# The phase diagram of the (extended) Agassi Model

José-Enrique García-Ramos

Departamento de Ciencias Integradas, Universidad de Huelva, Huelva, Spain
enrique.ramos@dfaie.uhu.es

*In collaboration with J.M. Arias, J. Dukelsky, and P. Pérez-Fernández*

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# Why the Agassi model?

- It is a solvable many-body model that allows to mimic the main characteristics of the pairing-plus-quadrupole model.
- It can be exactly solved even in the case of large systems.
- Nowadays, it is used to benchmark many-body approximations because of its great flexibility and simplicity to be solved for large systems.
- The model (and in particular its extension) owns a very rich phase diagram.
- The model is, somehow, an extension of the two-level Lipkin-Meshkov-Glick model that incorporates pairing interaction.

# Agassi model

## Dan Agassi

"Validity of the BCS and RPA approximations in the pairing-plus-monopole solvable model", Dan Agassi, Nuclear Physics A **116**, 49 (1968).

## The original Hamiltonian

$$H = \frac{1}{2}\epsilon \sum_{m\sigma} \sigma a^\dagger_{m\sigma} a_{m\sigma} + \frac{1}{2} V \sum_{mm'\sigma} a^\dagger_{m\sigma} a^\dagger_{m'\sigma} a_{m'-\sigma} a_{m-\sigma}$$
$$-\frac{1}{4} G \sum_{mm'\sigma\sigma'} a^\dagger_{m\sigma} a^\dagger_{-m\sigma} a_{-m'-\sigma'} a_{m'\sigma'}$$

$\sigma = +1, -1$ and $m = -j, ..., -2, -1, 1, 2, ..., j$. Degeneracy $\Omega = 2j$

# Agassi model

## The O(5) as spectrum generator algebra

$$J^+ = \sum_{m=-j}^{j} c^\dagger_{1m} c_{-1m} = \left(J^-\right)^\dagger; \quad J^0 = \frac{1}{2} \sum_{m=-j}^{j} \left(c^\dagger_{1m} c_{1m} - c^\dagger_{-1m} c_{-1m}\right)$$

$$A^\dagger_1 = \sum_{m=1}^{j} c^\dagger_{1m} c^\dagger_{1,-m}; \quad A^\dagger_{-1} = \sum_{m=1}^{j} c^\dagger_{-1m} c^\dagger_{-1,-m}; \quad A^\dagger_0 = \sum_{m=1}^{j} \left(c^\dagger_{-1m} c^\dagger_{1,-m} - c^\dagger_{-1-m} c^\dagger_{1,m}\right)$$

$$N_\sigma = \sum_{m=-j}^{j} c^\dagger_{\sigma m} c_{\sigma m}, \qquad N = N_1 + N_{-1}$$

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## The Hamiltonian

$$H = \varepsilon J^0 - g\sum_{\sigma\sigma'} A^\dagger_\sigma A_{\sigma'} - \frac{V}{2}\left[\left(J^+\right)^2 + \left(J^-\right)^2\right] - 2hA^\dagger_0 A_0$$

For convenience

$$V = \frac{\varepsilon\chi}{2j-1}, \quad g = \frac{\varepsilon\Sigma}{2j-1}, \quad h = \frac{\varepsilon\Lambda}{2j-1}$$

# The energy surfaces

$\varphi$ Hartree-Fock variational parameter. $\beta$ Bogoliubov variational parameter. **Half filling**

## The first energy surface

$$E_1 = -\varepsilon j \cos\varphi \cos\beta - g j^2 \sin^2\beta - V j^2 \sin^2\varphi \cos^2\beta$$

$$\frac{E_1}{j\varepsilon} = -\cos\varphi\cos\beta - \frac{\Sigma}{2}\sin^2\beta - \frac{\chi}{2}\sin^2\varphi\cos^2\beta$$

## The second energy surface

$$E_2 = -\varepsilon j \cos\varphi \cos\beta - 2h j^2 \sin^2\beta \sin^2\varphi - V j^2 \sin^2\varphi \cos^2\beta$$

$$\frac{E_2}{j\varepsilon} = -\cos\varphi\cos\beta - \Lambda\sin^2\beta\sin^2\varphi - \frac{\chi}{2}\sin^2\varphi\cos^2\beta$$

# The phases of the system

- The spherical phase: $\varphi = 0$ and $\beta = 0$.
- The Hartree-Fock deformed phase: $\varphi \neq 0$ and $\beta = 0$.
- The BCS deformed phase: $\varphi = 0$ and $\beta \neq 0$.
- The Hartree-Fock plus BCS deformed phase: $\varphi \neq 0$ and $\beta \neq 0$.

