)} \ , \tag{43}$$

so that the OPE coefficient C coincides with the three-point structure constant.

Inserting the OPE in a four-point function of primary fields, we find

$$\left\langle V_{\Delta_1}(z)V_{\Delta_2}(0)V_{\Delta_3}(\infty)V_{\Delta_4}(1)\right\rangle = \sum_{\Delta\in S} C_{\Delta_1,\Delta_2,\Delta}C_{\Delta,\Delta_3,\Delta_4}\mathcal{F}^{(s)}_{\Delta}(z)\mathcal{F}^{(s)}_{\Delta}(\bar{z}) \ . \tag{44}$$

Definition 15 (Conformal block)
The four-point conformal block on the sphere,

$$\mathcal{F}^{(s)}_{\Delta}(z) \underset{z\to 0}{=} z^{\Delta-\Delta_1-\Delta_2}\Big(1+O(z)\Big) \ , \tag{45}$$

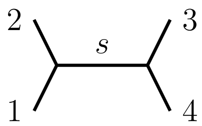
is the normalized contribution of the Verma module $\mathcal{V}_\Delta$ to a four-point function, obtained by summing over left-moving descendents. It is a locally holomorphic function of z. Its dependence on $c,\Delta_1,\Delta_2,\Delta_3,\Delta_4$ are kept implicit. The label (s) stands for for s-channel.

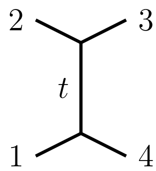
Conformal blocks are in principle known, as they are universal functions, entirely determined by conformal symmetry. This is analogous to characters of representations, also known as zero-point conformal blocks on the torus.

Our axiom 10 on the commutativity of fields implies that the OPE is associative, and that we can use the OPE of any two fields in a four-point function. In particular, using the OPE of the first and fourth fields, we obtain

$$\left\langle V_{\Delta_1}(z)V_{\Delta_2}(0)V_{\Delta_3}(\infty)V_{\Delta_4}(1)\right\rangle = \sum_{\Delta\in S} C_{\Delta,\Delta_1,\Delta_4}C_{\Delta_2,\Delta_3,\Delta}\mathcal{F}^{(t)}_{\Delta}(z)\mathcal{F}^{(t)}_{\Delta}(\bar{z}) \ , \tag{46}$$

where $\mathcal{F}^{(t)}_{\Delta}(z) \underset{z\to 1}{=} (z-1)^{\Delta-\Delta_1-\Delta_4}\Big(1+O(z-1)\Big)$ is a t-channel conformal block. The equality of our two decompositions (44) and (46) of the four-point function is called crossing symmetry, schematically

$\sum_s C_{12s}C_{s34}$ 

$= \sum_t C_{23t}C_{t41}$ 

. (47)

Given the spectrum S, crossing symmetry is a system of quadratic equations for the structure constant $C_{\Delta_1,\Delta_2,\Delta_3}$. Requiring that this system has solutions is a strong constraint on the spectrum.

Definition 16 (Conformal field theory)
A (model of) conformal field theory on the Riemann sphere is a spectrum S and a set of correlation functions $\left\langle\prod_{i=1}^N V_{|w_i\rangle}(z_i)\right\rangle$ with $|w_i\rangle\in S$ that obey all our axioms, in particular crossing symmetry.

Definition 17 (Defining and solving)
To define a conformal field theory is to give principles that uniquely determine its spectrum and correlation functions. To solve a conformal field theory is to actually compute them.

2.3 Hypergeometric four-point functions

Crossing symmetry equations typically involve infinite sums, which makes them difficult to solve. However, if at least one field is degenerate, then the four-point function belongs to the finite-dimensional space of solutions of a BPZ equation. For example, $G(z) = \left\langle V_{\langle 2,1\rangle}(z)V_{\Delta_1}(0)V_{\Delta_2}(\infty)V_{\Delta_3}(1)\right\rangle$ is a combination of only two s-channel conformal blocks. These blocks are a particular basis of solutions of the BPZ equation, characterized by their asymptotic behaviour near $z=0$ (45),

$$\mathcal{F}^{(s)}_{\alpha_1-\frac{b}{2}}(z) = z^{b\alpha_1}(1-z)^{b\alpha_3}F(A,B,C,z) \quad , \quad \mathcal{F}^{(s)}_{\alpha_1+\frac{b}{2}}(z) = \mathcal{F}^{(s)}_{\alpha_1-\frac{b}{2}}(z)\Big|_{\alpha_1\to Q-\alpha_1} \quad , \tag{48}$$

where $F(A,B,C,z)$ is the hypergeometric function with parameters

$$\begin{cases} A = \frac{1}{2} + b(\alpha_1+\alpha_3-Q) + b(\alpha_2-\frac{Q}{2}) \ , \\ B = \frac{1}{2} + b(\alpha_1+\alpha_3-Q) - b(\alpha_2-\frac{Q}{2}) \ , \\ C = 1 + b(2\alpha_1-Q) \ . \end{cases} \tag{49}$$

