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Shapes describing the fusion, binary and ternary fission, ! and cluster radioactivities, fragmentation and alpha molecules

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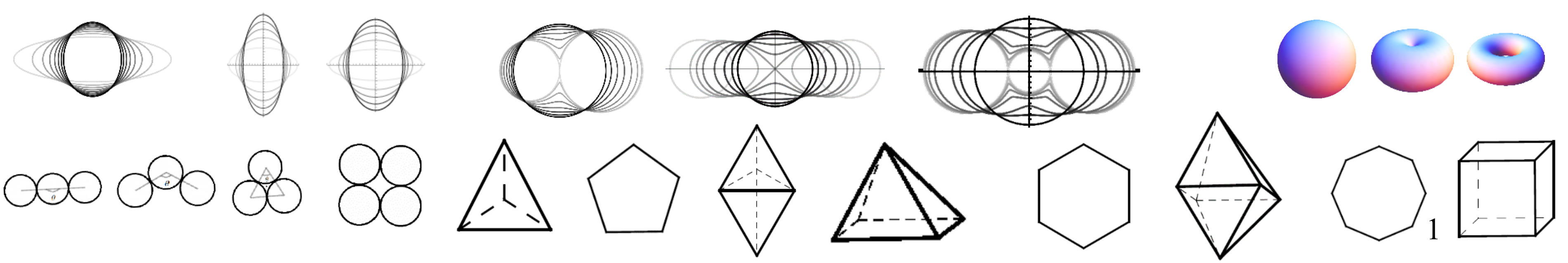
- Definitions
- Ellipsoids
- Elliptic and hyperbolic lemniscatoids
- Prolate ternary shapes
- Pumpkin-like shapes and tori
- Bubbles
- n-alphas : ^{8}Be, ^{12}C, ^{16}O, ^{20}Ne, ^{24}Mg, ^{32}S



General definitions

Relative (to the sphere of radius R_0) shape-dependent surface B_s, curvature B_k and Coulomb (or gravitational) B_c functions :

$$B_s = \int_\sigma \frac{d\sigma}{4\pi {R_0}^2} \qquad B_k = \int_\sigma k_l \frac{d\sigma}{8\pi R_0} \qquad B_C = \frac{15}{16\pi^2 {R_0}^5} \int d\tau \int \frac{d\tau'}{|\vec{r} - \vec{r}'|}$$

(B_s, B_k, and B_c = 1 for the sphere)

For axially symmetric shapes :

$$B_C = \frac{1}{2} \int \frac{v(\theta_i)}{v_0} \left[\frac{R(\theta_i)}{R_0}\right]^3 \sin(\theta_i) d\theta_i$$

$$\frac{v(\theta_i)}{v_0} = \frac{3}{2\pi {R_0}^2} \int \frac{\rho \left[(\rho_i + \rho)\frac{dz}{d\theta} + (z_i - z)\frac{d\rho}{d\theta}\right] K(k) - \frac{1}{2}[(\rho_i + \rho)^2 + (z_i - z)^2]\frac{dz}{d\theta} D(k)}{\sqrt{(\rho_i + \rho)^2 + (z_i - z)^2}} d\theta$$

(v(!) is the potential at the surface of the shape)

Relative (dimensionless) quadrupole moment :

$$Q = \frac{1}{{R_0}^5} \iiint (3z^2 - r^2) d\tau$$

Inverse effective moment of inertia :

$${I_{eff}}^{-1} = {I_\parallel}^{-1} - {I_\perp}^{-1}$$

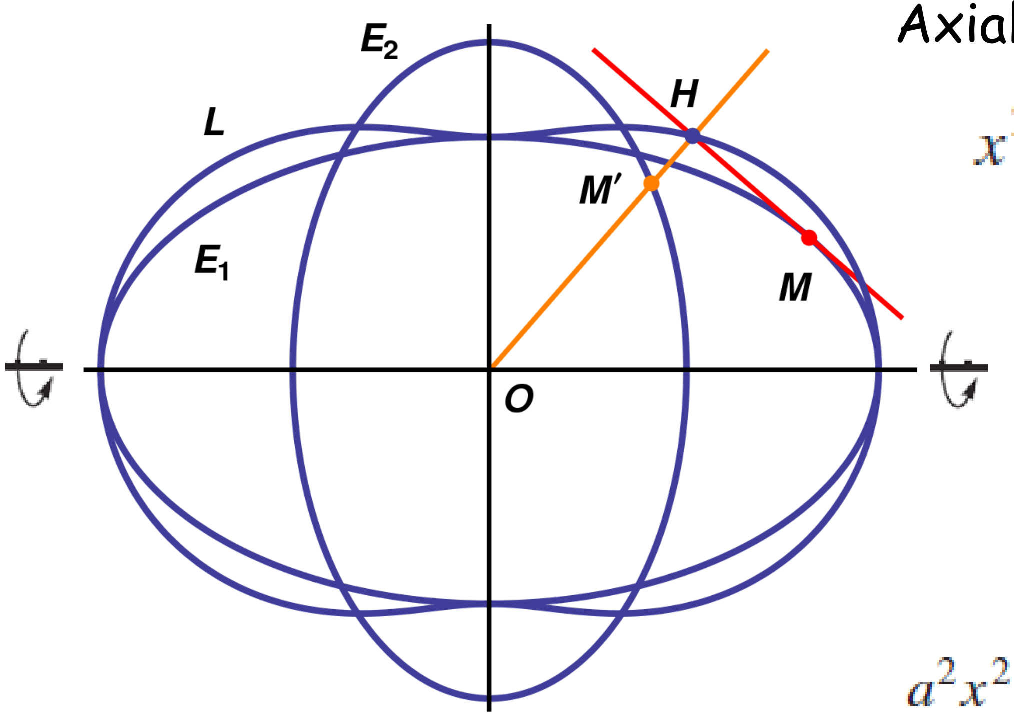
Deformation parameter

$$\beta = \frac{0.75}{\sqrt{5\pi}} Q {R_0}^2 \langle r^2 \rangle^{-1} \qquad \beta = 2\sqrt{\frac{\pi}{5}} \frac{I_\perp - I_\parallel}{I_\perp + 0.5 I_\parallel}$$

! ! !

Link between the elliptic lemniscatoid L and the prolate E_1 and oblate E_2 ellipsoids

When the point M generates the prolate ellipsoid E_1,the point H, projection of the origin onto the tangential plane in M, generates the elliptic lemniscatoid L. M', the inverse of the point H, describes the oblate ellipsoid E_2 .



Axially symmetric prolate ellipsoid:

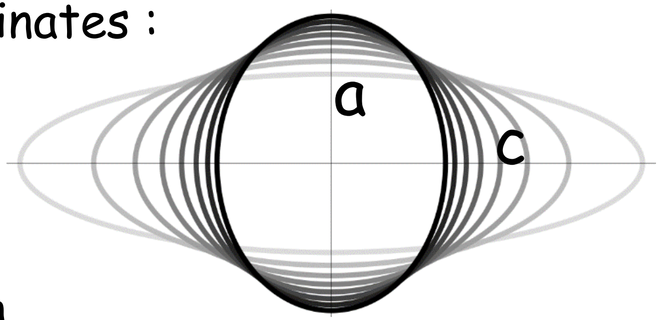
$$x^2/a^2 + y^2/a^2 + z^2/c^2 = 1$$

Elliptic lemniscatoid:

$$a^2x^2 + a^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2$$

Prolate and oblate ellipsoids

For axially symmetric ellipsoids and in polar coordinates :



$$1/R(\theta)^2 = \sin^2\theta/a^2 + \cos^2\theta/c^2$$

a is the transverse semiaxis and c is half the elongation.
s = a/c. (volume conservation: $a^2c = R_0^3$)
For prolate deformations s < 1 and the eccentricity $e^2 = 1 - s^2$.
For oblate deformations s > 1 and the eccentricity $e^2 = 1 - s^{-2}$.

In the prolate case :

$$B_s = \frac{(1-e^2)^{1/3}}{2}\left[1 + \frac{\sin^{-1}(e)}{e(1-e^2)^{1/2}}\right] \qquad B_C = \frac{(1-e^2)^{1/3}}{2e}\ln\left(\frac{1+e}{1-e}\right)$$

In the oblate case :

$$B_s = \frac{(1+\epsilon^2)^{1/3}}{2}\left[1 + \frac{\ln(\epsilon + (1+\epsilon^2)^{1/2})}{\epsilon(1+\epsilon^2)^{1/2}}\right] \qquad \epsilon^2 = s^2 - 1 \qquad B_C = \frac{(1+\epsilon^2)^{1/3}}{\epsilon}\tan^{-1}\epsilon$$

For the prolate ellipsoidal shapes :

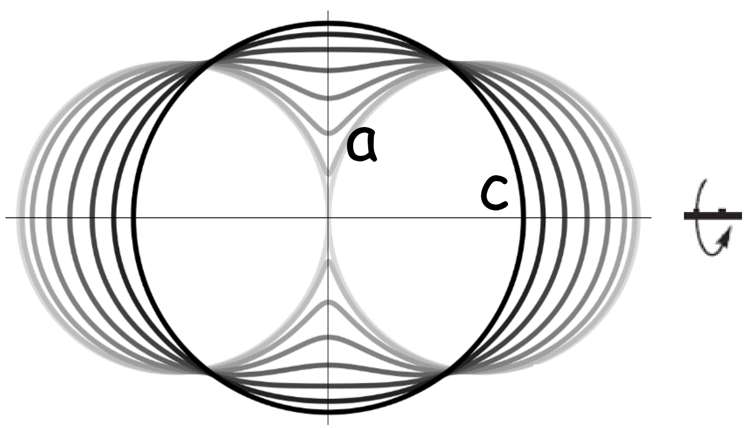
$$I_\perp = \frac{s^{-4/3} + s^{2/3}}{2} \qquad I_\parallel = s^{2/3} \qquad Q = \frac{8\pi}{15}(s^{-4/3} - s^{2/3})$$

