Generalized Gibbs Ensembles in field theories

Eric Vernier (SISSA)

A review of some recent advances, including :

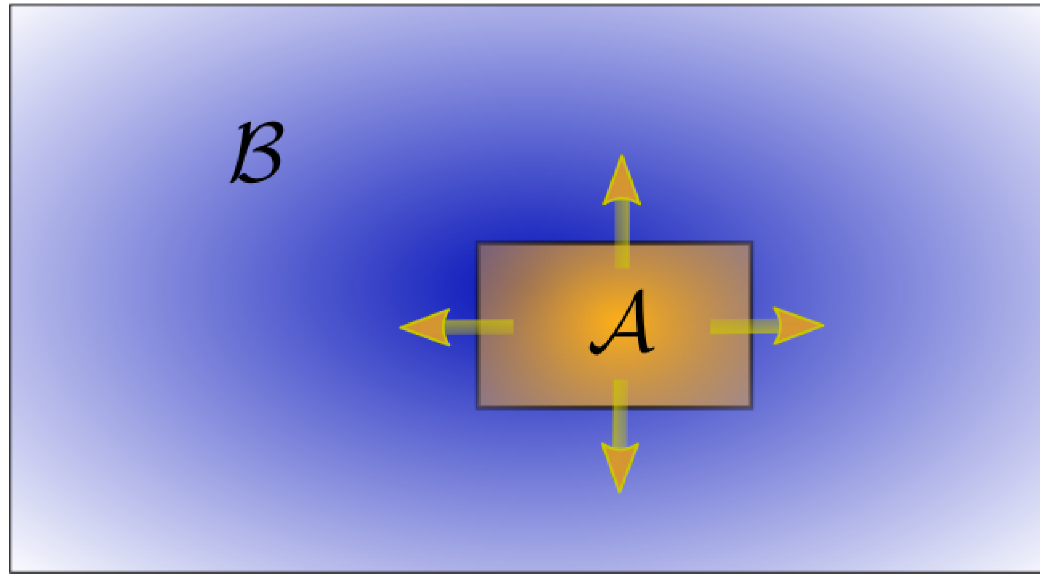
- **EV, A. Cortés-Cubero, arXiv:1609.03220**
- L. Piroli, EV, P. Calabrese, Phys. Rev. B 94 (2016)
- L. Piroli, B. Pozsgay, EV, arXiv:1611.06126
- L. Piroli, EV, P. Calabrese, M. Rigol, arXiv:1611.08859

Annecy, January 5th 2017

Relaxation in isolated quantum systems

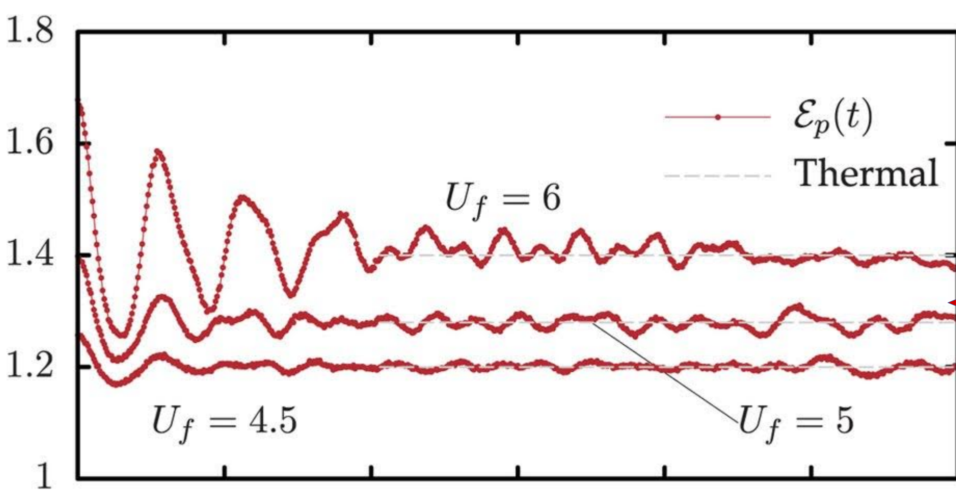
Simplest setup for non-eq. dynamics : **quantum quench**

$$|\psi\rangle = e^{-iHt}|\psi_0\rangle$$



Relaxation at long times :

The whole systems acts as its own heat bath



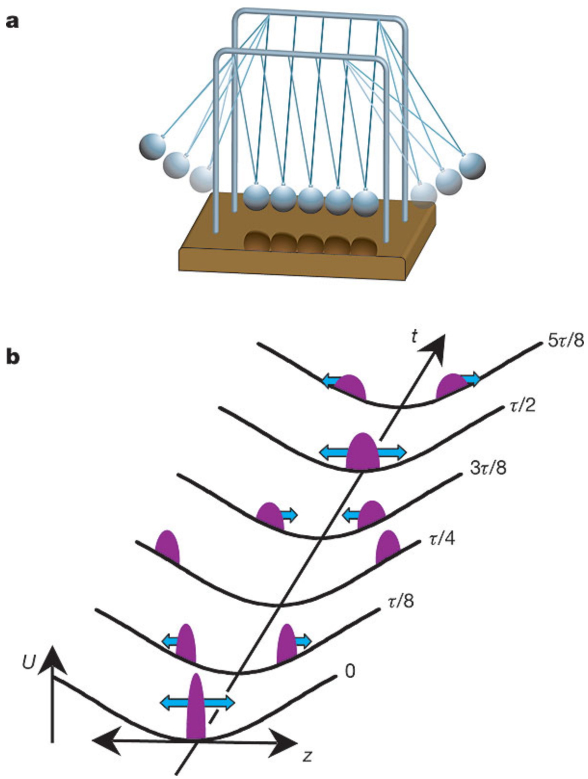
Carleo et. al., 2012

"thermalization"

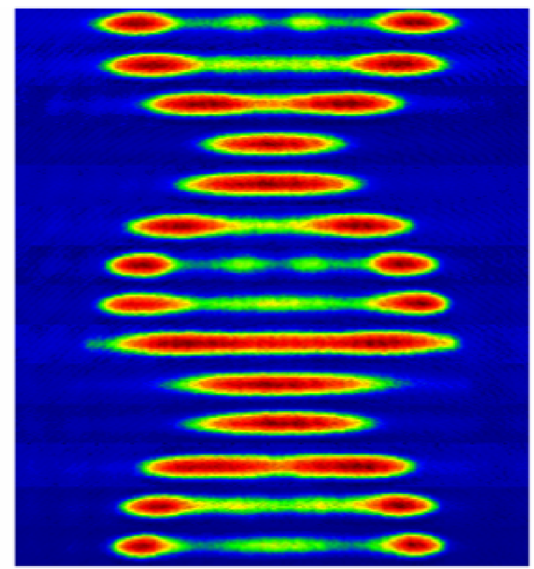
$$\rho = e^{-\beta H}$$

(Gibbs ensemble : equlibrium stat. mech)

Some systems do not thermalize...



Quantum Newton's cradle (with Rb atoms)
(Kinoshita, Wenger, Weiss, Nature 2006)



The reason is that such systems are **integrable**

Many *local* conserved charges,

$$\{Q_n\}$$

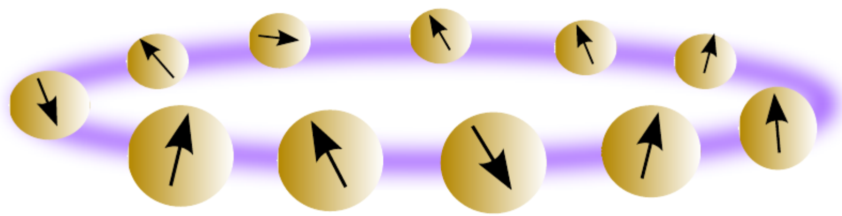
which highly constrain the dynamics

General expectation (Rigol '08) :
relaxation at long times towards a
Generalized Gibbs Ensemble

$$\rho_{\mathrm{GGE}} = e^{-\sum_n \beta_n Q_n}$$

Quantum integrable systems

Quantum spin chains



$$H = \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z$$

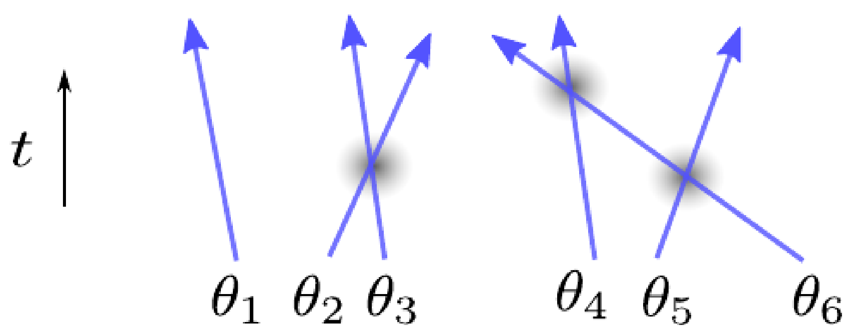
(Heisenberg XXZ chain)

1D Bose/ Fermi gases

$$H = -\sum_j \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<l} \delta(x_j - x_l)$$

(Lieb-Liniger model)

Quantum field theories



$$\mathcal{S} = \int \mathrm{d}^2 x \left(\frac{1}{2} (\partial_\nu \varphi)^2 - 2\mu \cos(\beta\varphi) \right)$$

(Sine-Gordon model)

Several questions

- is the GGE proposal correct ?

If so :

- **what charges** to include ? (remember statement about locality)

- how **quantitatively different** from the thermal ensemble ?