Let us build single-valued four-point functions as linear combinations of such blocks. Single-valuedness near $z=0$ forbids terms such as $\mathcal{F}^{(s)}_{\alpha_1-\frac{b}{2}}(z)\mathcal{F}^{(s)}_{\alpha_1+\frac{b}{2}}(\bar{z})$, and we must have

$$G(z) = \sum_{i=\pm} c_i^{(s)}\mathcal{F}^{(s)}_{\alpha_1-i\frac{b}{2}}(z)\mathcal{F}^{(s)}_{\alpha_1-i\frac{b}{2}}(\bar{z}) = \sum_{j=\pm} c_i^{(t)}\mathcal{F}^{(t)}_{\alpha_3-j\frac{b}{2}}(z)\mathcal{F}^{(t)}_{\alpha_3-j\frac{b}{2}}(\bar{z}) \ . \tag{50}$$

The s- and t-channel blocks are two bases of the same space of solutions of the BPZ equation, and they are linearly related,

$$\mathcal{F}^{(s)}_{\alpha_1-i\frac{b}{2}}(z) = \sum_{j=\pm} F_{ij}\mathcal{F}^{(t)}_{\alpha_3-j\frac{b}{2}}(z) \ . \tag{51}$$

In particular, this implies

$$\frac{c_+^{(s)}}{c_-^{(s)}} = -\frac{F_{-+}F_{--}}{F_{++}F_{+-}} = \frac{\gamma(A)\gamma(B)\gamma(C-A)\gamma(C-B)}{\gamma(C)\gamma(C-1)} \quad \text{with} \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \ . \tag{52}$$

We will later express $c_\pm^{(s)}$ in terms of three-point structure constants, and obtain equations for these structure constants.

3 Liouville theory

3.1 Definition

Definition 18 (Liouville theory)
For any value of the central charge $c \in \mathbb{C}$, Liouville theory is the conformal field theory whose space of states is

$$\int_{\frac{Q}{2}+i\mathbb{R}_+} d\alpha \ \mathcal{V}_\alpha \otimes \bar{\mathcal{V}}_\alpha \ , \tag{53}$$

and whose correlation functions are analytic functions of b and α, assuming it exists and its unique.

Why these values of α? If $c \in \mathbb{R}$ we want $\Delta \in \mathbb{R}$ i.e. $\alpha \in (\frac{Q}{2}+i\mathbb{R}) \cup \mathbb{R}$. But we also want Δ to be bounded from below, and the natural lower bound is $\Delta(\alpha = \frac{Q}{2}) = \frac{c-1}{24}$. Anyway, the most important part of the definition is the assumption of analyticity in α.

The integral is actually over the half-line $\frac{Q}{2}+i\mathbb{R}_+$ so as not to count representations twice. We indeed have $\Delta(\alpha)=\Delta(Q-\alpha)$ so $\mathcal{V}_\alpha=\mathcal{V}_{Q-\alpha}$. The fields $V_\alpha(z)$ and $V_{Q-\alpha}(z)$ then correspond to the same primary state, and we must have a reflection relation,

$$V_\alpha(z)=R(\alpha)V_{Q-\alpha}(z)\ , \tag{54}$$

where the function $R(\alpha)$ is called the reflection coefficient. Let us schematically write three-point functions and OPEs in Liouville theory,

$$\left\langle V_{\alpha_1}V_{\alpha_2}V_{\alpha_3}\right\rangle = C_{\alpha_1,\alpha_2,\alpha_3}\ , \tag{55}$$

$$V_{\alpha_1}V_{\alpha_2}=\int_{\frac{Q}{2}+i\mathbb{R}_+} d\alpha\ C_{\alpha_1,\alpha_2,Q-\alpha}\Big(V_\alpha+\cdots\Big)\ . \tag{56}$$

In order to have reasonably simple crossing symmetry equations, we need degenerate fields. But the spectrum is made of Verma modules, and the corresponding fields are not degenerate. So we need a special axiom.