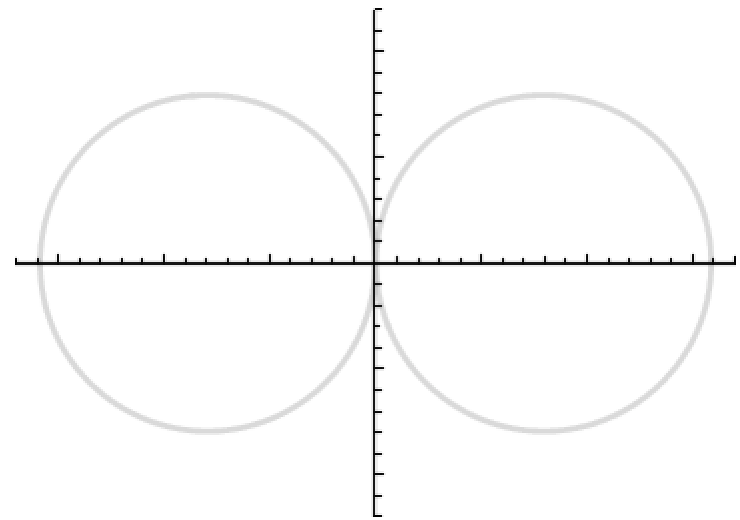
Elliptic Lemniscatoïds

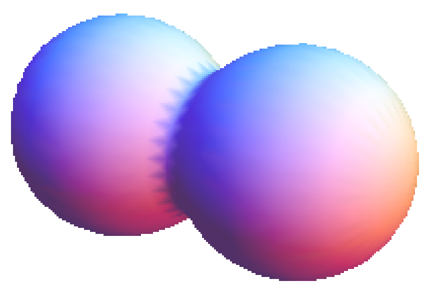
$R(\theta)^2 = a^2 \sin^2\theta + c^2 \cos^2\theta.$ $s = a/c.$

a and c are the radial and transverse semi-axes.

Assuming volume conservation, when s varies from 0 to 1 (or vice-versa) the elliptic lemniscatoids evolve from two tangent spheres to a sphere with the intermediate formation of a deep neck.







(for s = 1/3)

Elliptic Lemniscatoïds

Volume :

$$V = \frac{4}{3}\pi {R_0}^3 = \frac{\pi}{12}c^3\left[4 + 6s^2 + \frac{3s^4}{\sqrt{1-s^2}}\sinh^{-1}\left(\frac{2}{s^2}\sqrt{1-s^2}\right)\right]$$

Surface area :

$$S = 4\pi {R_0}^2 B_s = 2\pi c^2\left[1 + \frac{s^4}{\sqrt{1-s^4}}\sinh^{-1}\left(\frac{1}{s^2}\sqrt{1-s^4}\right)\right]$$

Distance r between the mass centers of the left and right parts of the system :

$$r = \frac{2\int_0^c z d^3r}{\int_0^c d^3r}$$

$$r = \pi c^4 \frac{1 + s^2 + s^4}{3V}$$

Relative perpendicular and parallel moments of inertia :

$$I_{\perp,rel} = \frac{c^5 s^2}{512(1-s^2){R_0}^5}\left[\frac{112}{s^2} + 8 + 30s^2 - 135s^4 + \frac{120s^4 - 135s^6}{\sqrt{1-s^2}}\sinh^{-1}\left(\sqrt{\frac{1-s^2}{s^2}}\right)\right]$$

$$I_{\parallel,rel} = \frac{c^5 s^2}{512(1-s^2){R_0}^5}\left[\frac{32}{s^2} + 48 + 100s^2 - 210s^4 + \frac{240s^4 - 210s^6}{\sqrt{1-s^2}}\sinh^{-1}\left(\sqrt{\frac{1-s^2}{s^2}}\right)\right]$$

Quadrupole moment :

$$Q = \frac{\pi c^5 s^2}{96(1-s^2){R_0}^5}\left[\frac{16}{s^2} - 8 - 14s^2 + 15s^4 - \frac{24s^4 - 15s^6}{\sqrt{1-s^2}}\sinh^{-1}\left(\sqrt{\frac{1-s^2}{s^2}}\right)\right]$$

Asymmetric quasimolecular shapes

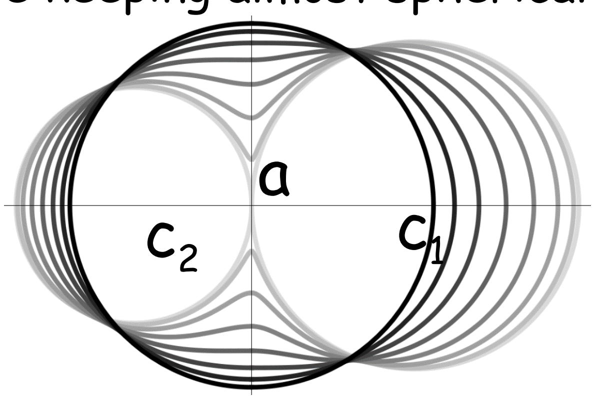
The transition from two unequal spheres to one sphere or vice versa can be described in joining two different elliptic lemniscatoids assuming the same transverse distance a and the volume conservation.

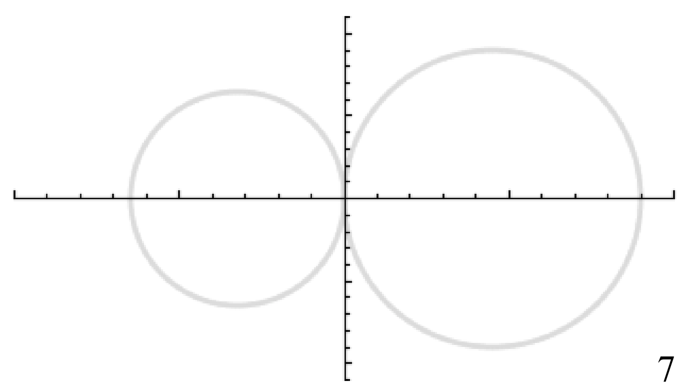
$$R(\theta)^2 = \begin{cases} a^2 \sin^2\theta + {c_1}^2 \cos^2\theta & 0 \le \theta \le \pi/2 \\ a^2 \sin^2\theta + {c_2}^2 \cos^2\theta & \pi/2 \le \theta \le \pi \end{cases} \qquad {R_0}^3 = {R_1}^3 + {R_2}^3$$

The two parameters $s_1 = a/c_1$ and $s_2 = a/c_2$ define the shape and the two radii R_1 and R_2 connect s_1 and s_2:

$${s_2}^2 = \frac{{s_1}^2}{{s_1}^2 + \left(1 - {s_1}^2\right)(R_2/R_1)^2}$$

When s_1 increases from 0 to 1, the shape evolves continuously from two touching different spheres to one sphere with the formation of a deep neck, while keeping almost spherical ends.





Asymmetric quasimolecular shapes

The distance r between the centers of mass of the two parts is $r = r_1+r_2$. The volume of the two parts being conserved, r_1 and r_2 depend on z_v, the distance from the origin to the separation plane.

$$r_1 = \frac{1}{\frac{4}{3}R_1^3}\left\{\frac{z_v^4 - a^2 z_v^2}{4} + \frac{c_1^4 + a^2 c_1^2 + a^4}{12} - \frac{a s_2^2}{3(1-s_2^2)}\left[\left(\frac{z_v^2(1-s_2^2)}{s_2^2} + \frac{a^2}{4}\right)^{3/2} - \frac{a^3}{8}\right]\right\}$$

$$r_2 = \frac{1}{\frac{4}{3}R_2^3}\left\{\frac{z_v^4 - a^2 z_v^2}{4} - \frac{a^4}{4}\left(\frac{1-s_2^2}{s_2^4}\right) + \frac{a s_2^2}{3(1-s_2^2)}\left[a^3\left(\frac{1}{s_2^2} - \frac{1}{2}\right)^3 - \left(\frac{z_v^2(1-s_2^2)}{s_2^2} + \frac{a^2}{4}\right)^{3/2}\right]\right\}$$

z_v is the solution of the equation :

$$\frac{1}{3}z^3 - \frac{1}{2}a^2 z + \frac{1}{12}(2c_2^3 + 3a^2 c_2) + \frac{1}{2}\sqrt{c_2^2 - a^2}\left[D^2 \sinh^{-1}\left(\frac{c_2}{D}\right) - D^2 \sinh^{-1}\left(\frac{z}{D}\right) - z\sqrt{z^2 + D^2}\right] = \frac{4}{3}R_2^3$$

Hyperbolic lemniscatoids (Cassinian ovaloids)

For one-body shapes, a and c are the radial and transverse semi-axes and s = a/c.