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In the original formulation of the Agassi model only the three first phases were present, but in the extended version of the model the four basis can be found and, moreover, there is coexistence of some of the phases.

# Critical points of the first energy surface

I: $\varphi = \beta = 0$ $(E_1/(j\varepsilon) = -1)$. Regardeless the values of $\Sigma$ and $\chi$.

- $\chi < 1$ and $\Sigma < 1$: it's a minimum
- $\chi > 1$ and $\Sigma > 1$: it's a Maximum
- $\chi > 1$ and $\Sigma < 1$ or $\chi < 1$ and $\Sigma > 1$: it's a saddle point.

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III: $\beta = 0, \cos\varphi = \frac{1}{\chi}$ $(E_1/(j\varepsilon) = -\frac{\chi^2+1}{2\chi})$. Valid for $\chi > 1$

- $\chi > \Sigma$: it's a minimum.
- $\chi < \Sigma$: it's a saddle point.

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- $\chi > \Sigma$: it's a saddle point.

**V:Particular case $\chi = \Sigma$. $\cos\beta\cos\varphi = \frac{1}{\chi}$**

This solution corresponds to a kind of closed valley in the $\varphi - \beta$ plane.

# Critical points of the second energy surface

I: $\varphi = \beta = 0$ ($E_1/(j\varepsilon) = -1$). Regardeless the values of $\Lambda$ and $\chi$.

- $\chi < 1$: it's a minimum
- $\chi > 1$: it's a saddle point.

# Critical points of the second energy surface

**I: $\varphi = \beta = 0$ ($E_1/(j\varepsilon) = -1$). Regardeless the values of $\Lambda$ and $\chi$.**

- $\chi < 1$: it's a minimum
- $\chi > 1$: it's a saddle point.

**II: $\beta = 0, \cos\varphi = \frac{1}{\chi}$ ($E_1/(j\varepsilon) = -\frac{\chi^2+1}{2\chi}$). Valid for $\chi > 1$**

- $\Lambda > \frac{1}{2}\frac{\chi^3}{\chi^2-1}$: it's a saddle point.
- $\Lambda < \frac{1}{2}\frac{\chi^3}{\chi^2-1}$: it's a minimum.

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- $\chi < 1$: it's a minimum
- $\chi > 1$: it's a saddle point.

**II: $\beta = 0, \cos\varphi = \frac{1}{\chi}$ $(E_1/(j\varepsilon) = -\frac{\chi^2+1}{2\chi})$. Valid for $\chi > 1$**

- $\Lambda > \frac{1}{2}\frac{\chi^3}{\chi^2-1}$: it's a saddle point.
- $\Lambda < \frac{1}{2}\frac{\chi^3}{\chi^2-1}$: it's a minimum.

**III: $|\varphi| = |\beta| = \frac{\pi}{2}$ $(E_1 = -\Lambda)$**

- $\Lambda > \frac{1}{4}(\chi + \sqrt{4+\chi^2})$: it's a minimum.
- $\frac{1}{4}(\chi + \sqrt{4+\chi^2}) > \Lambda > \frac{1}{4}(\chi - \sqrt{4+\chi^2})$: it's a saddle point.
- $\Lambda < \frac{1}{4}(\chi - \sqrt{4+\chi^2}) < 0$: it's a Maximum.