… intense effort these past years, and still some ongoing debate

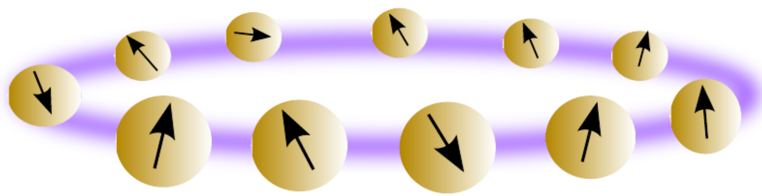
This talk

1. overview of the recent progress; most developments have focused on **spin chains**

2. and new results for **field theories** (Sine-Gordon)

The prototypical integrable quantum spin chain

Heisenberg XXZ chain :



$$H = \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \qquad \Delta = \cos\gamma$$

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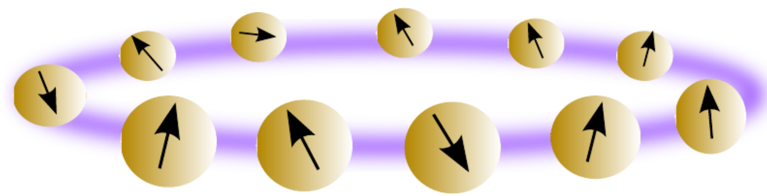
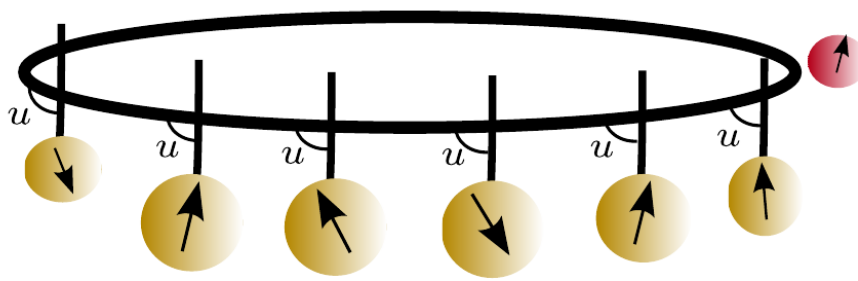


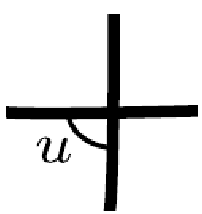
$$H = \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \qquad \Delta = \cos\gamma$$

Integrability : generated from a family of mutually commuting transfer matrices

$T(u) =$



$$= \mathrm{tr}\left(R_{0,L}(u) \ldots R_{0,1}(u)\right)$$



$$= R(u) = \begin{pmatrix} \sin(\gamma-u) & & & \\ & \sin u & \sin\gamma & \\ & \sin\gamma & \sin u & \\ & & & \sin(\gamma-u) \end{pmatrix}$$

(= six-vertex model of statistical mechanics)

$$[T(u), T(v)] = 0$$

The commuting transfer matrices may be used to generate a set of conserved charges :

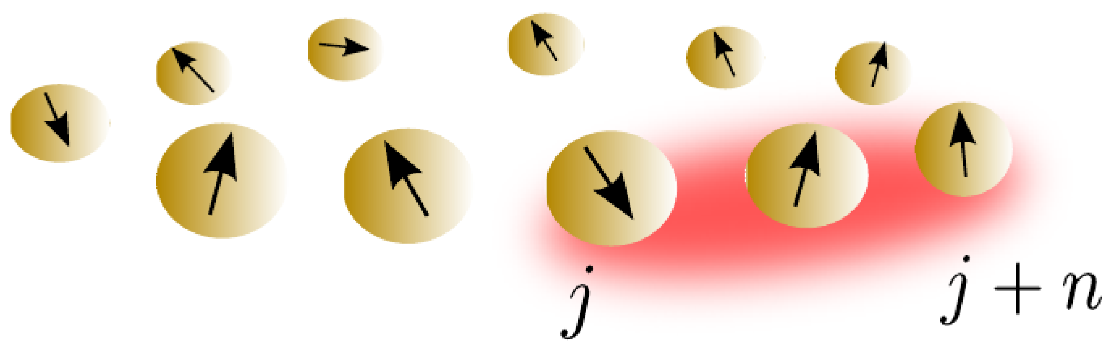
$$Q_n = \frac{\mathrm{d}^n}{\mathrm{d}u^n} \log T(u) \Big|_{u=0} \qquad n = 0, 1, \ldots$$

$$Q_0 = \text{momentum}$$
$$Q_1 = \text{Hamiltonian}$$
...

$$[Q_m, Q_n] = 0$$

Locality :

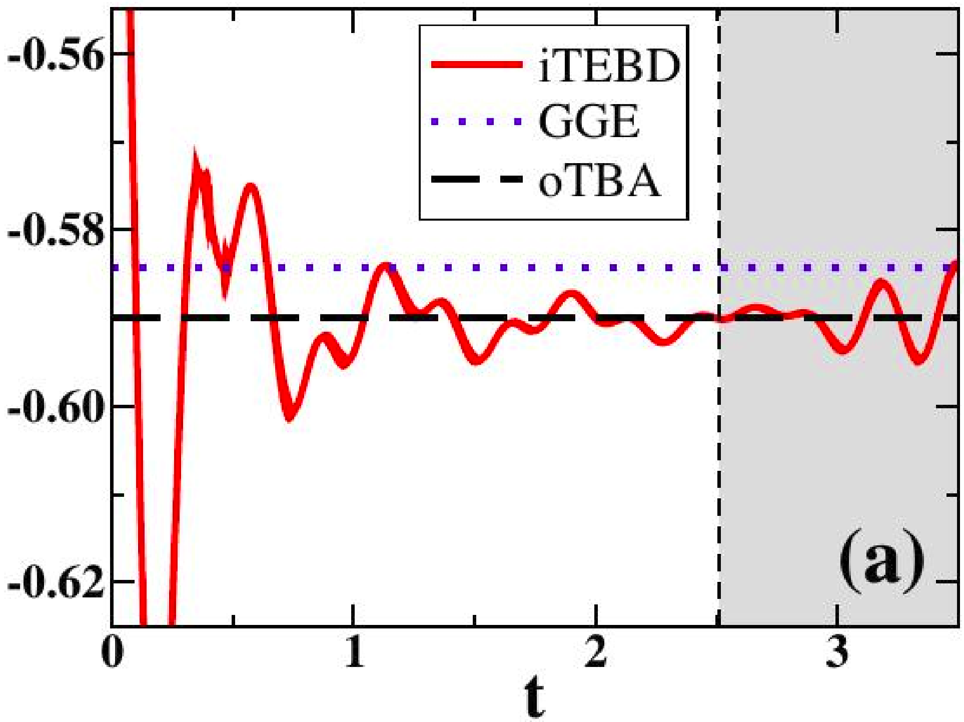
$$Q_n = \sum_j$$



So from there, “local GGE”

$$\rho_{\mathrm{GGE}} = e^{-\sum_n \beta_n Q_n}$$

GGE from the local charges vs numerics (DMRG, iTEBD)