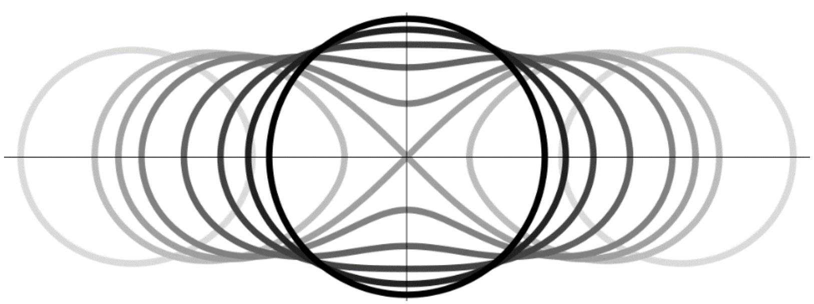
$$x^2 = -z^2 + 0.5c^2(s^2-1) + 0.5c\sqrt{8(1-s^2)z^2 + c^2(1+s^2)^2}$$

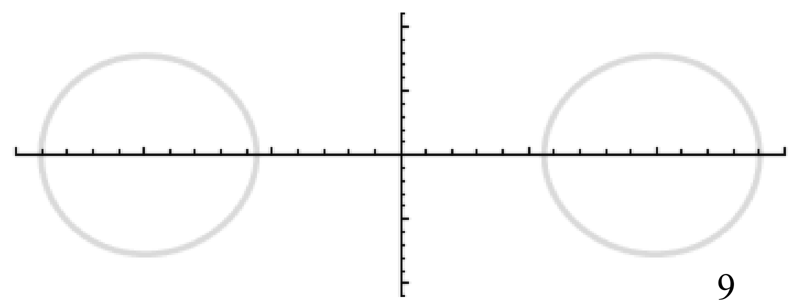
$$V = \frac{\pi c^3}{12}\left[-2+6s^2+\frac{3(1+s^2)^2}{\sqrt{2(1-s^2)}}\sinh^{-1}\left(\frac{2\sqrt{2(1-s^2)}}{1+s^2}\right)\right]$$

For two-body shapes s is the opposite of the ratio of the distance between the tips of the fragments and the system elongation. When s varies from 1 to -1 the shapes vary continuously from a sphere to two infinitely separated spheres, assuming volume conservation.

$$x^2 = -z^2 - 0.5c^2(s^2+1) + 0.5c\sqrt{8(1+s^2)z^2 + c^2(1-s^2)^2}$$

$$V = \frac{\pi c^3}{12}\left[-2(1+s)^3+\frac{3(1-s^2)^2}{\sqrt{2(1+s^2)}}\sinh^{-1}\left(\frac{2(1+s)\sqrt{2(1+s^2)}}{(1-s)^2}\right)\right]$$





Hyperbolic lemniscatoids, formulas for one-body shapes

Relative surface:

$$B_s = \frac{c^2}{4{R_0}^2} \times \left[4(1+s^2) + 2\sqrt{\frac{2(1+s^2)}{1-s^2}} s^2 F\left(\sin^{-1}\sqrt{1-s^2}, \frac{1}{\sqrt{1+s^2}}\right) - 2(1+s^2)\sqrt{\frac{2(1+s^2)}{1-s^2}} E\left(\sin^{-1}\sqrt{1-s^2}, \frac{1}{\sqrt{1+s^2}}\right)\right]$$

Distance r between the mass centers of each part:

$$r = \pi c^4 \frac{1+s^2+s^4}{3V}$$

Relative perpendicular and parallel moments of inertia:

$$I_\perp = \frac{c^5}{1024(1-s^2){R_0}^5}\left[269 + 251s^2 - 145s^4 - 255s^6 - \frac{15(1+s^2)^2(17-30s^2+17s^4)}{2\sqrt{2(1-s^2)}} \sinh^{-1}\left(\frac{2\sqrt{2(1-s^2)}}{1+s^2}\right)\right]$$

$$I_\parallel = \frac{c^5}{512(1-s^2){R_0}^5}\left[147 - 27s^2 - 15s^4 - 225s^6 - \frac{15(1+s^2)^2(15-34s^2+15s^4)}{2\sqrt{2(1-s^2)}} \sinh^{-1}\left(\frac{2\sqrt{2(1-s^2)}}{1+s^2}\right)\right]$$

Quadrupole moment:

$$Q = \frac{\pi c^5}{192(1-s^2){R_0}^5}\left[-5 + 61s^2 - 23s^4 + 39s^6 + \frac{3(1+s^2)^2(13-38s^2+13s^4)}{2\sqrt{2(1-s^2)}} \sinh^{-1}\left(\frac{2\sqrt{2(1-s^2)}}{1+s^2}\right)\right]$$

Hyperbolic lemniscatoids, formulas for two-body shapes

Distance r between the mass centers of each part:

$$r = \frac{c^4}{8{R_0}^3} \times \frac{(1-s^2)^3}{1+s^2}$$

Relative perpendicular and parallel moments of inertia and quadrupole moment:

$$I_{\parallel} = \frac{c^5}{512(1+s^2){R_0}^5}\left[147 + 225s + 27s^2 - 15s^3 - 15s^4 + 27s^5 + 225s^6 + 147s^7 \right.$$
$$\left. - \frac{15(1-s^2)^2(15+34s^2+15s^4)}{2\sqrt{2(1+s^2)}} \sinh^{-1}\left(\frac{2(1+s)\sqrt{2(1+s^2)}}{(1-s)^2}\right)\right],$$

$$I_{\perp} = \frac{c^5}{1024(1+s^2){R_0}^5} \times \left[269 + 255s - 251s^2 - 145s^3 - 145s^4 - 251s^5 + 255s^6 \right.$$
$$\left. +269s^7 - \frac{15(1-s^2)^2(17+30s^2+17s^4)}{2\sqrt{2(1+s^2)}} \sinh^{-1}\left(\frac{2(1+s)\sqrt{2(1+s^2)}}{(1-s)^2}\right)\right]$$

$$Q = \frac{\pi c^5}{192(1-s^2){R_0}^5}\left[-5 - 39s - 61s^2 - 23s^3 - 23s^4 - 61s^5 - 39s^6 - 5s^7 \right.$$
$$\left. + \frac{3(1-s^2)^2(13+38s^2+13s^4)}{2\sqrt{2(1+s^2)}} \sinh^{-1}\left(\frac{2(1+s)\sqrt{2(1+s^2)}}{(1-s)^2}\right)\right].$$

Symmetric prolate ternary shapes

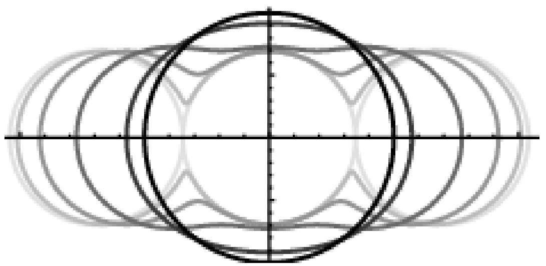
From the elliptic lemniscatoids one can generate symmetric prolate ternary shapes varying from one sphere to three aligned tangential identical spheres.

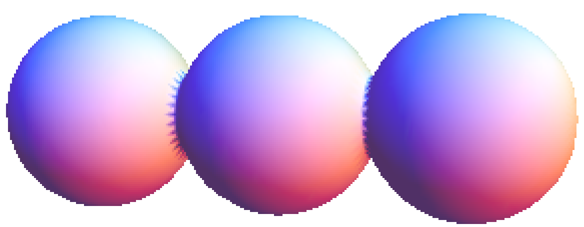
In the first quadrant: $x^2 = 0.5[a^2 - 2(z-d)^2 + \sqrt{a^4 + 4(z-d)^2(c^2-a^2)}]$

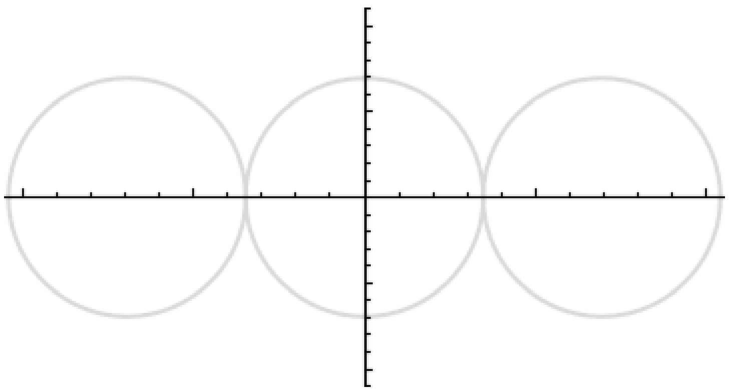
a is the neck radius, c half the elongation of the generating binary case. s = a/c varies from 1 to 0. d is the distance between the position of the crevice and the transverse axis and h is the maximal transverse radial distance.

$$d = \begin{cases} 0.5c\sqrt{\frac{1-2s^2}{1-s^2}} & \text{for } 0 \leqslant s < 0.5\sqrt{2} \\ 0 & \text{for } 0.5\sqrt{2} \leqslant s \leqslant 1 \end{cases}$$

$$h_{\max} = \begin{cases} 0.5c/(1-s^2)^{1/2} & \text{for } 0 \leqslant s < 0.5\sqrt{2} \\ a & \text{for } 0.5\sqrt{2} \leqslant s \leqslant 1 \end{cases}$$







Symmetric prolate ternary shapes

Volume:

$$V = \frac{4}{3}\pi {R_0}^3 = \frac{\pi c^3}{12}\left[4 + 6s^2 + g(\alpha) + \frac{3s^4}{\sqrt{1-s^2}} \ln\left(\frac{2 - s^2 + 2\sqrt{1-s^2}}{h(\alpha)}\right)\right]$$

! ! ! ! ! d/c and g and h functions of ! ! !

