Generalized detection algorithm for a stochastic background of gravitational waves with non-standard polarizations

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Probing the early Universe with Gravity

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Plan of the presentation

Detecting a stochastic background of gravitational waves

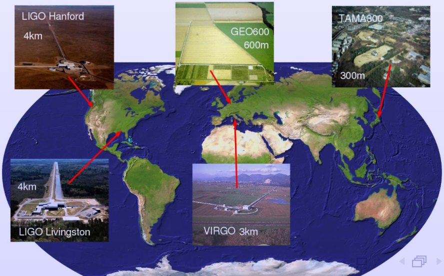
with non-standard polarizations

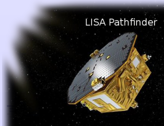
The first direct detection of a gravitational wave (GW) signal, **GW150914**, has started the new era of gravitational wave Astronomy.
Many kilometer-size Advanced laser interferometric GW detectors are currently operating all around the globe and several **upgrades** have been planned for the next decade, ~ 2021.

One of the most interesting targets for their search is a

stochastic background of gravitational waves (SGWB)

similar to the cosmic electromagnetic background radiation (CMB), generated by the incoherent superposition of a large number of independent and unresolved GW sources since the Big Bang (and before)!





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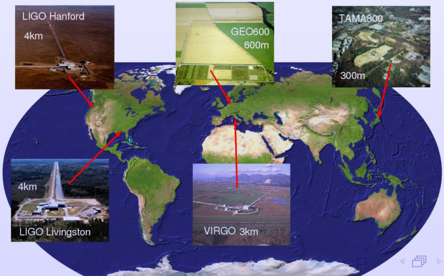

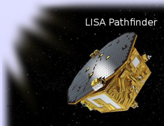

Motivations and possible targets

Why looking for an SGWB signal?

Basic idea: gravitational interaction is so weak that even the most energetic and catastrophic astrophysical processes may produce, individually, **GWs** that are too small to be detected. [Collins, 2010]

Fortunately, we can expect that there is a HUGE number of such sources in the Universe surrounding us.

What may have produced it?

Astrophysical sources: [Regimbau, 2011]

- bursts: supernova collapses and captures by supermassive black holes;
- coalescences of compact objects binary systems;
- periodic sources: turbulence and instabilities in rapidly rotating neutron stars.

Cosmological processes:

- Inflation: GWs produtction by the amplification of vacuum fluctuations; [Turner, 1997]
- Axion inflation: backreaction on the inflaton extends inflation; [Barnaby et al., 2012]
- Cosmic strings oscillations and decays; [Siemens et al., 2007]
- "Stiff ($w > 1/3$) energy", between inflation and RD era; [Boyle and Buonanno, 2008]
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- ??? ⇐ **other unknown production mechanisms**.

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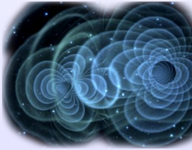

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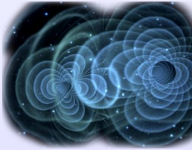

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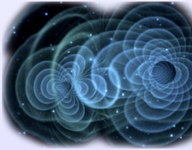

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General features of the SGWB

Astrophysical: [Regimbau and Mandic, 2008]

- possible contributions from the beginning of the stellar activity ($z \sim 10$), though most sources are expected to be located at red-shift $z \lesssim 5$;
- at lower z, sources are likely to be anisotropically distributed towards the Virgo Cluster or the Great Attractor.

Cosmological, "the Grail of GW astronomy": [Maggiore, 2000]

- characteristic decoupling temperature of gravitons from the primordial plasma:

$$\left(\frac{\Gamma}{H}\right)_{\text{gravitons}} \sim \frac{G_N^2 T^5}{T^2/M_{\text{Pl}}} \sim \left(\frac{T}{M_{\text{Pl}}}\right)^3, \qquad \Rightarrow \qquad T_{\text{dec}} \sim M_{\text{Pl}} \simeq 2.24 \times 10^{18}\ \text{GeV}$$

- assuming a causal source of GWs operating at sub-Hubble scales: $f_* = \varepsilon^{-1} H_*$:
  - temperature of production T_*:

$$T_* \simeq 6.1 \times 10^7 \varepsilon \left(\frac{f_0}{1\ \text{Hz}}\right)\left(\frac{100}{g_S(T_*)}\right)^{1/6} \text{GeV}$$