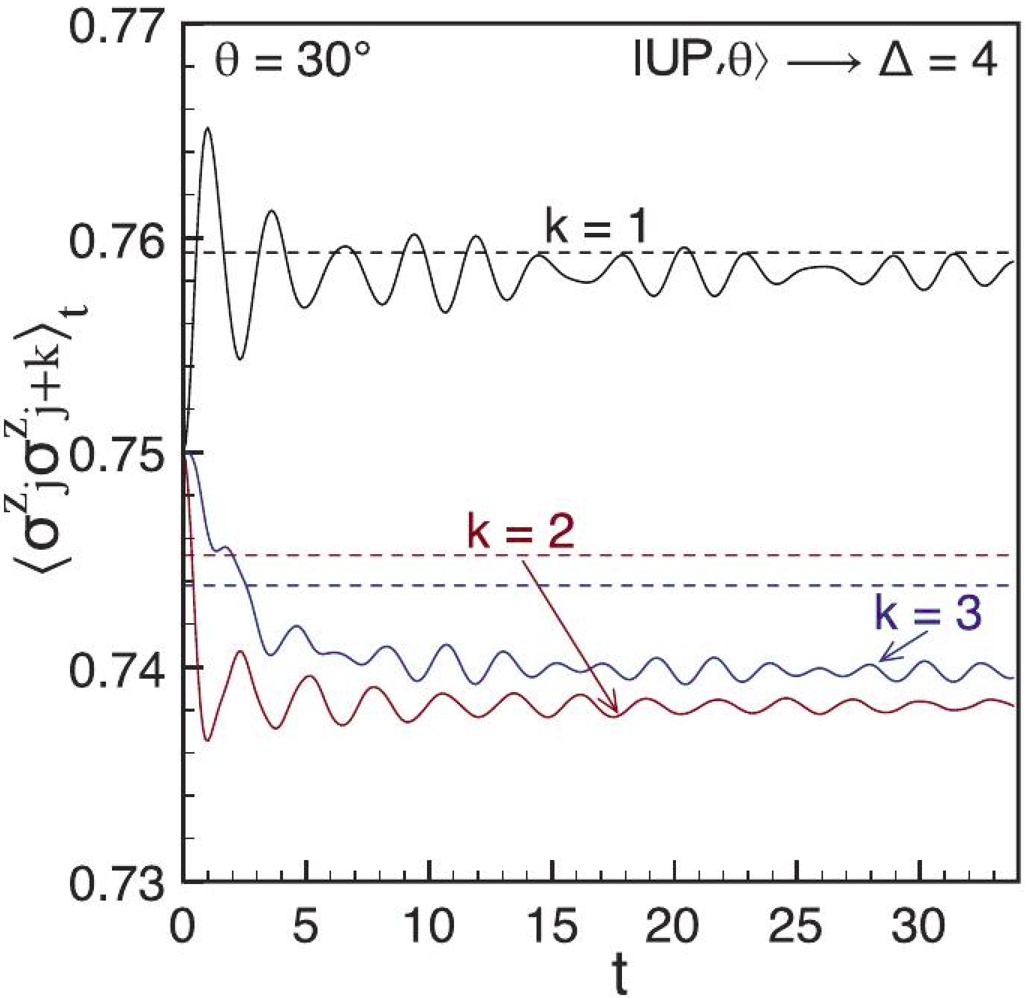


B. Pozsgay, M. Mestyán, M. A. Werner,
M. Kormos, G. Zaránd, G. Takács
PRL 113 (2014)

B. Wouters et. al. PRL 113 (2014)

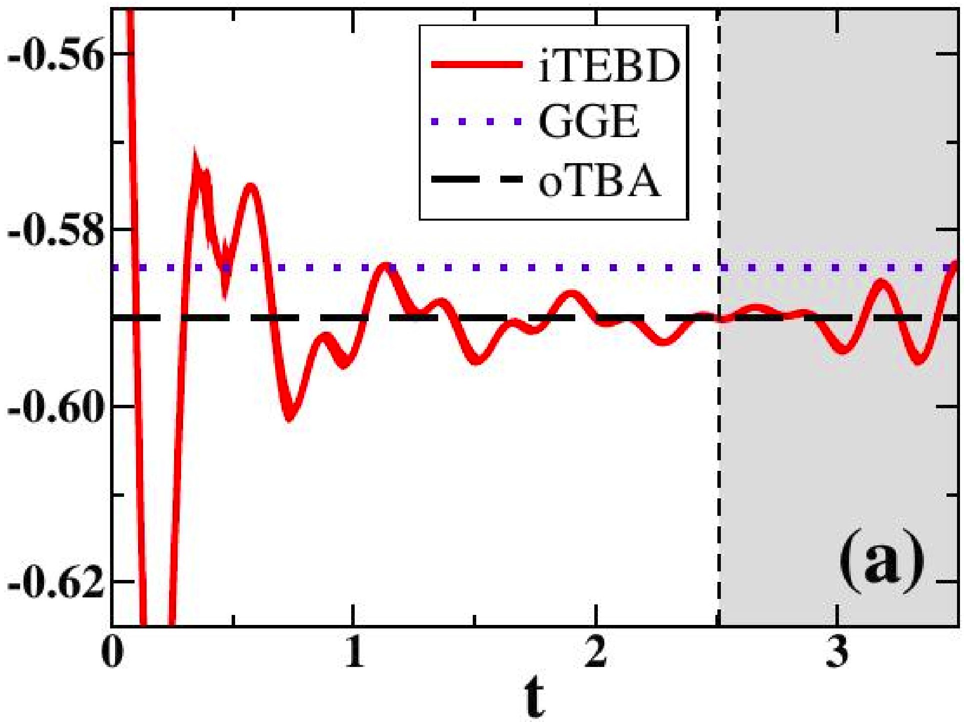
B. Pozsgay, JSTAT 2014

Fagotti Collura Essler Calabrese 2014



GGE from the local charges vs numerics (DMRG, iTEBD)

... not there yet !!



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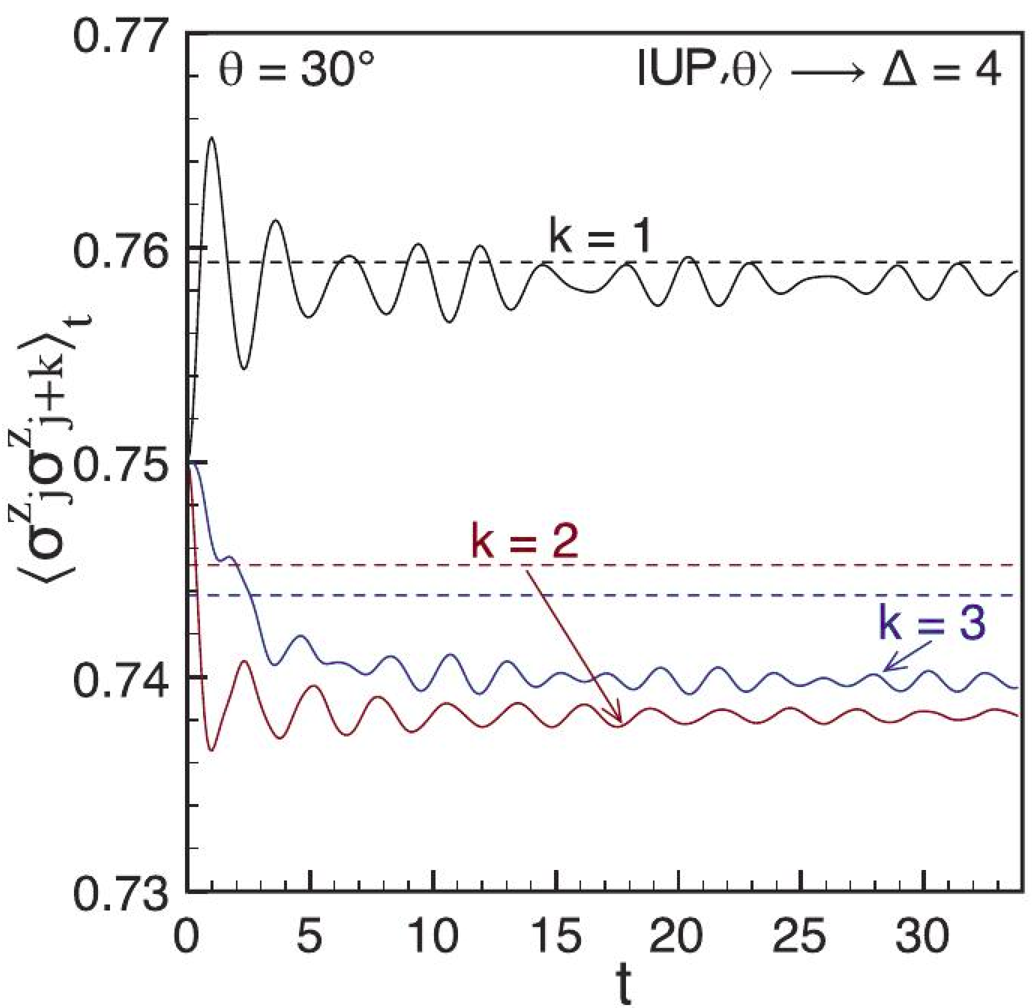
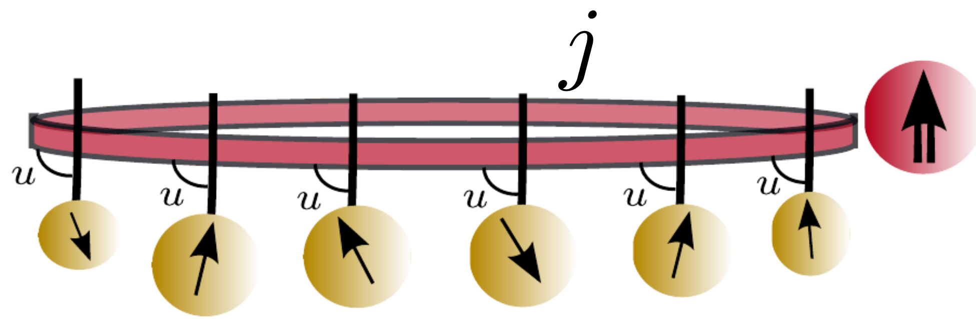


Solution : more charges required (Prosen, Ilievski,… 2015)

More conserved charges exist (Prosen 2011), derived from transfer matrices with **higher auxilliary spin j/2**