Relative surface function:

$$B_s = \begin{cases} \frac{c^2}{2R_0^2}\left[1 + \frac{\sqrt{1-2s^2}}{2} + \frac{s^4}{\sqrt{1-s^4}} \ln\left(\frac{\sqrt{2}(1+\sqrt{1-s^4})}{\sqrt{1-s^2} - \sqrt{(1+s^2)(1-2s^2)}}\right)\right] & \text{for } 0 \leqslant s \leqslant 0.5\sqrt{2} \\ \frac{c^2}{2R_0^2}\left[1 + \frac{s^4}{\sqrt{1-s^4}} \ln\left(\frac{1+\sqrt{1-s^4}}{s^2}\right)\right] & \text{for } 0.5\sqrt{2} \leqslant s < 1 \end{cases}$$

Distance between the left and right parts:

$$r = \begin{cases} c\left(2\alpha + \frac{\pi c^3(11-8s^2)}{48V(1-s^2)^2}\right) & \text{for } 0 \leqslant s < 0.5\sqrt{2} \\ \frac{\pi c^4}{3V}(1 + s^2 + s^4) & \text{for } 0.5\sqrt{2} \leqslant s \leqslant 1 \end{cases}$$

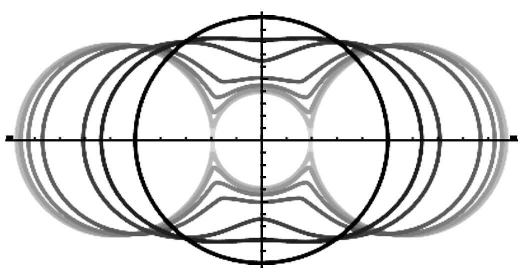
Asymmetric prolate ternary shapes

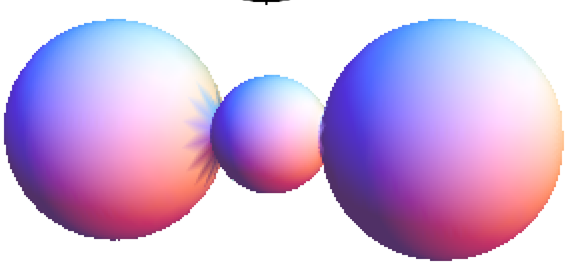
A symmetry plane cut the smallest fragment along its maximal orthogonal distance. $s_1 = a/c_1$ and $s_2 = a/c_2$. For $s_1 = s_2 = 1$, the shape is a sphere and for $s_1 = s_2 = 0$, two external spheres of radius R_1 are aligned and in contact with a smaller central sphere of radius R_2. s_1 and s_2 may be linked by:

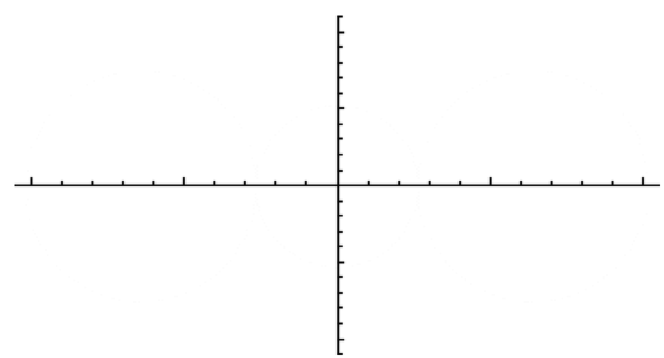
$$s_2{}^2 = \frac{s_1{}^2}{s_1{}^2 + \left(1 - s_1{}^2\right)(R_2/R_1)^2}$$

In the first quadrant ($i = 1$ for $z > d$ and $i = 2$ for $z < d$):

$$x^2 = -(z-d)^2 + 0.5 s_i{}^2 c_i{}^2 + 0.5 c_i \sqrt{4(1 - s_i{}^2)(z-d)^2 + s_i{}^4 c_i{}^2}$$







Asymmetric prolate ternary shapes

Volume:

$$V = \frac{\pi {c_1}^3}{12}\left[4 + 6{s_1}^2 + \frac{3{s_1}^4}{\sqrt{1-{s_1}^2}}\sinh^{-1}\left(\frac{2\sqrt{1-{s_1}^2}}{{s_1}^2}\right)\right]$$

$$+\frac{\pi {c_2}^3}{12}\left[6\alpha + 6\alpha {s_2}^2 - 8\alpha^3 + \frac{3{s_2}^4}{\sqrt{1-{s_2}^2}}\sinh^{-1}\left(\frac{2\alpha\sqrt{1-{s_2}^2}}{{s_2}^2}\right)\right]$$

Surface:

$$S = 2\pi {c_1}^2\left[1 + \frac{{s_1}^4}{\sqrt{1-{s_1}^4}}\sinh^{-1}\left(\frac{\sqrt{1-{s_1}^4}}{{s_1}^2}\right)\right]$$

$$+2\pi {c_2}^2\left[\alpha\sqrt{1-{s_2}^2} + \frac{{s_2}^4}{\sqrt{1-{s_2}^4}}\sinh^{-1}\left(\frac{\alpha\sqrt{2(1-{s_2}^4)}}{{s_2}^2}\right)\right]$$

Distance between the centers of mass of the two halves of the system:

$$r = \frac{\pi {c_1}^4}{3V}(1 + {s_1}^2 + {s_1}^4) + c_2\left[2\alpha + \frac{\pi {c_2}^3\alpha^4}{3V}.\frac{-5 + 8{s_2}^2 + 16{s_2}^6 - 16{s_2}^8}{(1-2{s_2}^2)^2}\right]$$

For three aligned separated spherical fragments:

$$B_s = \frac{2 + (R_2/R_1)^2}{(2 + (R_2/R_1)^3)^{2/3}} \qquad r = \frac{3}{2 + (R_2/R_1)^3}\left(\frac{(R_2/R_1)^4 R_1}{4} + \frac{4}{3}D\right)$$

Two ellipsoids

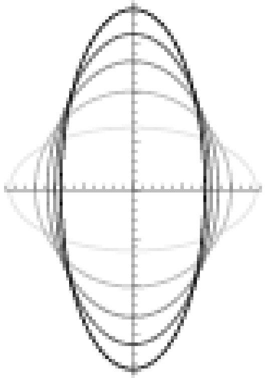
Coulomb interaction energy between two coaxial prolate or oblate ellipsoids:

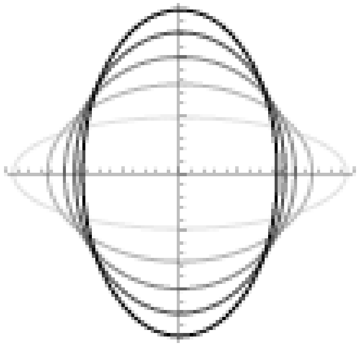
$$E_{C,\text{int}}(r) = \frac{Q_1 Q_2}{r}[s(\lambda_1) + s(\lambda_2) - 1 + S(\lambda_1,\lambda_2)] \qquad \lambda_i{}^2 = \frac{c_i{}^2 - a_i{}^2}{r^2}$$

For the prolate case: $$s(\lambda_i) = \frac{3}{4}\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_i{}^3}\right)\ln\left(\frac{1+\lambda_i}{1-\lambda_i}\right) + \frac{3}{2\lambda_i{}^2}$$

For the oblate case: $$s(\lambda_i) = \frac{3}{2}\left(\frac{1}{\omega_i} + \frac{1}{\omega_i{}^3}\right)\tan^{-1}\omega_i - \frac{3}{2\omega_i{}^2}, \quad \omega_i{}^2 = -\lambda_i{}^2$$

$$S(\lambda_1,\lambda_2) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{3}{(2j+1)(2j+3)} \times \frac{3}{(2k+1)(2k+3)} \times \frac{(2j+2k)!}{(2j)!(2k)!}\lambda_1{}^{2j}\lambda_2{}^{2k}$$