# The phase diagram

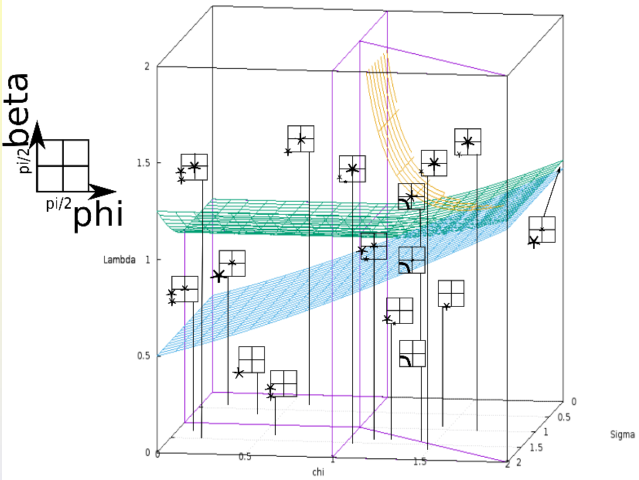


Figure: Phase transition for the extended Agassi model

# Numerical calculations

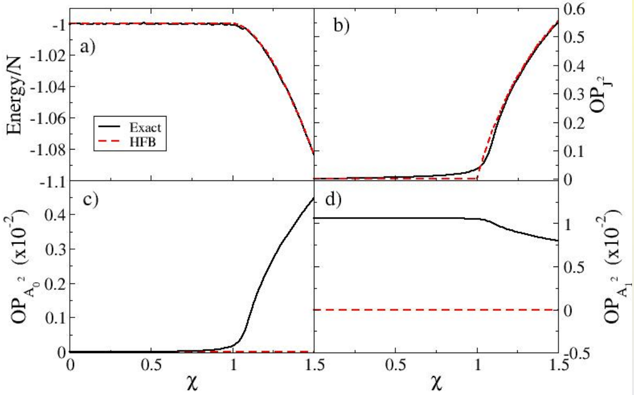


Figure: Comparison of exact and HFB calculations for ground state energy and order parameters with a Hamiltonian with parameters $\Sigma = 1/2$ and $\Lambda = 0$, as a function of $\chi$ ($2j = 100$).

# Numerical calculations

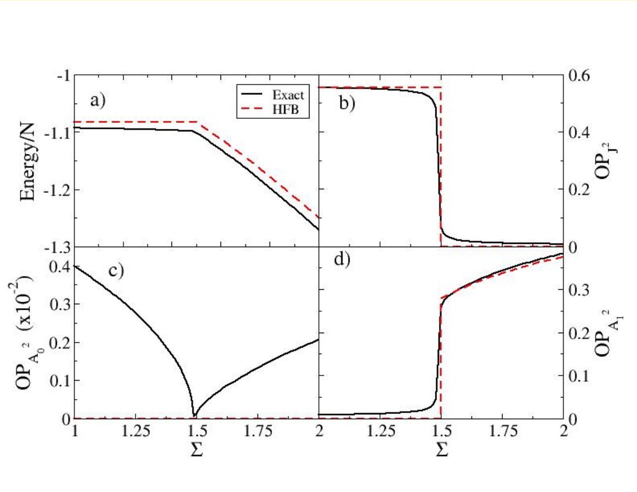


Figure: Comparison of exact and HFB calculations for ground state energy and order parameters with a Hamiltonian with parameters $\chi = 3/2$ and $\Lambda = 1/2$, as a function of $\Sigma$ ($2j = 100$).

# Numerical calculations

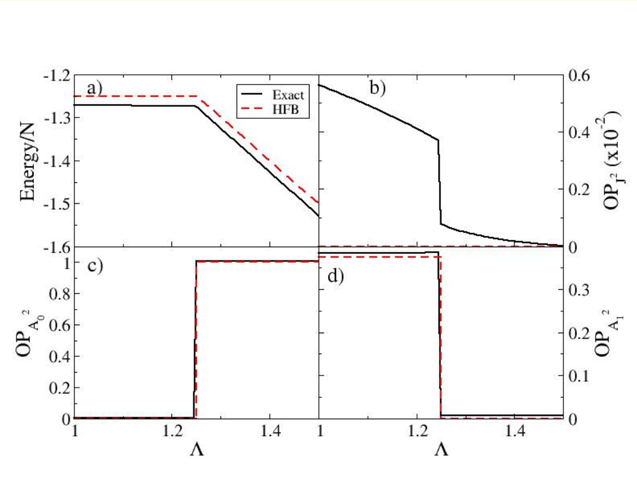


Figure: Comparison of exact and HFB calculations for ground state energy and order parameters with a Hamiltonian with parameters $\chi = 3/2$ and $\Sigma = 2$, as a function of $\Lambda$ ($2j = 100$).