$$T_j(u) =$$

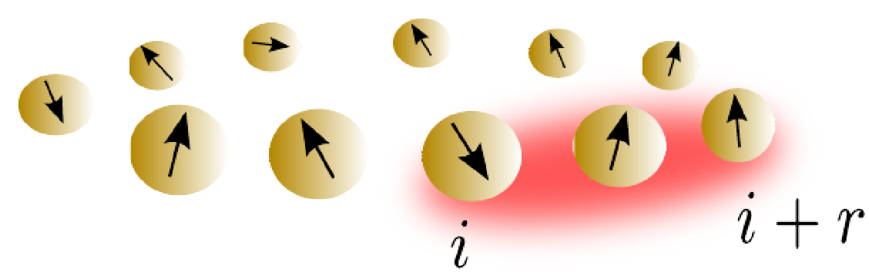


$$T_1(u) = T(u)$$

$$Q_{j,n} = \frac{\mathrm{d}^n}{\mathrm{d}u^n} \log T_j(u)\Big|_{u=0}$$

These satisfy a weaker form of locality : **quasilocality**

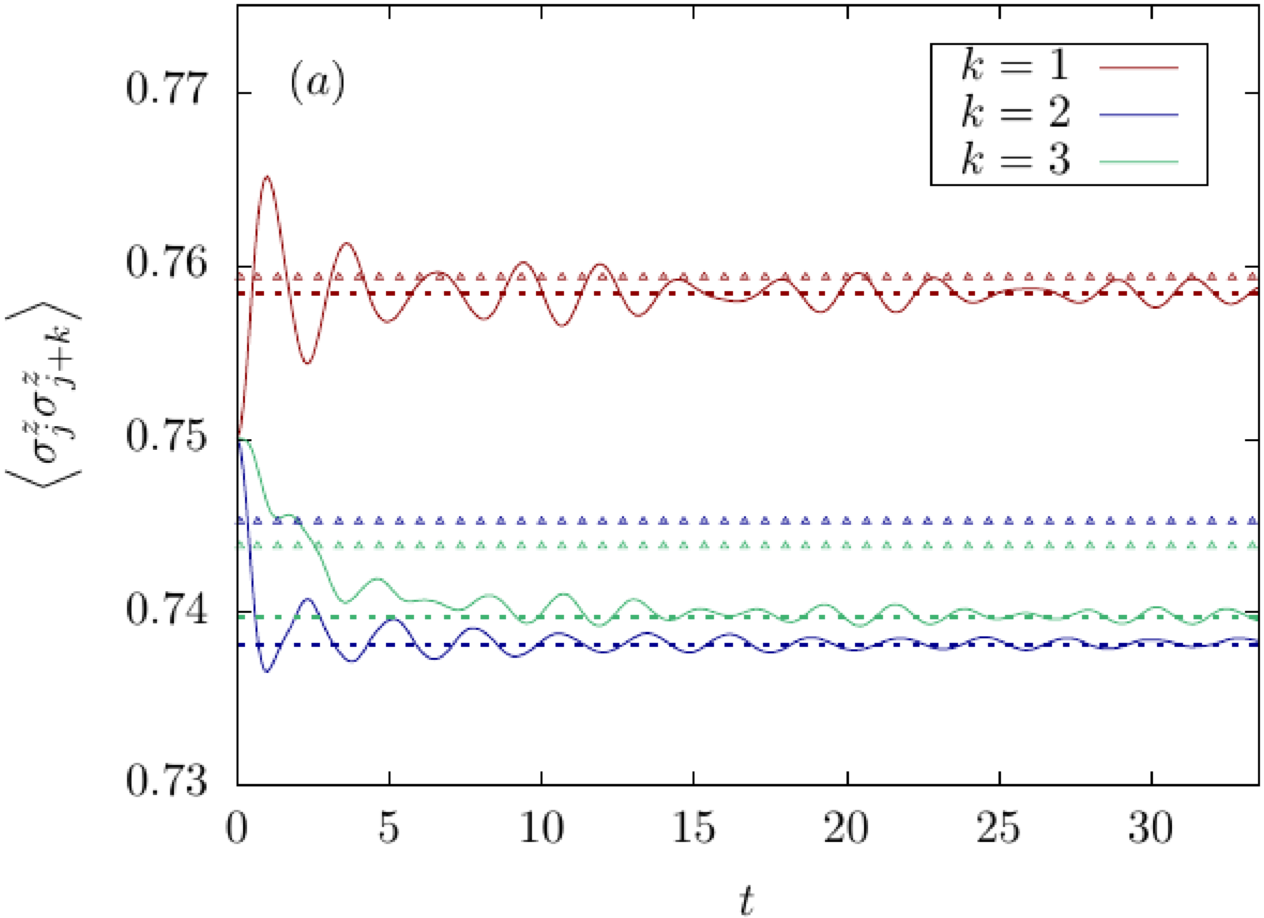
$$Q_{j,n} = \sum_r e^{-r/\xi} \sum_i$$



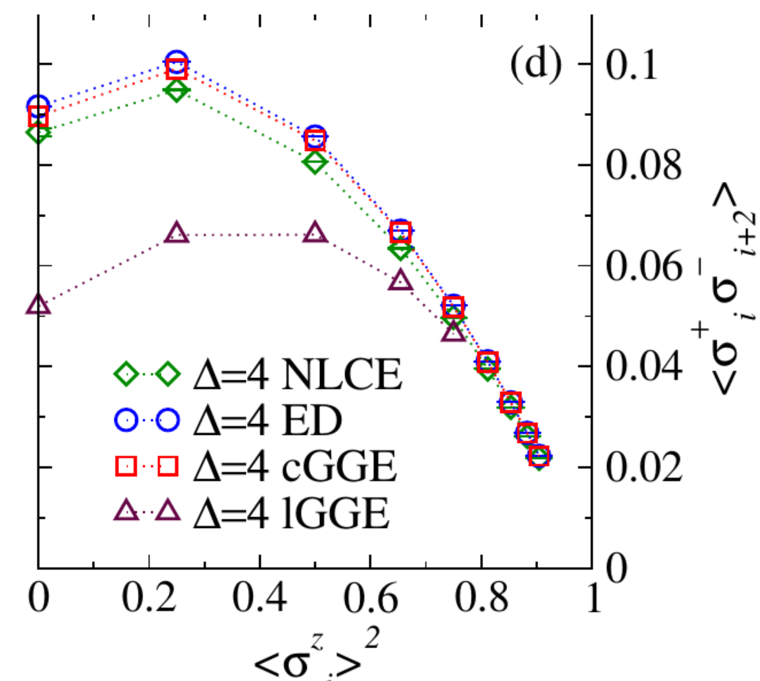
“Complete GGE”: $\rho_{\mathrm{GGE}} = e^{-\sum_j \sum_n \beta_{j,n} Q_{j,n}}$

... works !

Ilievski et. al. PRL 115, 2015



(L. Piroli, EV, P. Calabrese 2016)



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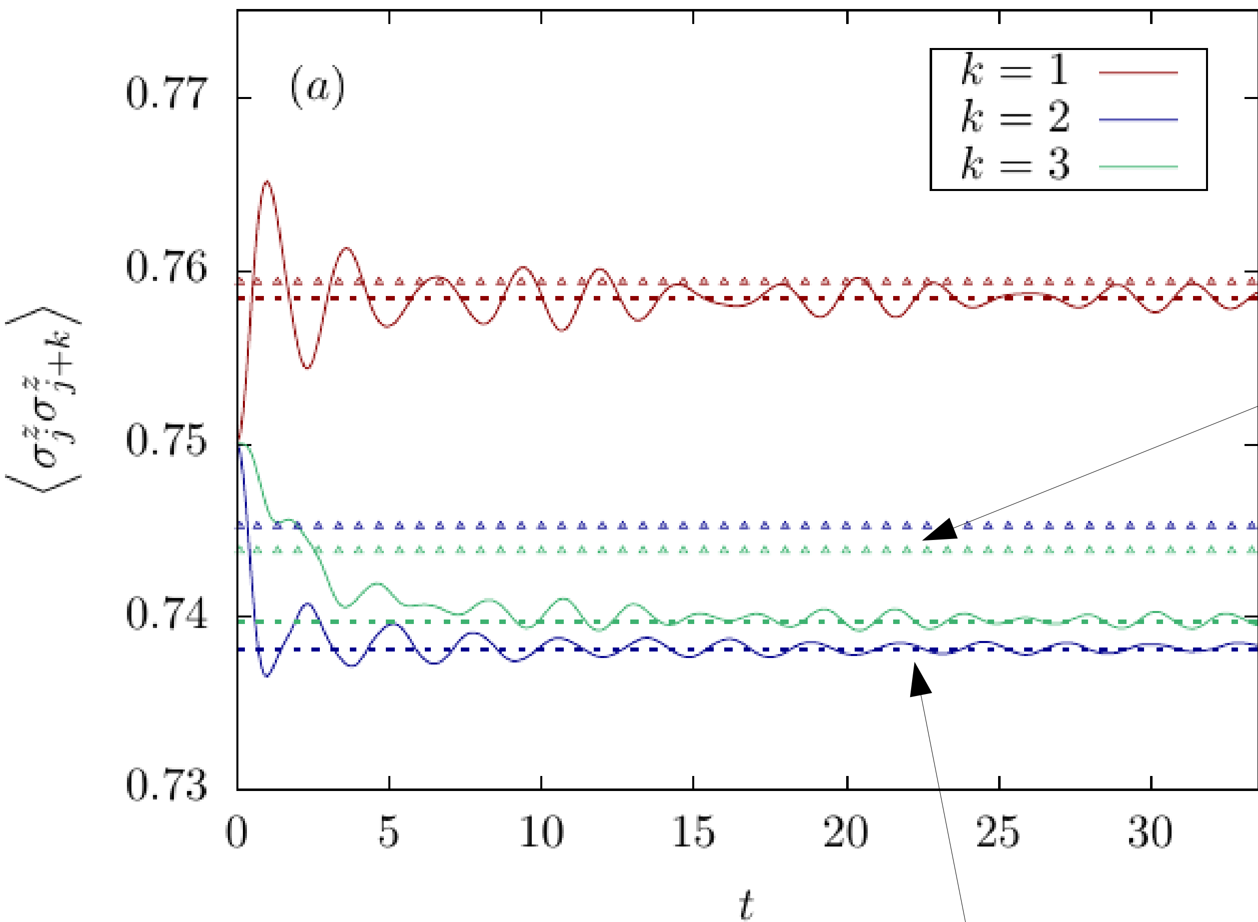
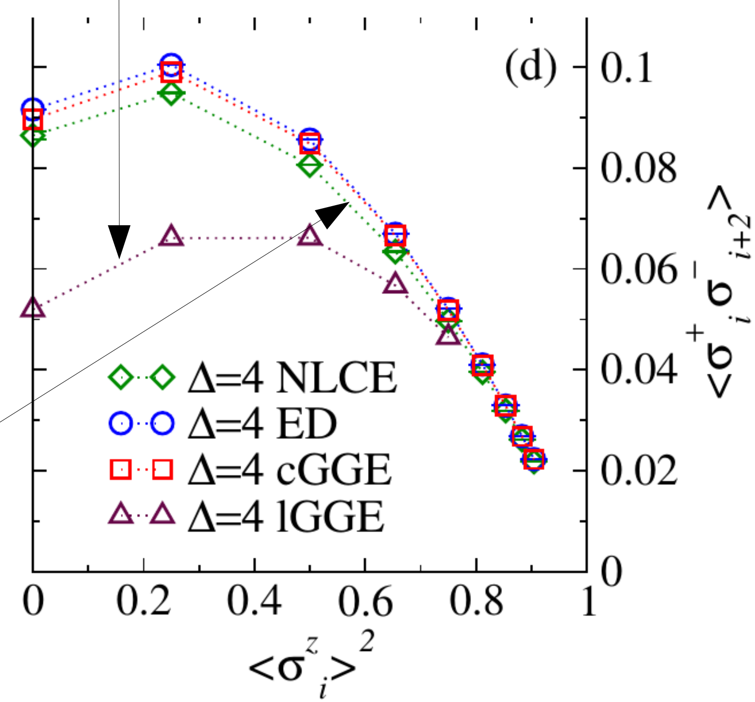


(L. Piroli, EV, P. Calabrese 2016)

Local GGE

Complete GGE



(L. Piroli, EV, P. Calabrese, M. Rigol 2016)

Interpretation from “representative eigenstates”

Quench-action approach (Caux Essler ‘13) : the local stationary state can be described by a single **representative eigenstate** $|\Phi\rangle$

$$\lim_{t\to\infty} \frac{\langle\Psi_0|\mathcal{O}(t)|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} = \frac{\mathrm{Tr}(\mathcal{O}\rho_{\mathrm{GGE}})}{\mathrm{Tr}\rho_{\mathrm{GGE}}} = \frac{\langle\Phi|\mathcal{O}|\Phi\rangle}{\langle\Phi|\Phi\rangle}$$

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Bethe ansatz construction of eigenstates (for the XXZ chain) :

$$\left|\{\lambda_i\}_{i=1,\dots r}\right\rangle \qquad \left(\frac{\sinh\left(\lambda_k + i\frac{\gamma}{2}\right)}{\sinh\left(\lambda_k - i\frac{\gamma}{2}\right)}\right)^L = \prod_{j(\neq k)} \frac{\sinh\left(\lambda_k - \lambda_j + i\gamma\right)}{\sinh\left(\lambda_k - \lambda_j - i\gamma\right)}$$