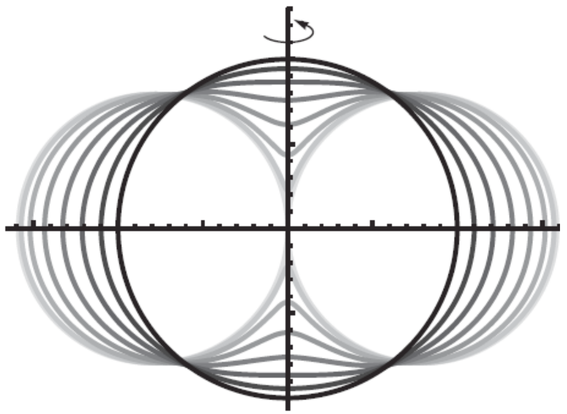


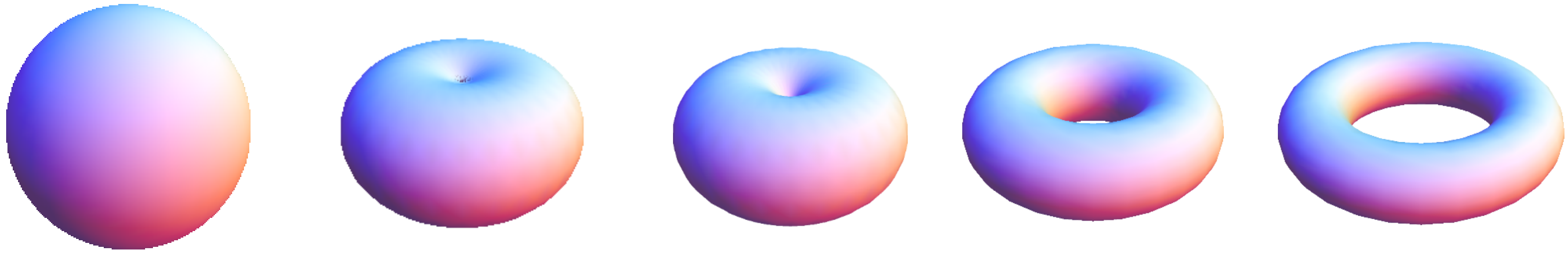


Pumpkin-like shapes and tori

A pumpkin-like configuration may be simulated using elliptic lemniscates and taking the vertical axis as axis of revolution. $s = a/c$ is also sufficient to define the shape. When s decreases from 1 to 0 an hollow progressively appears in this oblate lemniscatoid leading to a ring torus for which the upper and lower hollows are just linked.

Later on, the evolution of the ring torus can be governed by: $s_t = (r_t - r_s)/2r_s$.
r_t and r_s are the torus and sausage radii.





Pumpkin-like shapes and tori

For the **oblate elliptic lemniscatoids**:
volume, surface, perpendicular moment of inertia and mean square radius

$$V = \frac{4\pi {R_0}^3}{3} = \frac{4\pi c^3}{3}\left[\frac{s^3}{4} + \frac{3}{8}\left(s + \frac{\sin^{-1}\left(\sqrt{1-s^2}\right)}{\sqrt{1-s^2}}\right)\right] \qquad B_s = \frac{s}{4\pi {R_0}^2} = \frac{c^2}{2{R_0}^2}\left(s^2 + \frac{\sin^{-1}\left(\sqrt{1-s^4}\right)}{\sqrt{1-s^4}}\right)$$

$$I_{\perp,rel} = \frac{3c^5}{2{R_0}^5(1-s^2)}\left(-\frac{s^7}{24} - \frac{s^5}{16} - \frac{25s^3}{192} + \frac{35s}{128} - \frac{5}{16}\left(s^2 - \frac{7}{8}\right)\frac{\sin^{-1}\left(\sqrt{1-s^2}\right)}{\sqrt{1-s^2}}\right)$$

$$\langle r^2\rangle_{rel} = \frac{\langle r^2\rangle}{\frac{3}{5}{R_0}^2} = \frac{5c^5}{4{R_0}^5}\left[\frac{2s^5}{15} + \frac{s^3}{6} + \frac{1}{4}\left(s + \frac{\sin^{-1}\left(\sqrt{1-s^2}\right)}{\sqrt{1-s^2}}\right)\right]$$

For the **holed torus** :
volume, surface, perpendicular moment of inertia and mean square radius

$$V = \frac{4\pi {R_0}^3}{3} = 2\pi^2 r_t {r_s}^2 = \frac{\pi^2 {c_t}^3}{4}(1+2s_t) \qquad B_s = \frac{4\pi^2 r_s r_t}{4\pi {R_0}^2} = \frac{\pi {c_t}^2}{4{R_0}^2}(1+2s_t)$$

$$I_{\perp,rel} = \frac{35}{32}(1+3s_t+3{s_t}^2)\left(\frac{16}{3\pi(1+2s_t)}\right)^{2/3} \qquad \langle r^2\rangle_{rel} = \frac{5}{6}(1+2s_t+2{s_t}^2)\left(\frac{16}{3\pi(1+2s_t)}\right)^{2/3}$$

(thick skin) Bubbles

Assuming volume conservation, the bubble characteristics can be expressed in terms of a single parameter, the ratio p = r_1/r_2 of the inner radius r_1 and the outer one r_2.

$$V = \frac{4\pi {R_0}^3}{3} = \frac{4\pi}{3}({r_2}^3 - {r_1}^3)$$

$$r_1 = R_0 p(1-p^3)^{-1/3} \qquad r_2 = R_0(1-p^3)^{-1/3}$$

$$B_s = \frac{1+p^2}{(1-p^3)^{2/3}} \qquad B_C = \frac{1-2.5p^3+1.5p^5}{(1-p^3)^{5/3}}$$

$$I_{\perp,rel} = (1-p^5)(1-p^3)^{-5/3}$$

$$\langle r^2 \rangle_{rel}{}^{1/2} = \frac{\langle r^2 \rangle^{1/2}}{\sqrt{3/5}R_0} = (1-p^5)^{1/2}(1-p^3)^{-5/6}$$

Liquid Drop Model energy

$$E_{GLDM}=E_{vol}+E_{surface}+E_{Coulomb}+E_{proximity}$$

$$E_{vol} \,!\, !\, a_v(1 \,!\, k_v I^2)A$$

$$E_{surf} \,!\, a_s(1 \,!\, k_s I^2)A^{2/3} \,!\, \frac{Surf}{4\pi R_0^2}$$

$$E_{Coul} \,!\, \frac{9e^2Z^2}{16\pi^2R_0^6} \,!\, \frac{d\tau d\tau'}{\left|\vec{r} \,!\, \vec{r}'\right|}$$

$$I \,!\, (N \,!\, Z)/A$$

$$a_v \,!\, 15.494 \text{ MeV}$$

$$a_s \,!\, 17.9439 \text{ MeV}$$

$$k_v \,!\, 1.8$$

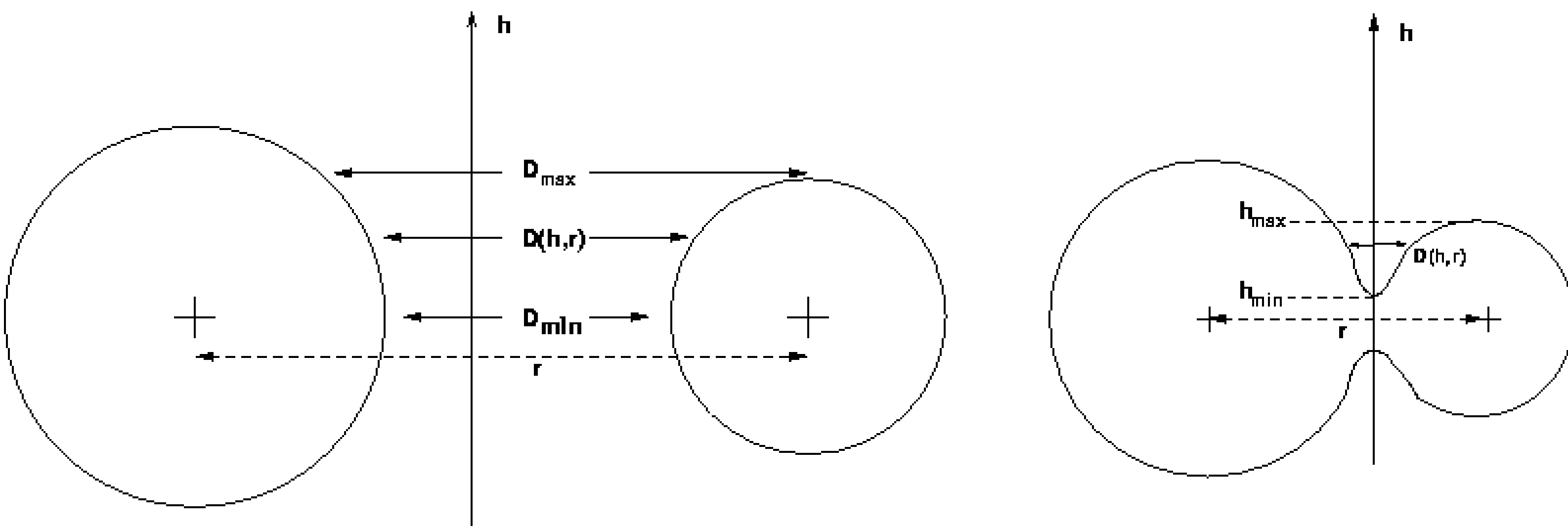
$$k_s \,!\, 2.6$$

$$R_0 \,!\, 1.28A^{1/3} \,!\, 0.76 \,!\, 0.8A^{!\,1/3}$$

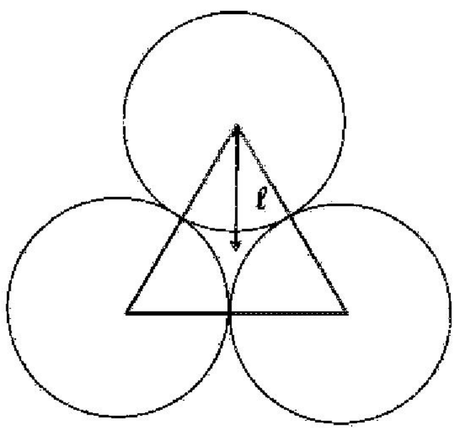
Proximity energy

Additional energy to the surface energy taking into account the finite range of the nuclear interaction between opposite nucleons in a gap between incoming nuclei or in a neck in one-body compact shapes (~ - 9.4 MeV at the contact point of two !   )

$$E_{proximity}(r) = 2\gamma \int_{h_{min}}^{h_{max}} \varphi\left[D(r,h)/b\right] 2\pi h\,dh$$