  - epoch of production t_*:

$$t_* \simeq 6.6 \times 10^{-21} \varepsilon^{-2} \left(\frac{1\ \text{Hz}}{f_0}\right)^2 \left(\frac{100}{g_S(T_*)}\right)^{1/6} \text{sec}$$

where $g_S(T)$ is a measure of the effective number of d.o.f. at temperature T, as far as the entropy S is concerned: $g_S(T_0) \simeq 3.91$, $g_S(\gtrsim 300\ \text{GeV}) \simeq 106.75$. $\varepsilon \sim 10^{-3} \div 1$ [Binetruy et al., 2012].

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f_0	$t_* \varepsilon^2$	$T_* \varepsilon^{-1}$
10 Hz	7×10^{-23} sec	6×10^7 GeV
10^3 Hz	7×10^{-27} sec	6×10^9 GeV

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Importance of studying the SGWB

Any SGWB takes trace of the process(es) that produced it. Studying it will shed light on:

- very early Universe cosmology ($t \sim 10^{-27}$ sec) and the "fairly close" ($z \lesssim 5$) structure of the Universe, never accessible with any other means (*e.g.* EM radiation and neutrinos);

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- correspondingly high-energy physics ($\sim 10^{10}$ GeV), beyond the Standard Model of particle physics: strings, supersymmetries, higher dimensions, quantum gravity...;

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- **Alternative Theories of Gravity**, different from Einstein's General Relativity (GR).

There are several reasons to introduce (and test) these theories: [Capoziello, 2010]

- testing GR itself, considering these theories as alternatives in an hypothesis test;
- they "emerge" in effective actions describing the low energy limit of models for the unification of fundamental interactions (like superstrings, supregravity, GUTs);
- they can correct some issues with the standard FRW cosmological model, in particular at extreme regimes: dark matter and dark energy, Big Bang singularity, etc.;
- a step toward the solution for the gravity quantization problem.

Therefore, a coherent way to test the SGWB should necessarily pass through alternative theories of gravity, and, conversely, the study of an SGWB can be a very valuable testing ground for these theories, as well as for GR.

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Gravitational Waves in Alternative Theories of Gravity

An enormous variety of proposals for Alternative theories of Gravity can be found in literature (see for example [Capozziello and De Laurentis; Clifton et al.] or [Will]). They can differ mainly through:

- the gravitational action and the equations of motion, ($f(R)$ theories)
- the presence of additional dynamical gravitational fields, (Brans-Dicke, vector-tensor theories)
- higher spatial dimensions, (Kaluza-Klein, DGP braneworld)
- prior geometries, (bimetric theories, stratified theories)
- etc...

Despite their differences, **ALL** of the “viable” theories have in common that

they admit wave-like solutions (GWs)

to be **consistent with GW150914 and GW151226** (and as a possible consequence of being based on second order differential equations and to incorporate the request of local Lorentz invariance [Will]).

GWs predicted by different theories could differ through:

- the propagation speed (*e.g.* for massive gravitons or extra-dimensions); ← current GW150914 model dependent bound: $m_g < 10^{-22}$ eV, or $\lambda_g > 10^{13}$ km [Abbott et al. 2016].
- the waveform (depending on both the source and on the field equations of the theory);
- **the polarization modes** ⇐ **model independent test!**

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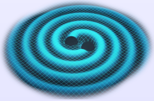

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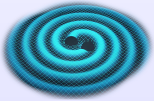

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- the waveform (depending on both the source and on the field equations of the theory);
- the **polarization modes** $\Leftarrow$ **model independent test**!

Non-standard polarization modes in alternative theories

It can be shown that any generic theory of gravity, with additional fields, degrees of freedom, massive gravitons, or extra dimensions (once projected on our 3-space) can predict, at most, **six** polarization modes of a GW.