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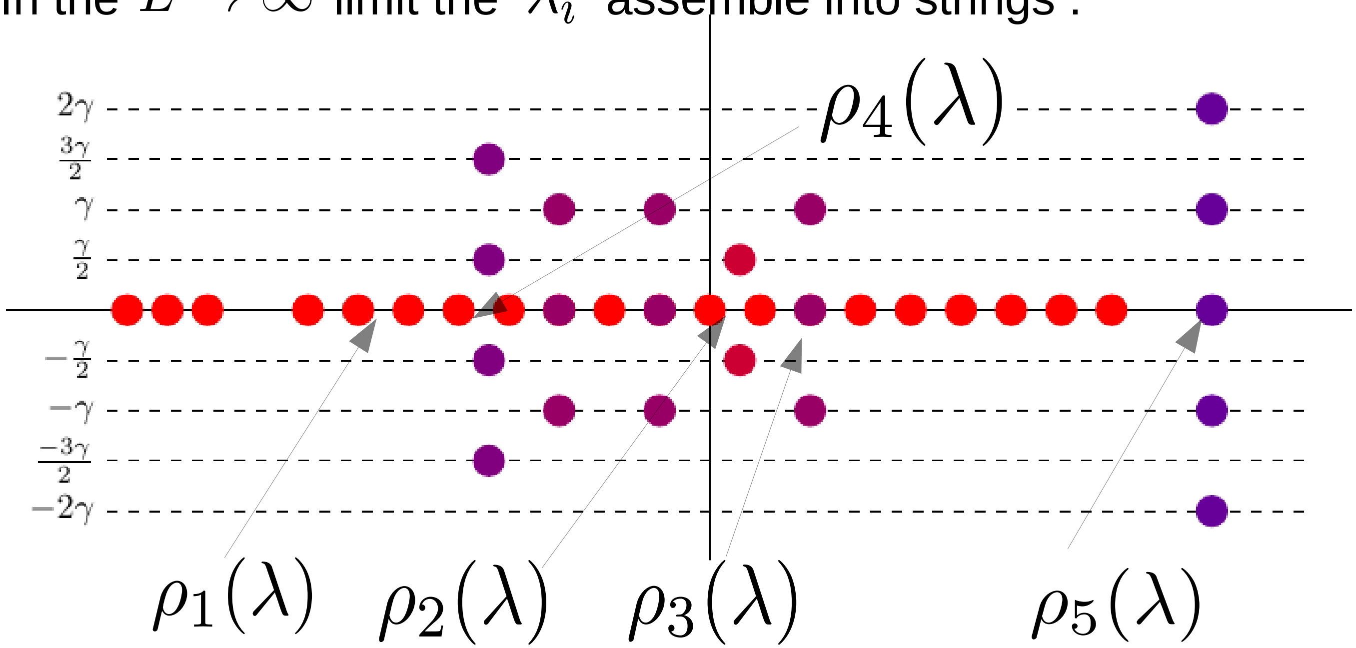
In the $L \to \infty$ limit the λ_i assemble into strings :



So the stationary state is specified by a set of densities

$$\{\rho_n(\lambda), \rho_n^h(\lambda)\}$$

“String-charge duality” (Ilievski et al 2015) :

$$X_j(\lambda) = \frac{1}{2i\pi}\partial_\lambda \log T_j(i\lambda)$$ generators of the $Q_{j,n}$, quasilocal for $|\mathrm{Im}\lambda| < \frac{\gamma}{2}$

$$\boxed{\begin{aligned} \rho_j(\lambda) &= X_j\left(\lambda + i\frac{\gamma}{2}\right) + X_j\left(\lambda - i\frac{\gamma}{2}\right) - X_{j+1}(\lambda) - X_{j-1}(\lambda) \\ \rho_j^h(\lambda) &= \frac{1}{\pi}\frac{\sin(j\gamma)}{\cos(j\gamma) - \cosh(2\lambda)} - X_j\left(\lambda + i\frac{\gamma}{2}\right) - X_j\left(\lambda - i\frac{\gamma}{2}\right) \end{aligned}}$$

Linear relations between charges $X_j(\lambda)$ and densities $\rho_j(\lambda), \rho_j^h(\lambda)$

(so the local charges (j=1) were only sensitve to one type of strings)

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Linear relations between charges $X_j(\lambda)$ and densities $\rho_j(\lambda), \rho_j^h(\lambda)$

(so the local charges (j=1) were only sensitve to one type of strings)

In practice, starting from a state $|\psi_0\rangle$,

1. Evaluate expectation values of charges on $|\psi_0\rangle$
2. Compute densities
3. Compute local observables, entropies, etc...

Technical but important aspect: for certain initial states the densities satisfy a **Y-system**

$$\eta_n\left(\lambda + i\frac{\gamma}{2}\right)\eta_n\left(\lambda - i\frac{\gamma}{2}\right) = (1 + \eta_{n-1}(\lambda))(1 + \eta_{n+1}(\lambda)) \qquad \eta_n(\lambda) = \frac{\rho_n^h(\lambda)}{\rho_n(\lambda)}$$

see L. Piroli, B. Pozsgay, EV, arXiv:1611.06126

Summary so far

For integrable spin chains, complete GGE

$$\rho_{\mathrm{GGE}} = e^{-\sum_j \sum_n \beta_{j,n} Q_{j,n}}$$

$j = 1$ local charges

$j > 1$ quasilocal charges, in correspondence with the densities of strings

...what about field theories ?

2nd part of this talk : integrable quantum field theories

EV and A. Cortés-Cubero, arXiv:1609.03220

sine-Gordon model :

$$\mathcal{S} = \int \mathrm{d}^2 x \left(\frac{1}{2} \left(\partial_\nu \varphi \right)^2 - 2\mu \cos(\beta \varphi) \right)$$

$$\frac{\beta^2}{8\pi} = \frac{p}{p+1}$$

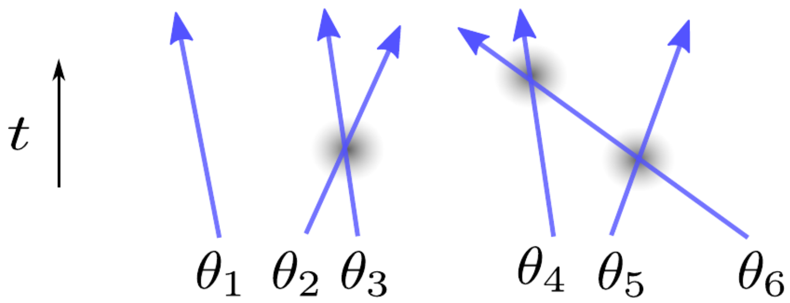
Paradigmatic model of integrable interacting quantum field theory

Relevant in many physical contexts : low-energy description of spin chains, spin ladders, cold atomic gases

Quantum description involves 2 kinds of particles : **soliton, antisoliton**

We consider the "repulsive regime" $p > 1$: no bound states !

For theories with only one type of particle or diagonal scattering (free theories, sinh-Gordon…), quantization of the momenta $\{\theta_i\}$



$$e^{iLm\sinh\theta_j} = \pm \prod_{k(\neq j)} S(\theta_j - \theta_k)$$

$$E = \sum_i m_i \cosh\theta_i$$

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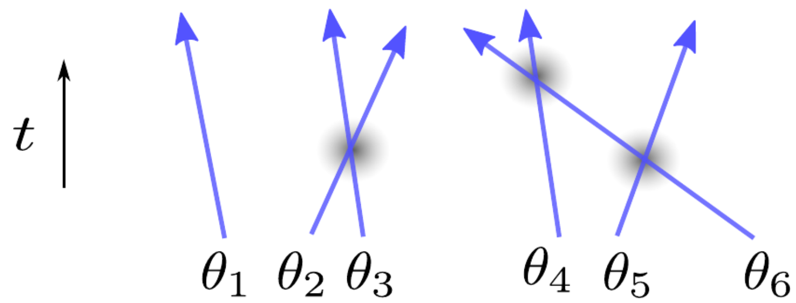

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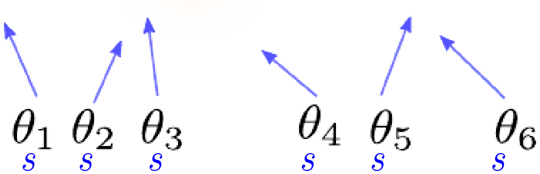
$$E = \sum_i m_i \cosh\theta_i$$

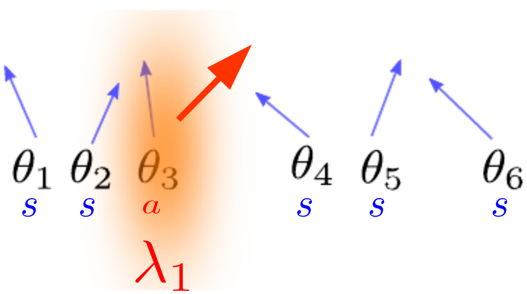
However in SG, **two types of particles** (soliton, antisoliton) with **non-diagonal scattering**

the states can be parametrized in terms of auxilliary particles (**magnons**) :

$$|\theta_1, \theta_2 \ldots\rangle =$$



$$|\theta_1, \theta_2 \ldots \lambda_1\rangle =$$



$$\mathrm{e}^{imL\sinh\theta_j} = \prod_{k(\neq j)}^{N} S_0(\theta_j - \theta_k) \prod_{k=1}^{Q} \frac{\sinh\frac{1}{p}(\theta_j - \lambda_k + i\pi/2)}{\sinh\frac{1}{p}(\theta_j - \lambda_k - i\pi/2)}$$

etc...

$$\prod_{k=1}^{N} \frac{\sinh\frac{1}{p}(\lambda_j - \theta_k + i\pi/2)}{\sinh\frac{1}{p}(\lambda_j - \theta_k - i\pi/2)} = \prod_{k(\neq j)}^{Q} \frac{\sinh\frac{1}{p}(\lambda_j - \lambda_k + i\pi)}{\sinh\frac{1}{p}(\lambda_j - \lambda_k - i\pi)},$$

A GGE in sine-Gordon ?