Axiom 19 (Degenerate fields in Liouville theory)
The degenerate fields $V_{\langle 2,1\rangle}$ and $V_{\langle 1,2\rangle}$, and their correlation functions, exist.

Let us introduce notations for OPE coefficients of $V_{\langle 2,1\rangle}$:

$$V_{\langle 2,1\rangle}V_\alpha \sim C_+(\alpha)V_{\alpha-\frac{b}{2}}+C_-(\alpha)V_{\alpha+\frac{b}{2}}\ . \tag{57}$$

It can be shown that there exists a field renormalization $V_\alpha(z)\to\lambda(\alpha)V_\alpha(z)$ such that $C_+(\alpha)=1$. Then, using the reflection relation, we can deduce

$$C_+(\alpha)=1\quad ,\quad C_-(\alpha)=\frac{R(\alpha)}{R(\alpha+\frac{b}{2})}\ . \tag{58}$$

3.2 Three-point structure constants

In the case of the four-point function $\left\langle V_{\langle 2,1\rangle}(z)V_\alpha(0)V_{Q-\alpha}(\infty)V_{\langle 2,1\rangle}(1)\right\rangle$, we have

$$c_+^{(s)}=C_+(\alpha)C_-(\alpha-\tfrac{b}{2})=\frac{R(\alpha-\frac{b}{2})}{R(\alpha)}\quad ,\quad c_-^{(s)}=C_-(\alpha)C_+(\alpha+\tfrac{b}{2})=\frac{R(\alpha)}{R(\alpha+\frac{b}{2})}\ , \tag{59}$$

and eq. (52) boils down to

$$\frac{R(\alpha-\frac{b}{2})R(\alpha+\frac{b}{2})}{R(\alpha)^2}=\frac{\gamma(2b\alpha)\gamma(b(Q-2\alpha))}{\gamma(b(2Q-2\alpha))\gamma(b(2\alpha-Q))}\ . \tag{60}$$

We could get the same equation with $b\to\frac{1}{b}$ by having $V_{\langle 1,2\rangle}$ instead of $V_{\langle 2,1\rangle}$ in the four-point function. The analytic solution of these equations is

$$R(\alpha)=-b^2\frac{\gamma(b(2\alpha-Q))}{\gamma(\frac{1}{b}(Q-2\alpha))}\ . \tag{61}$$

This also determines $C_\pm(\alpha)$.

In a more general hypergeometric four-point function the coefficients $c_\pm^{(s)}$ are

$$c_\pm^{(s)}=C_\pm(\alpha_1)C_{\alpha_1\mp\frac{b}{2},\alpha_2,\alpha_3}\ . \tag{62}$$

Using eq. (52), we obtain the shift equation

$$\frac{C_{\alpha_1+b,\alpha_2,\alpha_3}}{C_{\alpha_1,\alpha_2,\alpha_3}} = b^{-4}\frac{\gamma(2b\alpha_1)\gamma(2b\alpha_1+b^2)}{\prod_{\pm,\pm}\gamma\left(b\alpha_1 \pm b(\alpha_2-\frac{Q}{2}) \pm b(\alpha_3-\frac{Q}{2})\right)} \ . \tag{63}$$

Again, the equation with $b \to \frac{1}{b}$ should hold too. In order to solve these equations, we need a function that produces Gamma functions when its argument is shifted by b or $\frac{1}{b}$.

Definition 20 (Upsilon function)
For $b > 0$, let $\Upsilon_b(x)$ be the unique (up to a constant factor) holomorphic function that obeys the shift equations

$$\frac{\Upsilon_b(x+b)}{\Upsilon_b(x)} = b^{1-2bx}\gamma(bx) \qquad \textit{and} \qquad \frac{\Upsilon_b(x+\frac{1}{b})}{\Upsilon_b(x)} = b^{\frac{2x}{b}-1}\gamma(\tfrac{x}{b}) \ . \tag{64}$$

For $ib > 0$, the meromorphic function

$$\hat{\Upsilon}_b(x) = \frac{1}{\Upsilon_{ib}(-ix+ib)} \ , \tag{65}$$

then obeys the same shift equations.