We can classify them according to the relative displacement they produce on test masses with respect to their propagation direction ($E(2)$ classification scheme [Eardley et al., 1973]):

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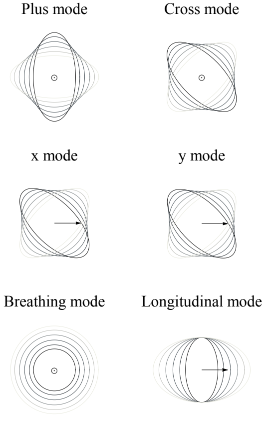

Theoretical Model	e^+_{ij}	$e^\times_{ij}$	e^b_{ij}	e^ℓ_{ij}	e^x_{ij}	e^y_{ij}
Einstein General Relativity	*	*				
Scalar-tensor (Brans-Dicke) theories	*	*	*	*		
$f(R)$ theories	*	*	*	*	*	*
Vector-tensor theories	*	*	*	*	*	*
GR in a noncompactified 5D sp.	*	*	*	*	*	*
GR in a noncompactified 6D sp.	*	*	*	*	*	*
5D Kaluza-Klein theory	*	*	*		*	*
Randall-Sundrum braneworld	*	*				
DGP braneworld (normal branch)	*	*				
DGP braneworld (acceler. branch)	*	*	*	*		
Bimetric (Rosen's, Rastall's)	*	*	*	*	*	*

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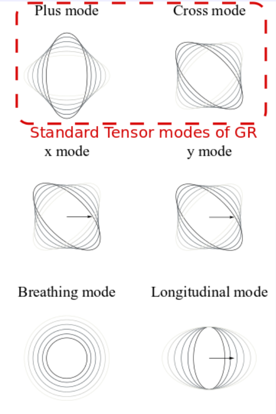

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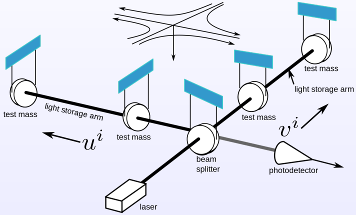
The detector signal

A GW laser interferometric detector measures the phase shift of the light after traveling into its two arms.
The resulting signal is a time series $s(t)$ containing the **strain** $h(t)$, produced by GWs, and a **noise** component $n(t)$:

$$s(t) = h(t) + n(t), \qquad \text{where} \quad h(t) \equiv h_{ij}(t, \mathbf{x}) D^{ij}$$

where $D^{ij} = \frac{1}{2}\left[u^i u^j - v^i v^j\right]$ is the detector tensor, fixed by the geometry of the detector.



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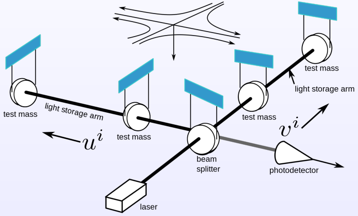

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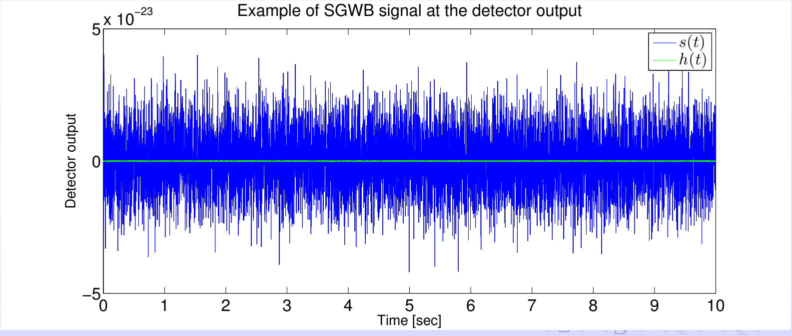

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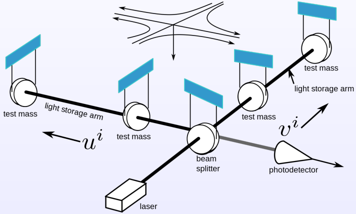
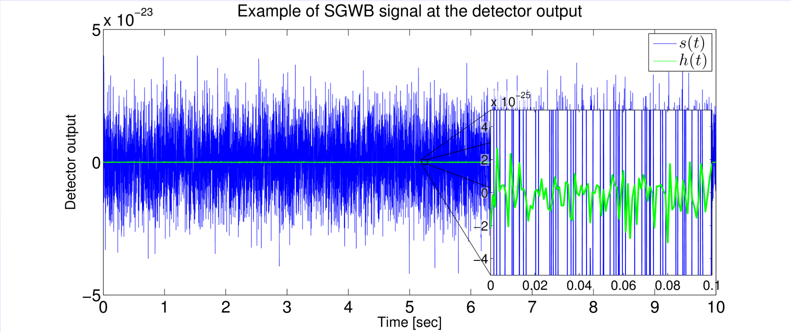
$$s(t) = h(t) + n(t), \qquad \text{where} \quad h(t) \equiv h_{ij}(t, \mathbf{x}) D^{ij}$$

where $D^{ij} = \frac{1}{2}\left[u^i u^j - v^i v^j\right]$ is the detector tensor, fixed by the geometry of the detector.