Local conserved charges in SG :

$$H|\theta_1,\ldots\rangle = \left(\sum_i m\cosh(\theta_i)\right)|\theta_1,\ldots\rangle$$

$$P|\theta_1,\ldots\rangle = \left(\sum_i m\sinh(\theta_i)\right)|\theta_1,\ldots\rangle$$

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$$Q_s^+|\theta_1,\ldots\rangle = \left(\sum_i m\cosh(s\theta_i)\right)|\theta_1,\ldots\rangle$$

$$Q_s^-|\theta_1,\ldots\rangle = \left(\sum_i m\sinh(s\theta_i)\right)|\theta_1,\ldots\rangle$$

$$s = 1, 3, 5, \ldots$$

(Lorentz spin)

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(Lorentz spin)

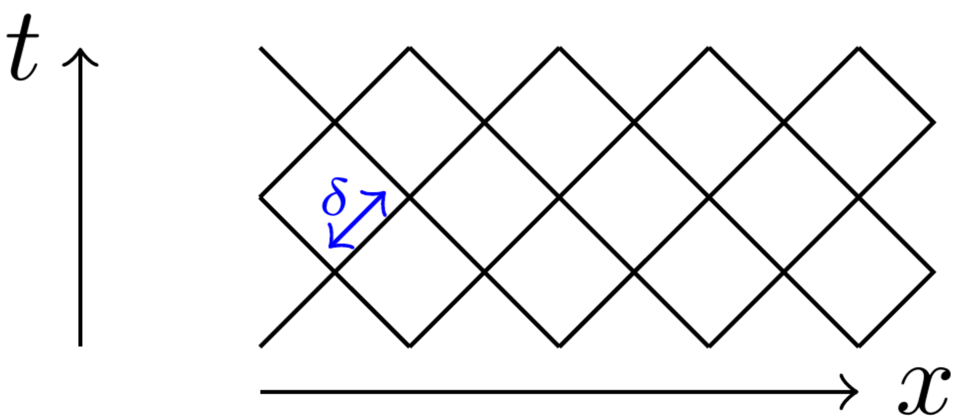
However these charges are blind to the configurations of magnons

So a GGE built out of them cannot fully describe the stationary state
(e.g: expectation values of charged vertex operators)

This of course does not prevent the existence of elevant results for neutral operators :
V. Gritsev, E. Demler, M. Lukin and A. Polkovnikov 2007
B. Bertini, D. Schuricht, F. Essler 2014
M. Kormos, G. Zaránd 2015
C. P. Moca, M. Kormos, G. Zaránd, 2016

“Light-cone discretization”

(Destri & De Vega 1987, starting from the equivalent massive Thirring model)

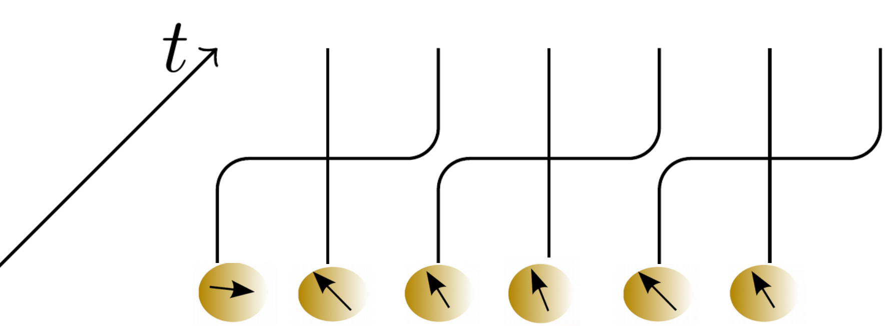


On each link, either one fermion (>) or empty (<)

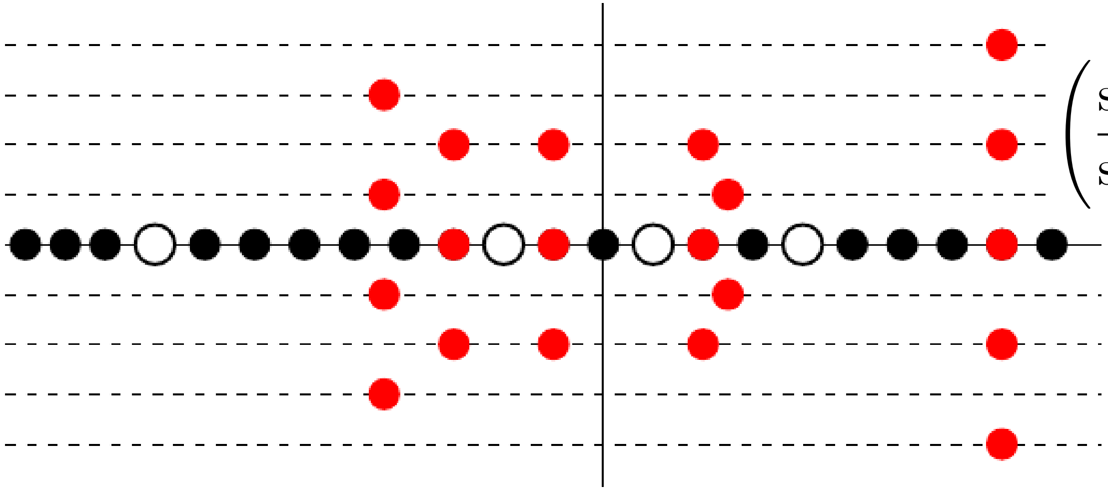
Weights are those of the six-vertex model

This is nothing but an inhomogeneous version of the XXZ transfer matrix introduced previously !

$$T(u) = \mathrm{tr}\left(R_{0,N}(u + i\Lambda/2) R_{0,N-1}(u - i\Lambda/2) \ldots R_{0,1}(u - i\Lambda/2)\right)$$



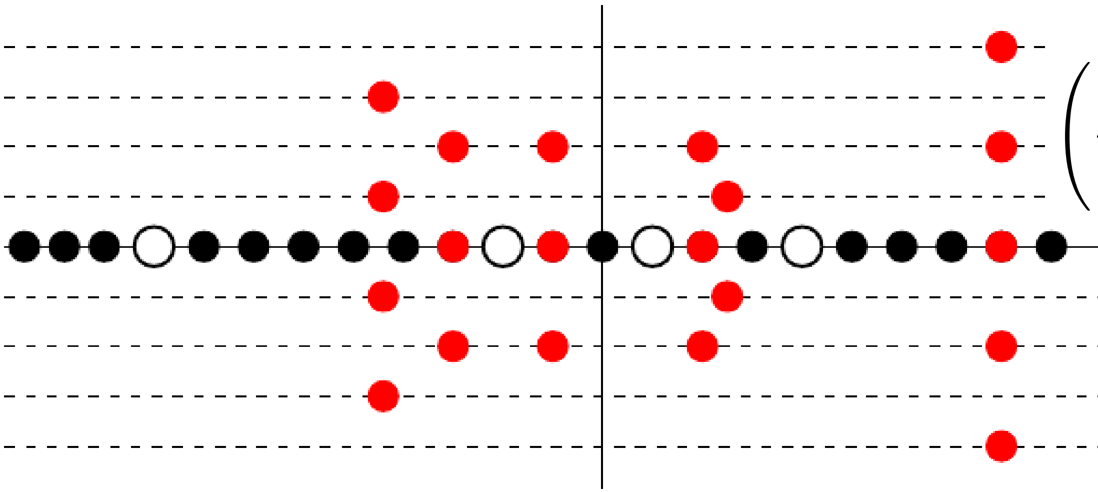
$$\begin{aligned} P_{\text{light cone}} &= \delta^{-1} i \log\left(T\left(i\frac{\Lambda}{2}\right) T\left(-i\frac{\Lambda}{2}\right)\right) \\ H_{\text{light cone}} &= \delta^{-1} i \log\left(T\left(i\frac{\Lambda}{2}\right) T\left(-i\frac{\Lambda}{2}\right)^{-1}\right) \end{aligned}$$



$$\left(\frac{\sinh\left(\lambda_i - \frac{\Lambda}{2} + i\frac{\gamma}{2}\right)}{\sinh\left(\lambda_i - \frac{\Lambda}{2} - i\frac{\gamma}{2}\right)}\right)^{\frac{N}{2}} \left(\frac{\sinh\left(\lambda_i + \frac{\Lambda}{2} + i\frac{\gamma}{2}\right)}{\sinh\left(\lambda_i + \frac{\Lambda}{2} - i\frac{\gamma}{2}\right)}\right)^{\frac{N}{2}}$$

$$= \prod_{j(\neq i)} \frac{\sinh\left(\lambda_i - \lambda_j - i\gamma\right)}{\sinh\left(\lambda_i - \lambda_j + i\gamma\right)}$$

$$\gamma = \frac{\pi}{p+1}$$



$$\left(\frac{\sinh\left(\lambda_i-\frac{\Lambda}{2}+i\frac{\gamma}{2}\right)}{\sinh\left(\lambda_i-\frac{\Lambda}{2}-i\frac{\gamma}{2}\right)}\right)^{\frac{N}{2}}\left(\frac{\sinh\left(\lambda_i+\frac{\Lambda}{2}+i\frac{\gamma}{2}\right)}{\sinh\left(\lambda_i+\frac{\Lambda}{2}-i\frac{\gamma}{2}\right)}\right)^{\frac{N}{2}}=\prod_{j(\neq i)}\frac{\sinh\left(\lambda_i-\lambda_j-i\gamma\right)}{\sinh\left(\lambda_i-\lambda_j+i\gamma\right)}$$