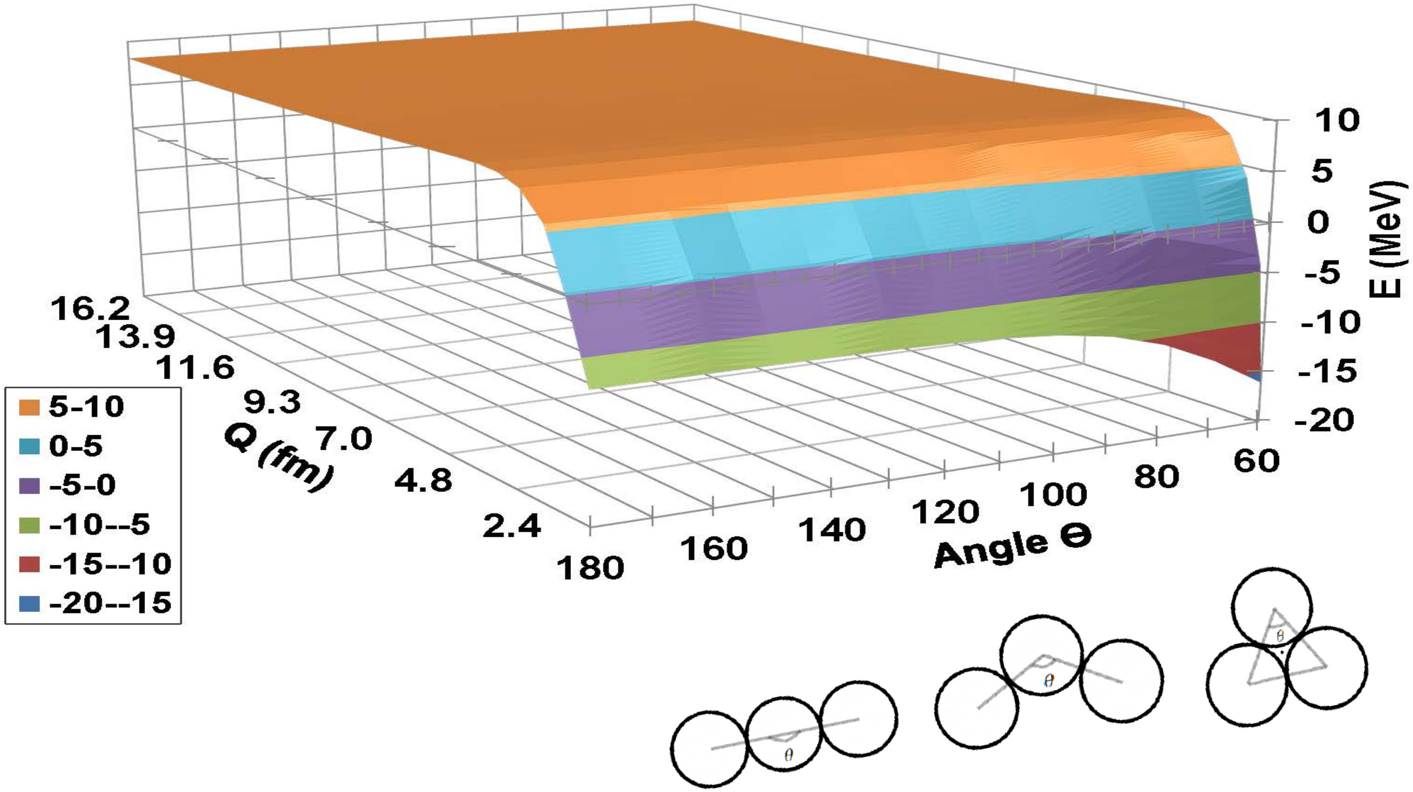
^{12}C nucleus (triangular configuration)



$\langle r^2 \rangle^{1/2}$ (gs, exp) = 2.47 fm
$\langle r^2 \rangle^{1/2}$ (GLDM) = 2.43 fm
(for a linear chain 3.16 fm)

Electric quadrupole moment (gs):
Q_0 (exp) = -22+-10 e fm^2
Q_0 (GLDM) = -24.4 e fm^2

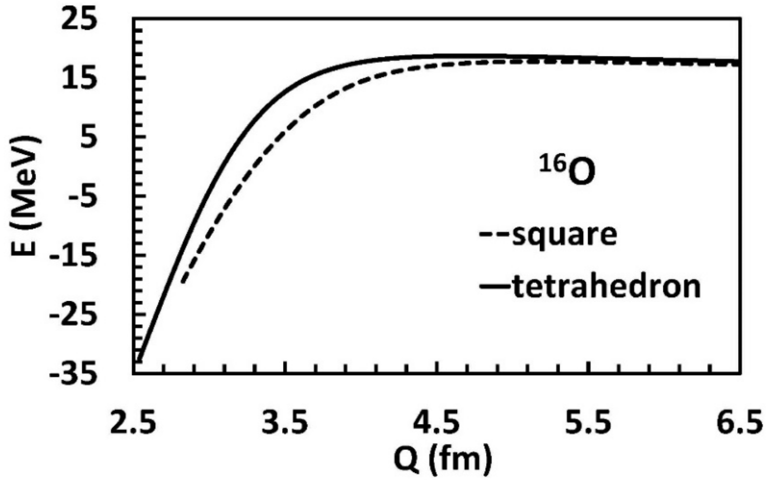
Two data compatible with an equilateral triangular shape of the ^{12}**C** ground state, but not with a linear chain.



Deformation barrier versus the three-alphas configuration.

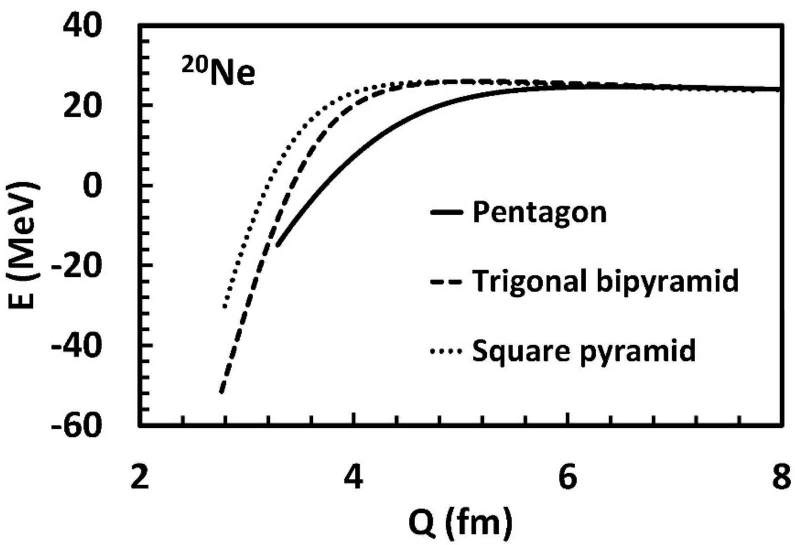
The difference between the energies of the minima of the linear chain configuration and the minima of the oblate equilateral configuration is 7.36 MeV, close to the energy 7.65 MeV of the excited Hoyle state.

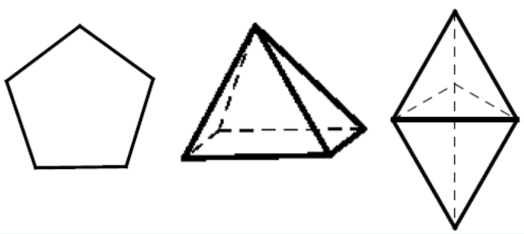
^{16}O and ^{20}Ne nuclei



4 and 6 bonds

rms radius (fm):
Exp: 2.70
Square: 2.83
Tetrahedron: 2.54
Linear chain: 4.15

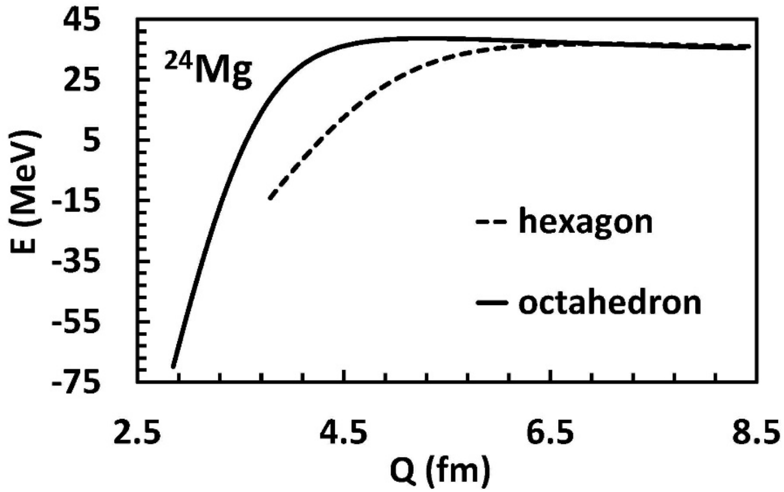


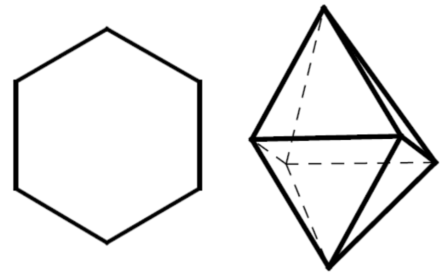


5, 8 and 9 bonds

rms radius (fm):
Exp: 3.01
Pentagon: 3.29
Square pyr.: 2.79
Trigonal bipyr.: 2.76