Cross-correlation analysis

The **strain signal** $h(t)$ is expected to be (in the most favorable case: designed Advanced Ligo strain sensitivity [LIGO Scientific Collaboration et al., 2013] and LIGO S5 SGWB sensitivity [Abbott et al., 2009]) 10^2 **times smaller** than the **noise** $n(t)$.

Then, how can we extract information about the SGWB?

⇒ **Cross-correlation analysis**: [Michelson, 1987]

> if the noises of two, or more, detectors far apart are uncorrelated, correlating their outputs the noise contributions cancel and **it remains only the signal**;

This procedure allows to get rid of the dominant noise contribution and improve, by several orders of magnitude ($\sim 10^4$, as for LIGO S5), the sensitivity that a single detector may have to the SGWB.

For the validity of this approach it is important to:

- check the assumptions for the absence of correlated noise (e.g. Schumann resonances [Thrane et al., 2014]);
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Assumptions about the signal and the noise

Noise:

- absence of correlations between the noises of different detectors; ← to be checked!
- stationary: the total data stream is broken into periods where the detector is assumed to be so;
- Gaussian distributed: assuming the noise to be given by the superposition of a large number of contribution, this is guaranteed by the Central Limit Theorem (CLT). ← to be checked!

and therefore it can be fully described by the value of its correlation:

$$\langle \tilde{n}_i^*(f)\tilde{n}_j(f')\rangle = \delta_{ij}\delta(f-f')\frac{1}{2}P_i(f), \qquad \text{where } i,j = 1,\dots,N \text{ detectors}$$

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- Gaussian distributed: CLT for large number of pointlike astrophysical sources or causally independent cosmological production horizons; [Allen, 1997]
- isotropic: justified (in first approximation) for the cosmological SGWB by analogy with the CMB. Astrophysical sources at $z \gtrsim 0.1$ are isotropic, while those at $z \lesssim 10^{-2}$ are not. ← to be checked!
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SGWB signal cross-correlator

$$\langle \tilde{h}_i^*(f)\tilde{h}_j(f')\rangle = \delta(f-f')\frac{1}{2}\sum_A \Gamma_{ij}^A(f)\, S_h^A(f), \qquad \text{where } A = \pm 2, \pm 1, 0$$

where it can be shown from invariance principles that the most general isotropic SGWB allows, at most, 6 independent polarization components, but only 5 of them can be distinguished with an orthogonal arms, interferometric detector. Some equivalent definitions:

$$S_h^{\pm 2} \equiv \frac{1}{\sqrt{2}}\left(S_h^{+} \pm i S_h^{\times}\right) \equiv S_h^T \pm V^T, \quad S_h^{\pm 1} \equiv \frac{1}{\sqrt{2}}\left(S_h^{x} \pm i S_h^{y}\right) \equiv S_h^V \pm V^V, \quad S_h^S = S_h^b + S_h^{\ell},$$

where $V^{T,V}$ is the parity violation Stokes parameter (non-vanishing in those theories of gravity that violate parity. [Crowder et al., 2013; Lue et al., 1999; Seto and Taruya, 2007]).
The two scalar polarization modes, b and ℓ, are completely degenerate at the detector output.

In the previous equation:

$\Gamma_{ij}^A(f)$ is the A-th component overlap reduction function (ORF) between the i-th and the j-th detector. It takes into account the distance and different orientation of the two detector and it measures the corresponding loss of coherence between their signals:

$$\Gamma_{ij}^A(f) \equiv \int_{S^2} \frac{d^2\hat{\Omega}}{4\pi} e^{2\pi i f \hat{\Omega}\cdot(\mathbf{x}_i - \mathbf{x}_j)/c} D_i^{ab} e_{ab}^{A*}(\hat{\Omega})\, D_j^{cd} e_{cd}^{A}(\hat{\Omega}), \qquad a,b,c,d = 1,2,3.$$

$S_h^A(f)$ is the A-th component of the SGWB power spectrum density, similar to $P_i(f)$ but several orders of magnitude smaller ($\sim 10^2$ in the most favorable case, but probably much more...).