The functions $\Upsilon_b(x)$ and $\hat{\Upsilon}_b(x)$ can respectively be defined for $\Re b > 0$ and $\Re ib > 0$ by analytic continuation. And we have

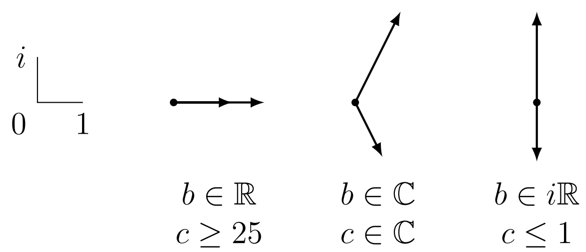
$$\Upsilon_b(x) = \prod_{m,n=0}^{\infty} f\left(\frac{\frac{Q}{2}-x}{\frac{Q}{2}+mb+nb^{-1}}\right) \quad \text{with} \quad f(x) = (1-x^2)e^{x^2} \ . \tag{66}$$

Using the functions $\Upsilon_b(x)$, we can write a solution C of the shift equations (63) for three-point structure constants,

$$C_{\alpha_1,\alpha_2,\alpha_3} = \frac{\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)} \ . \tag{67}$$

This solution is called the DOZZ formula for Dorn, Otto, A. Zamolodchikov and Al. Zamolodchikov (1995). It holds if $c \notin]-\infty, 1]$ i.e. $b \notin i\mathbb{R}$. On the other hand, doing the replacement $\Upsilon_b \to \hat{\Upsilon}_b$, we obtain a solution $\hat{C}$ that holds if $c \notin [25, \infty[$ i.e. $b \notin \mathbb{R}$ (2000s).

Using the analyticity of correlation functions, the solution is unique if b and b^{-1} are aligned, i.e. if $b^2 \in \mathbb{R}$:



(68)

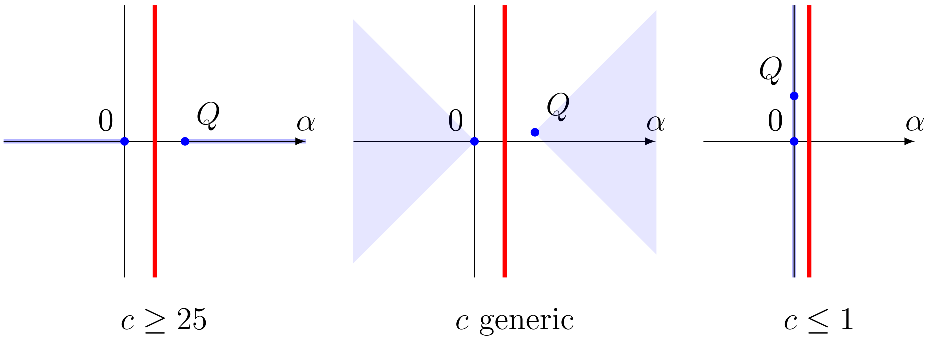
However, for general values of c, both C and $\hat{C}$ are solutions. In order to prove the existence and uniqueness of Liouville theory, we have to determine which solutions lead to crossing-symmetric four-point functions.

3.3 Four-point functions

The s-channel decomposition of a Liouville four-point function reads

$$\left\langle V_{\alpha_1}(z)V_{\alpha_2}(0)V_{\alpha_3}(\infty)V_{\alpha_4}(1)\right\rangle = \frac{1}{2}\int_{\frac{Q}{2}+i\mathbb{R}} d\alpha\ C_{\alpha_1,\alpha_2,Q-\alpha}C_{\alpha,\alpha_3,\alpha_4}\mathcal{F}^{(s)}_{\alpha}(z)\mathcal{F}^{(s)}_{\alpha}(\bar{z})\ , \qquad (69)$$

where the structure can be C or $\hat{C}$. Let us accept that the conformal blocks $\mathcal{F}^{(s)}_{\alpha}(z)$ have poles when $\alpha = \alpha_{\langle r,s\rangle}$ (14), the momentums for which the s-channel representation becomes reducible. We now plot the positions of these poles (blue regions) relative to the integration line (red), depending on the central charge:



(70)

The four-point function built from C is analytic on $c \notin]-\infty, 1]$. So if Liouville theory exists for $c \geq 25$, then it also exists for c generic, with the same structure constant C. On the other hand, the limit $c \to]-\infty, 1]$ is singular. Actually, for $c \leq 1$, the integration line has to be slightly shifted in order to avoid the poles. So the structure constant $\hat{C}$ is expected to be valid only for $c \leq 1$.

That is how far we can easily get with analytic considerations. Numerical tests of crossing symmetry confirm that Liouville theory exists for all values of c, with the three-point structure constants $\hat{C}$ if $c \leq 1$, and C otherwise [3] (2015).

References

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