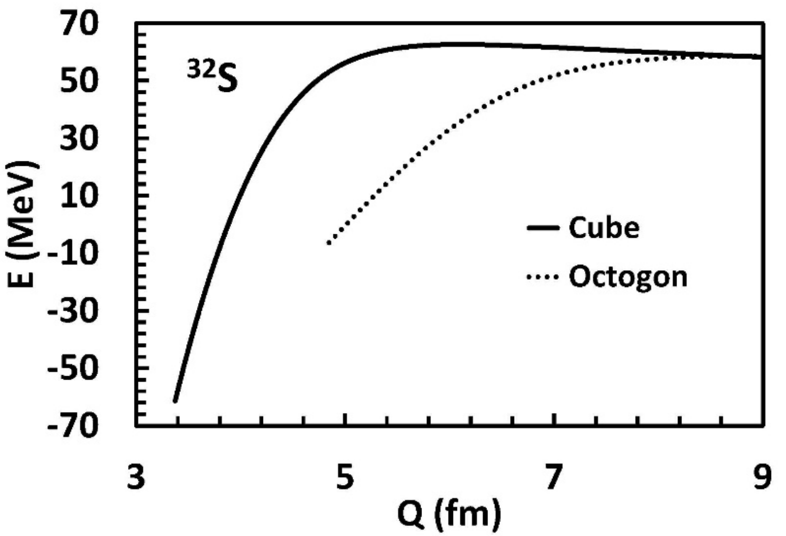
^{24}Mg and ^{32}S nuclei

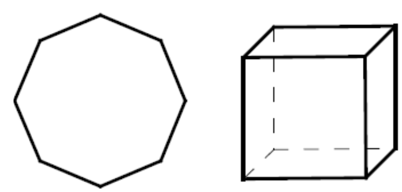




6 and 12 bonds

rms radius (fm):
Exp: 3.06
Hexagon: 3.79
Octahedron.: 2.85





8 and 12 bonds

rms radius (fm):
Exp: 3.26
Octagon: 4.85
Cube.: 3.37

Conclusion

- Different axially symmetric shape sequences are proposed to describe ground or excited states of leptodermous nuclear matter distributions and to follow their evolution in the entrance or decay channels of nuclear reactions such as fusion, fission, alpha decay and cluster radioactivities. These shapes are derived from the generalized lemniscate families.
- The energies of the ^{12}C, ^{16}O, ^{20}Ne, ^{24}Mg and ^{32}S 4n nuclei have been determined assuming different planar and three-dimensional shapes of the ! molecules: linear chain, triangle, square, tetrahedron, pentagon, trigonal bipyramid, square pyramid, hexagon, octahedron, octagon and cube. These calculations suggest that an oblate equilateral triangular configuration is compatible with the ground state shape of ^{12}C and a prolate almost aligned shape for the excited Hoyle state shape. The three dimensional shapes are favored for the heavier nuclei.

Thank you for your attention

Shape review: Phys. Rev. C 95 (2017) 054610

Multibody shapes: Phys. Rev. C 92 (2015) 054308

Fission: Phys. Rev. C 86 (2012) 044326

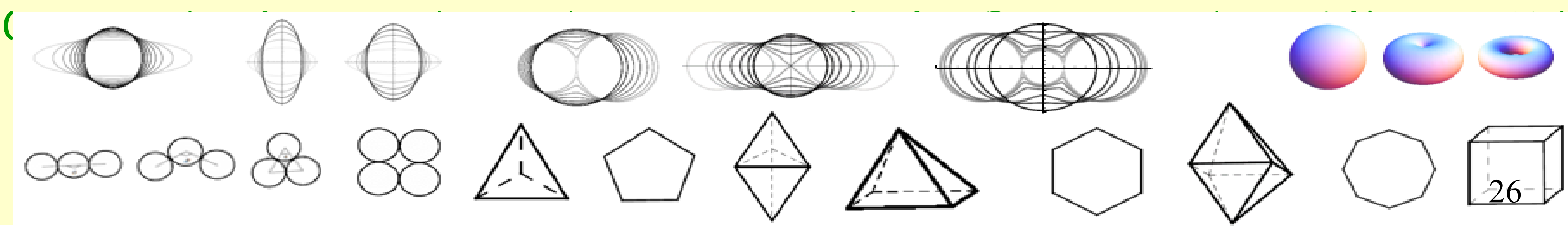
Cluster radioactivity: Nucl. Phys. A 683 (2001) 182

Alpha emission; J. Phys. G 26 (2000) 1149

Pumpkin-like and torus: Nucl. Phys. A 598 (1996) 125

Ternary fission: J. Phys. G: 15 (1989) L1

Fusion: Nucl. Phys. A 444 (1985) 477



Recent studies on light n-alpha clusters

' Evidence for Triangular D_{3h} Symmetry in ^{12}C ' ,
D.J. Marin-Lambarri *et al*, PRL 113, 012502, (2014)

' Further improvement of the upper limit on the direct 3α Decay from the Hoyle State in ^{12}C ',
M. Itoh *et al*, PRL 113, 102501, (2014)

' Decay and structure of the Hoyle state ',
→ There appear to be some peaks in the interior density distribution corresponding to configurations of equilateral and isosceles triangles
S. Ishikawa, PRC 90, 061604 (R), (2014)

' Giant Dipole Resonance as a Fingerprint of *α* Clustering Configurations in ^{12}C and ^{16}O ',
W.B. He *et al*, PRL 113, 032506, (2014)

' One-Dimensional α Condensation of α-Linear-Chain States in ^{12}C and ^{16}O ',
T. Suhara *et al*, PRL 112, 062501, (2014)

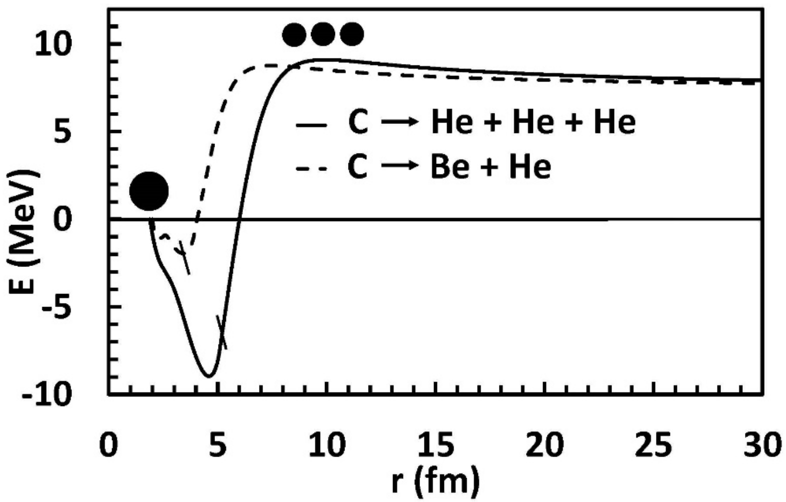
' Evidence for tetrahedral symmetry in ^{16}O ',
R. Bijker *et al*, PRL 112, 152501, (2014)

' *Ab initio* Calculation of the spectrum and structure of ^{16}O ',
→ For the ground state …tetrahedral configuration, for the first excited spin-0 state …square configuration of alpha clusters
E. Epelbaum *et al*, PRL 112, 102501, (2014)

' Signatures of α clustering in light nuclei from relativistic nuclear collisions '
W. Broniowski *et al*, PRL 112, 112501, (2014)

and other papers …

^{12}C nucleus

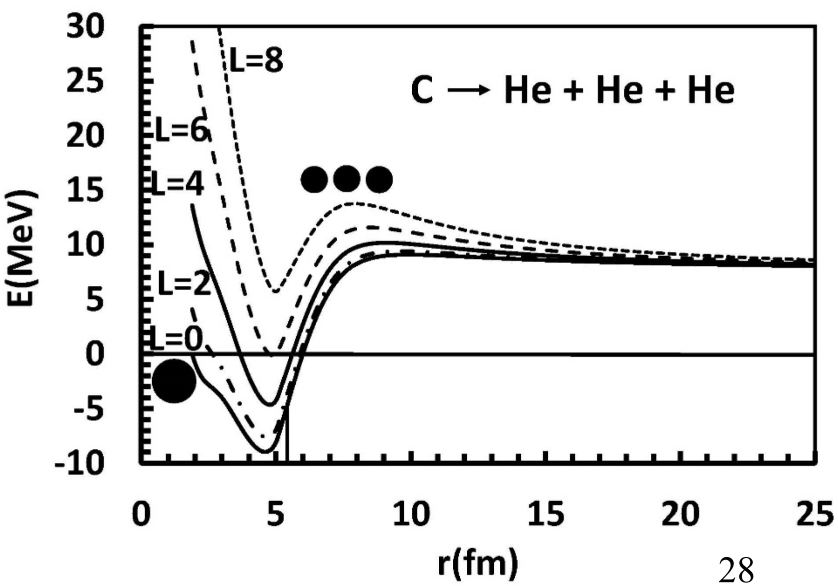


Potential barriers governing the binary ^{12}C<-> $^{8}Be+^{4}He$ and prolate ternary ^{12}C<-> $^{4}He+^{4}He+^{4}He$ reactions.