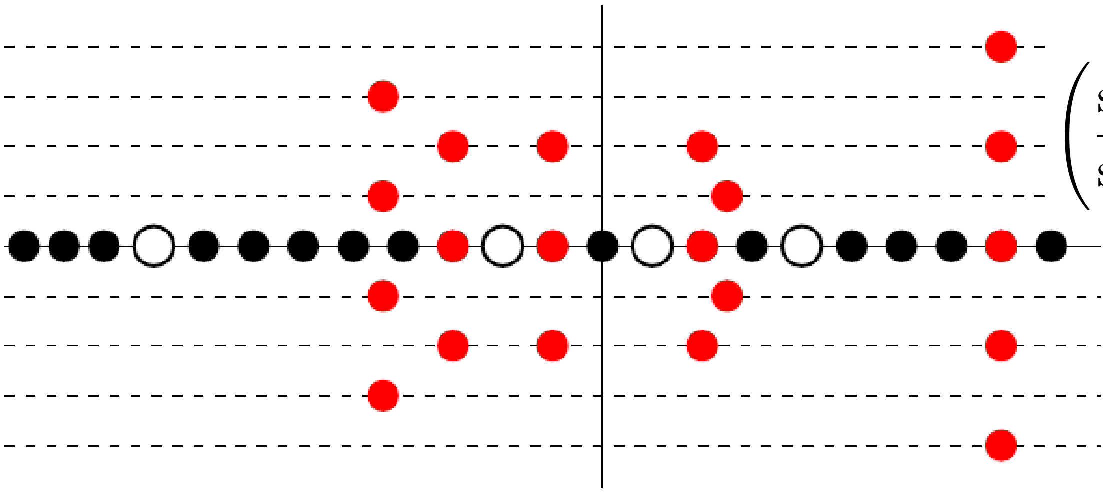
$$\gamma=\frac{\pi}{p+1}$$

Scaling limit

$N\to\infty\,,\delta\to 0\,,\Lambda\to\infty$

$N\delta=L$ fixed

$m=\delta^{-1}e^{-(p+1)\frac{\Lambda}{2}}$ fixed



$$\left(\frac{\sinh\left(\lambda_i - \frac{\Lambda}{2} + i\frac{\gamma}{2}\right)}{\sinh\left(\lambda_i - \frac{\Lambda}{2} - i\frac{\gamma}{2}\right)}\right)^{\frac{N}{2}} \left(\frac{\sinh\left(\lambda_i + \frac{\Lambda}{2} + i\frac{\gamma}{2}\right)}{\sinh\left(\lambda_i + \frac{\Lambda}{2} - i\frac{\gamma}{2}\right)}\right)^{\frac{N}{2}}$$

$$\gamma = \frac{\pi}{p+1} \qquad = \prod_{j(\neq i)} \frac{\sinh\left(\lambda_i - \lambda_j - i\gamma\right)}{\sinh\left(\lambda_i - \lambda_j + i\gamma\right)}$$

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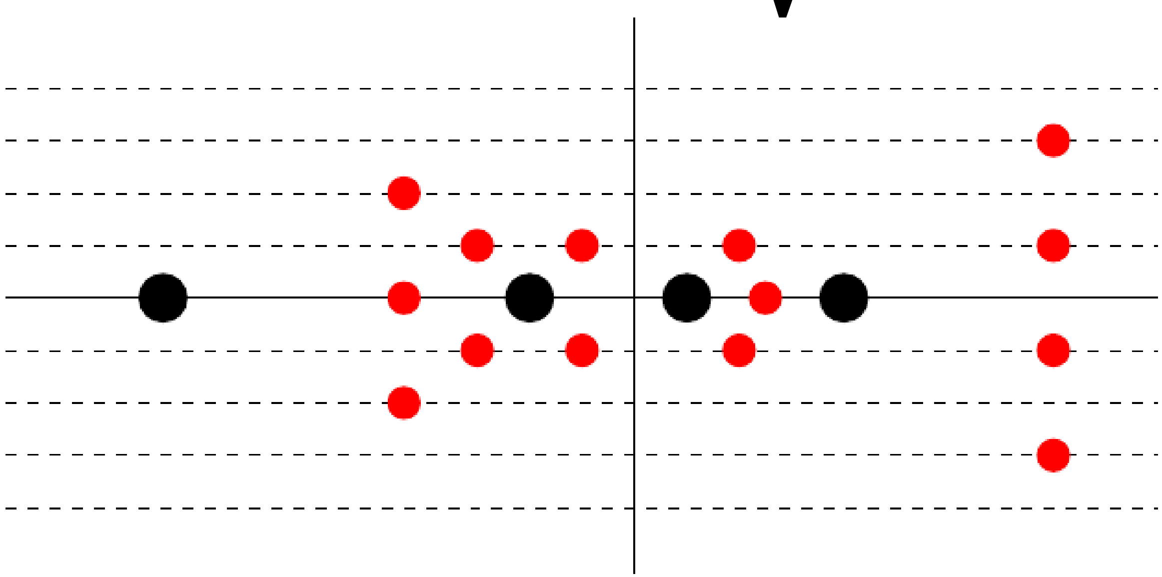


XXZ ground state → SG vacuum

XXZ real holes → SG particles

XXZ complex roots → SG magnons

Building the charges from the light-cone lattice

The XXZ construction of (quasi)local charges can be generalized to the light-cone lattice

$$X_j(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \log T_j(i\lambda)$$

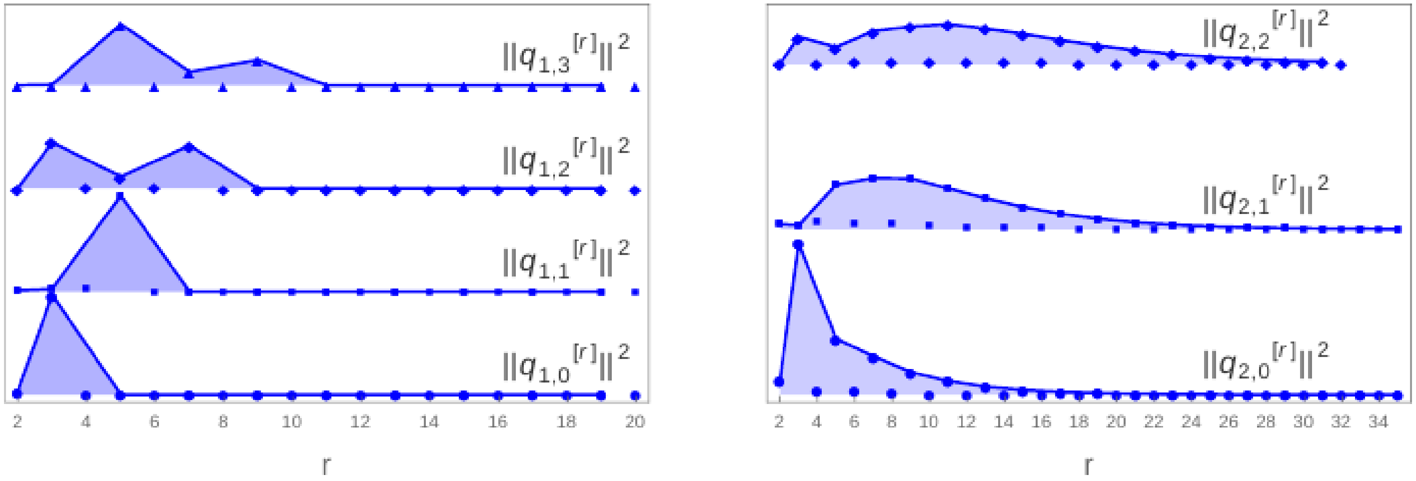
$$Q^{\pm}_{j,n} = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left(X_j\left(\lambda + \frac{\Lambda}{2}\right) \mp X_j\left(-\lambda - \frac{\Lambda}{2}\right)\right)\Bigg|_{\lambda=0}$$

$Q^{+}_{1,0}$ = Hamiltonian
$Q^{-}_{1,0}$ = Momentum

(quasi)local on the lattice

$$Q^{\pm}_{j,n} = \sum_{r=2}^{N} \sum_{i} \left(q^{\pm}_{j,n}\right)^{[r]}$$

$$||(q_{j,n})^{[r]}||_{\mathrm{HS}} \sim e^{-r/\xi_j}$$



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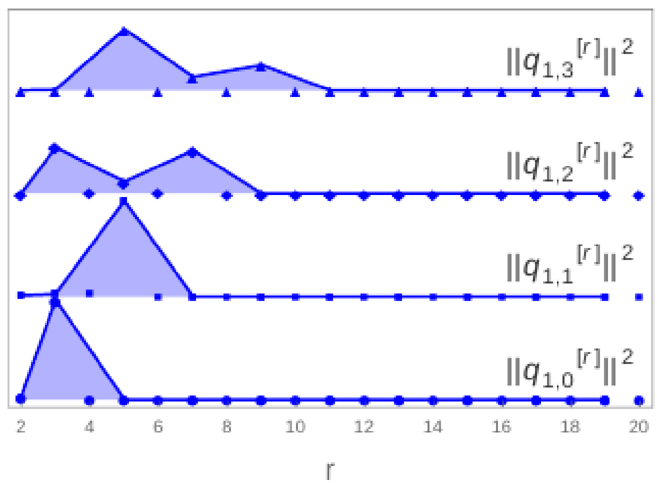

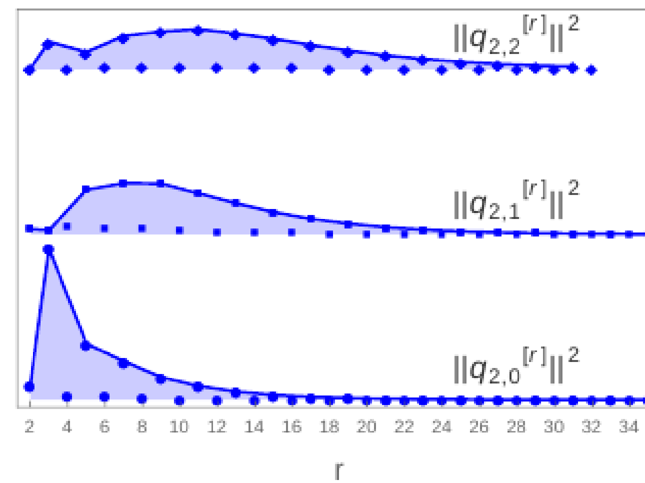

Scaling limit : $N \to \infty \,, \delta \to 0 \,, \Lambda \to \infty \qquad N\delta = L$ fixed

ξ_j remains finite as $\Lambda \to \infty$

So $\xi_j^{\mathrm{SG}} = \delta \xi_j \to 0$

local field theory charges !