# La Rábida Summer School (2018)



La Rábida (Spain)

18th-22th June 2018

http://institucional.us.es/rabida

Organized by U. of Huelva
and U. of Seville

# La Rábida Summer School (2018)

Rábida 18

International Scientific Meeting on Nuclear Physics

- Alexandre Obertelli. CEA Saclay Service de Physique nucleaire (France).Topic: "Nuclear Reaction Experiments"
- Prof. Tommi Eronen. University of Jyväskylä (Finland). Topic: "Experimental techniques for mass measurements"
- Prof. Alex Brown. Michigan State University (USA). Topic: "Shell model"
- Prof. Pierre Capel. Université Libre de Bruxelles (Belgium). Topic: "Nuclear reaction theory"
- Prof. José Gómez Cadenas. Instituto de Física Corpuscular, IFIC-CSIC (Spain).Topic: "Neutrino physics and NEXT experiment"
- Prof. Katia Parodi. LMU Munich Physics (Germany). Topic: "Medical image processing, treatment planning, PET applications"

# Thank you for your attention

# A pictorial view

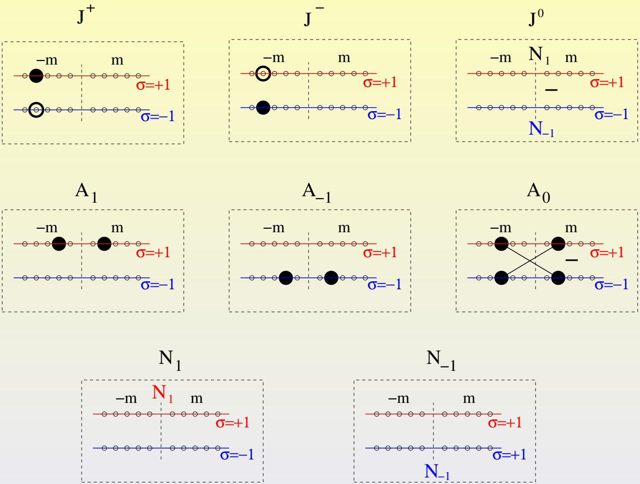


Figure: A pictorial view of the O(5) generators in the Agassi model Hilbert space

# Hartree-Fock-Bogoliubov transformation

## The transformation

Hartree-Fock transformation

$$a^\dagger_{\eta,m} = \sum_\sigma D_{\eta\sigma} c^\dagger_{\sigma,m}$$

Bogoliubov transformation

$$\alpha^\dagger_{\eta,m} = u_\eta a^\dagger_{\eta,m} - sig(m)\, v_\eta a_{\eta,-m}$$

with the constraint (at half filling)

$$u^2_{-1} = v^2_1,\ u^2_1 = v^2_{-1},\ v^2_\eta + u^2_\eta = 1$$

(E.D. Davis and W.D. Weiss, J. Phys. G **12**, 805 (1986).)

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## A convenient parametrization

$$D_{1,1} = D_{-1,-1} = \cos\frac{\varphi}{2}\ ,\ D_{-1,1} = -D_{1,-1} = \sin\frac{\varphi}{2}$$

$$v_1 = \sin\frac{\beta}{2},\ v_{-1} = \cos\frac{\beta}{2}$$

# Numerical calculations

## Order parameters in the laboratory frame

$$OP_{J^2} = \frac{< J_+^2 > + < J_-^2 >}{2j^2} \rightarrow \sin^2\varphi\cos^2\beta \; (\sin^2\varphi\cos^2\beta)$$

$$OP_{A_0^2} = \frac{< A_0^+ A_0 >}{j^2} \rightarrow 0 \; (\sin^2\beta\sin^2\varphi)$$

$$OP_{A_1^2} = \frac{< A_1^+ A_1 > + < A_{-1}^+ A_{-1} >}{2j^2} \rightarrow \frac{1}{2}\sin^2\beta \; (0)$$