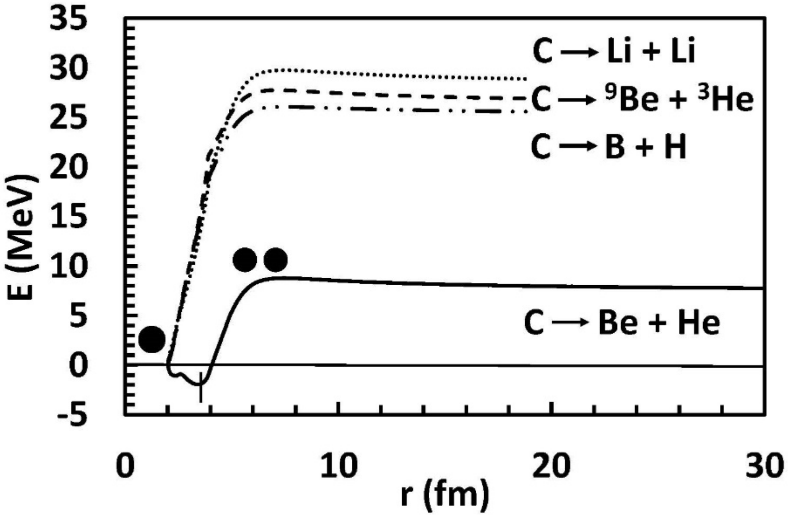
$Q_{3!}$! = 7.27 MeV

Q_{Be+He} = 7.37 MeV

L-dependent barriers for the prolate ternary ^{12}C<-> $^{4}He+^{4}He+^{4}He$ reaction.



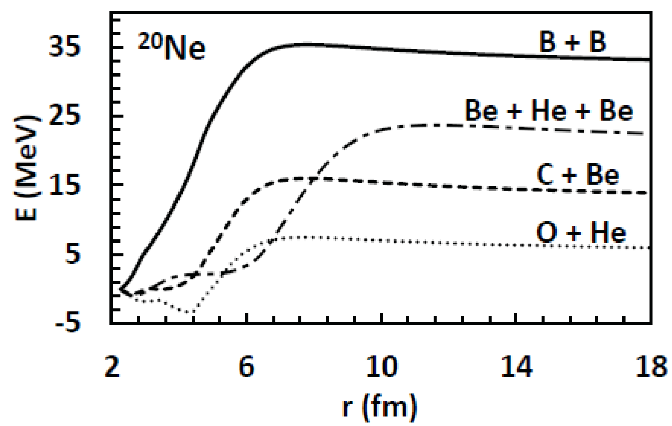
^{12}C nucleus



Potential barriers governing the $^{12}C \leftrightarrow {}^{8}Be+{}^{4}He$, $^{12}C \leftrightarrow {}^{10}B+{}^{2}H$, $^{12}C \leftrightarrow {}^{9}Be+{}^{3}He$ and $^{12}C \leftrightarrow {}^{6}Li+{}^{6}Li$ reactions.

	r_{sph}	$r_{E_{min}}$	r_{cont}	$r_{E_{max}}$	∞
$^{12}C \leftrightarrow {}^{8}Be + {}^{4}He$					
r (fm)	1.91	3.41	3.96	7.43	
E (MeV)	0.00	−1.95	−0.63	8.77	7.365
$^{12}C \rightarrow {}^{10}B + {}^{2}H$					
r (fm)	1.98		3.81	7.49	
E (MeV)	0.00		18.26	26.06	25.19
$^{12}C \rightarrow {}^{9}Be + {}^{3}He$					
r (fm)	1.94		3.90	7.12	
E (MeV)	0.00		20.97	27.73	26.28
$^{12}C \rightarrow {}^{6}Li + {}^{6}Li$					
r (fm)	1.89		4.0	7.48	
E (MeV)	0.00		19.63	29.44	28.17

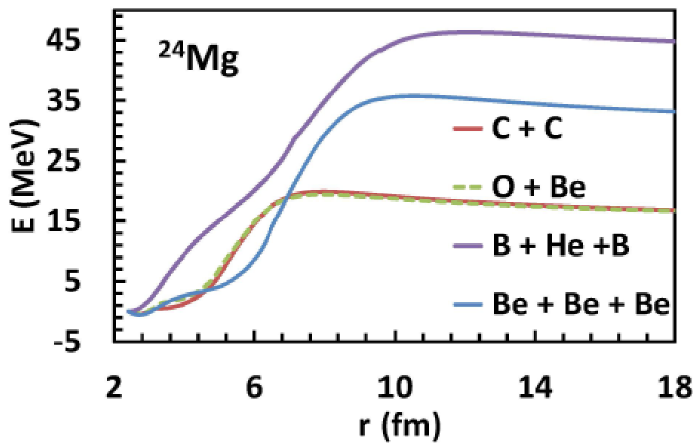
^{20}Ne nucleus



$Q_{5!}$! ! ! !
! ! ! ! ! !
eV

Reaction	$^{16}O+^{4}He$	$^{12}C+^{8}Be$	$^{8}Be+^{4}He+^{8}Be$	$^{10}B+^{10}B$
Q(MeV)	4.73	11.98	19.35	31.14

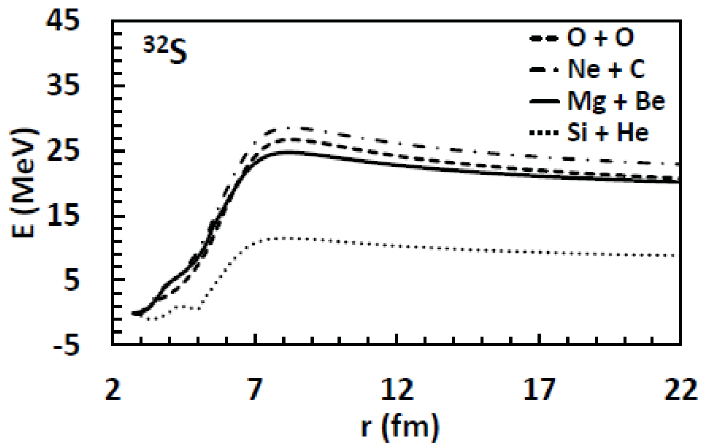
^{24}Mg nucleus

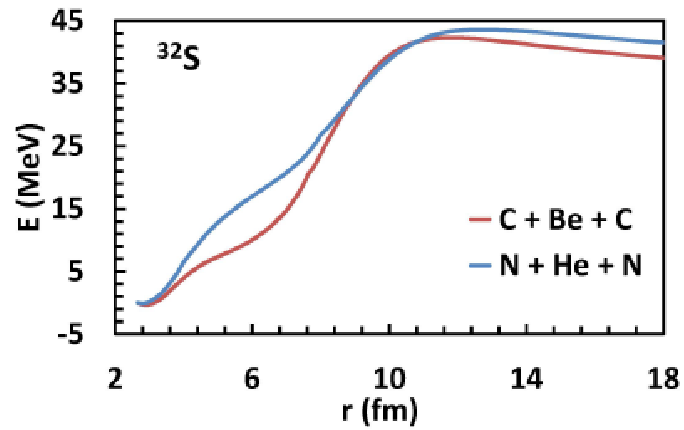


Q6! ! ! ! ! !
! ! ! ! ! eV

Reaction	^{20}Ne+^{4}He	^{12}C+^{12}C	^{16}O+^{8}Be	^{8}Be+^{8}Be+^{8}Be	^{10}B+^{4}He+^{10}B
Qreaction (MeV)	9.32	13.93	14.14	28.76	40.46

^{32}S nucleus





Reaction	^{28}Si+^{4}He	^{16}O+^{16}O	^{24}Mg+^{8}Be	^{20}Ne+^{12}C	^{12}C+^{8}Be+^{12}C	^{14}N+^{4}He+N
$Q_{reaction}$ (MeV)	6.95	16.54	17.02	18.97	30.96	34.17

$Q_{8!}$! !

! ! ! !

! !

Ground state rms radius

^{16}O	Square	Tetrahedron	Linear config.
rms radius (fm) Exp : 2.70 fm	2.83	2.54	4.15
Q_{elec} (e.fm^2)	-49.17 (Oblate)	0	
^{20}Ne	Pentagon	Trigonal bipyramid	Square pyramid
rms radius (fm) Exp : 3.01 fm	3.29	2.76	2.79
Q_{elec} (e.fm^2)	-89.63 (Oblate)	41.29 (Prolate)	-29.73 (Oblate)
^{24}Mg	Hexagon	Octahedron	
rms radius (fm) Exp : 3.06 fm	3.79	2.85	
Q_{elec} (e.fm^2)	-149.75 (Oblate)	0	
^{32}S	Octogon	Cube	
rms radius (fm) Exp : 3.26 fm	4.85	3.37	
Q_{elec} (e.fm^2)	-345.3 (Oblate)	0	