What is the action of our charges on the SG states ?

$$X_j(\lambda) = \sum_{\{\lambda_k^{\mathrm{XXZ}}\}} \frac{\sin j\gamma}{\cosh(2(\lambda - \lambda_k^{\mathrm{XXZ}})) - \cos\frac{j\gamma}{2}}$$

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$$\Big\downarrow \text{ scaling limit}$$

$$\frac{Q^{\pm}_{j,n} - Q^{\pm\mathrm{vac}}_{j,n}}{L} \xrightarrow{L\to\infty} \underbrace{\int \mathrm{d}\theta \rho(\theta) q^{\pm}_{j,n}(\theta)}_{\text{particles}} + \sum_m \underbrace{\int \mathrm{d}\lambda \rho_m(\lambda) q^{\pm}_{j,n,m}(\lambda)}_{\text{magnons}}$$

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scaling limit

$$\frac{Q_{j,n}^{\pm} - Q_{j,n}^{\pm\mathrm{vac}}}{L} \xrightarrow{L\to\infty} \int \mathrm{d}\theta \rho(\theta) q_{j,n}^{\pm}(\theta) + \sum_m \int \mathrm{d}\lambda \rho_m(\lambda) q_{j,n,m}^{\pm}(\lambda)$$

particles magnons

1. Local charges (j=1)

$$q_{1,n}^{\pm}(\theta) = \frac{4\pi}{\gamma} \sum_{k=0}^{\infty} (-1)^k (2k+1)^n (\delta M)^{2k+1} c_{\pm}\left((2k+1)\theta\right) \qquad \begin{aligned} c_+(\theta) &= \cosh\theta \\ c_-(\theta) &= \sinh\theta \end{aligned}$$

Leading term : $q_{1,n}^{\pm}(\theta) \sim m\cos\theta\,,\, m\sin\theta$ **= energy, momentum**

By forming linear combinations, $m^3\cos(3\theta)\,,\, m^3\sin(3\theta)\,,m^5\cos(5\theta)\,,\, m^5\sin(5\theta)\,,$ ...

= all known SG local conserved charges

(Reshetikhin & Saleur '94)

$$q_{1,n,m}^{\pm} = 0$$

Blind to the magnon structure

2. “Quasilocal” charges (j>1)

$$q^{\pm}_{j,n}(\theta) = \frac{4\pi}{\gamma}\sum_{k=0}^{\infty}(-1)^k(2k+1)^n \frac{\sin\left(\frac{\pi(\pi-j\gamma)}{2\gamma}(2k+1)\right)}{\sin\left(\frac{\pi(\pi-\gamma)}{2\gamma}(2k+1)\right)}(\delta M)^{2k+1}c_{\pm}\left((2k+1)\theta\right)$$

$$+ \frac{4\pi}{\pi-\gamma}\sum_{k=0}^{\infty}(-1)^k\left(\frac{2k\gamma}{\pi-\gamma}\right)^n \frac{\sin\left(\frac{\pi(\pi-j\gamma)}{\pi-\gamma}k\right)}{\sin\left(\frac{\pi(\pi-\gamma)}{\pi-\gamma}k\right)}(\delta M)^{\frac{2k\gamma}{\pi-\gamma}}c_{\pm}\left(\frac{2k\gamma}{\pi-\gamma}\theta\right)$$

fractional Lorentz spin !

$$q^{\pm}_{j,n,m}(\lambda) = \frac{4\pi}{\pi-\gamma}\sum_{k=0}^{\infty}(-1)^k\left(\frac{2k\gamma}{\pi-\gamma}\right)^n \frac{\sin\left(\frac{\pi(\pi-j\gamma)}{\pi-\gamma}k\right)\sin\left(\frac{\pi(\pi-(m+1)\gamma)}{\pi-\gamma}k\right)}{\sin\left(\frac{\pi^2}{\pi-\gamma}k\right)}(\delta M)^{\frac{2k\gamma}{\pi-\gamma}}c_{\pm}\left(\frac{2k\gamma}{\pi-\gamma}\lambda\right)$$

not blind to the magnon structure

Charges of fractional Lorentz spin were known in the sine-Gordon litterature :

- Bernard LeClair 1990 : non-commuting, non-local quantum group charges

- Bazhanov Lukyanov A.B. Zamolodchikov 1996 : commuting, non local charges from field theory transfer matrices

We verify **same set of fractional Lorentz spins**, however in both previous constructions there is a **manifest non-locality**

Precise comparison will require some more work (following inspiring suggestions by G. Takács & H. Saleur)

Towards the GGE

The “string-charge duality” can be extended to the light-cone lattice :
linear relations between the charges and densities of particles/magnons

$$
\begin{aligned}
\rho(\theta) &= -\frac{1}{L}X_1\left(\theta+i\frac{\pi}{2}\right)-\frac{1}{L}X_1\left(\theta-i\frac{\pi}{2}\right)\,, \\
\rho_j(\lambda) &= \frac{1}{L}X_{j+1}\left(\lambda+i\frac{\pi}{2}\right)+\frac{1}{L}X_{j+1}\left(\lambda-i\frac{\pi}{2}\right)-\frac{1}{L}X_{j+2}\left(\lambda\right)-\frac{1}{L}X_j\left(\lambda\right) \\
\rho_j^h(\lambda) &= -\frac{1}{L}X_{j+1}\left(\lambda+i\frac{\pi}{2}\right)-\frac{1}{L}X_j\left(\lambda-i\frac{\pi}{2}\right)\,.
\end{aligned}
$$

so our charges should be enough to build a GGE

However, this still represents some work : we have computed the action on SG eigenstates, but for application to quenches, **action on generic initial states** would be needed

Conclusions

Complete GGE for **integrable spin chains**: $\rho_{\mathrm{GGE}} = e^{-\sum_j \sum_n \beta_{j,n} Q_{j,n}}$

$j = 1$ local charges

$j > 1$ quasilocal charges, in correspondence with the densities of strings

Field theories with non-diagonal scattering: the lattice quasilocal charges become field-theory local charges and encode the distributions of magnons

=> All the ingredients for a complete GGE are there

… except for the evaluation of our charges on arbitrary initial states ! (we only provided their eigenvalues)

New **local** charges with **fractional Lorentz spin…**. Quite non usual !
Probable relation with existing “non-local” charges : needs investigation (eg: comparing analytic properties with those of Bazhanov Lukyankov Zamolodchikov)

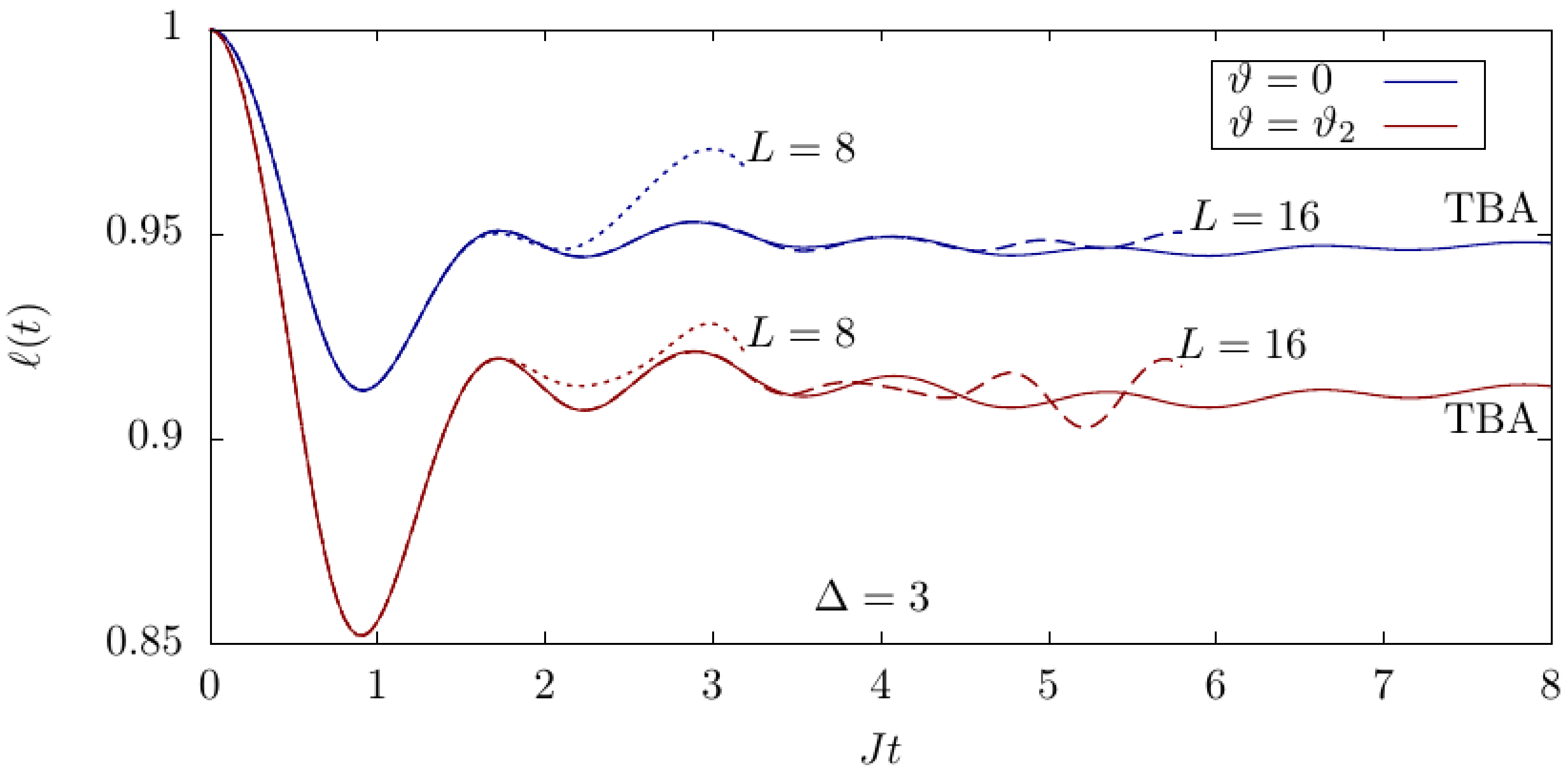
What' next

Analytical study of the full time evolution (dynamical phase transitions, etc...)

First analytical results : Loschmidt echo $\mathscr{L}(t) = \left| \langle \Psi_0 | e^{-iHt} | \Psi_0 \rangle \right|^2$

B. Pozsgay, JSTAT 2013

L. Piroli, B. Pozsgay, EV, 2016



Thank you for your attention !