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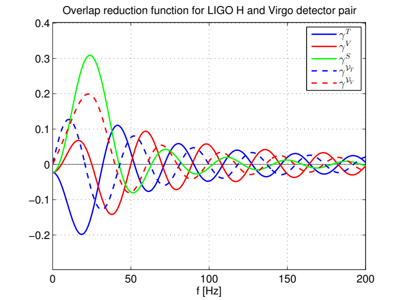
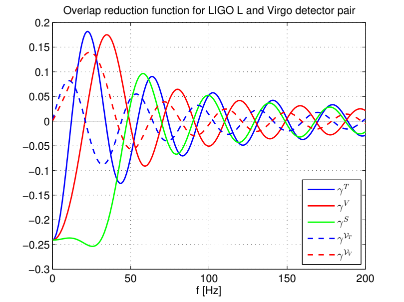
Overlap Reduction Functions





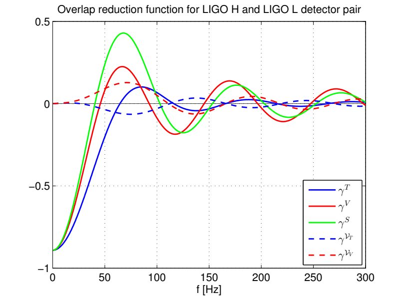


Figure: Normalized ORFs,

$$\gamma_{ij}^{A}(f) \equiv \Gamma_{ij}^{A}(f)/\Gamma_{ij\,\text{aligned and colocated}}^{A}$$

for the different detector pairs Virgo - LIGO(L), Virgo - LIGO(H) and LIGO(H) - LIGO(L). Notice how the differences of behavior between the polarization modes are manifest at around the characteristic frequency $f_c \equiv c/|\Delta\mathbf{x}|$, above of which the ORFs rapidly decrease to 0.

Standard cross-correlation analysis: a review

Currently there is no evidence for the presence of an SGWB signal in the recorded data sets.

Then: How can we claim a detection of an SGWB signal?

Assuming only standard GR, unpolarized tensor modes, the usual algorithm [Allen and Romano, 1999; Flanagan, 1993] consists into construct the optimal cross-correlation statistic:

$$Y(\tilde{s}) = \iint df\, df'\, \tilde{s}_i^*(f)\, \tilde{s}_j(f')\, \tilde{Q}_{ij}(f,f'),$$

where the optimal filter function $\tilde{Q}_{ij}(f,f')$ is found with a matching filter procedure, maximizing the SNR of $Y(\tilde{s})$. It results that:

$$\tilde{Q}_{ij}(f,f') = \delta(f-f')\, \frac{\Gamma_{ij}^{(T)}(f)\, S_h^{(T)}(f)}{P_i(f)\, P_j(f)}.$$

The SNR of this statistic is then compared with a fixed threshold (*e.g.* SNR > 5) [Maggiore, 2008] or it is performed a Neyman-Pearson test (more details later) [Allen and Romano, 1999].

Issues with this approach:

- It excludes the possibility for testing alternative theories of gravity, ignoring non-standard polarization modes;
- $Y(\tilde{s})$ is NOT an actual statistic (that is, an observable function of the data $\tilde{s}$) for the presence of the unknown function: $S_h(f)$. One solve this last point testing only power-law models:

$$S_h(f) = S_\nu f^\nu, \quad \text{with typical values of } \nu = -3, -1, 0.$$

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Generalized detection algorithm

and the purposes of the present thesis work

We decided to improve the standard cross-correlation analysis algorithm in the following ways:

- including the possibility for non-standard polarizations, as it seems to be adequate for studying an SGWB produced at very high energy-densities by some theory of gravity different from GR;
- relaxing the description by means of power-law templates, because a frequency-by-frequency study of its spectrum will be, after the first detection, the key point in order to understand what may have produced the detected SGWB;
- finding the maximum likelihood estimators (MLEs) for the SGWB spectral density components, $S_h^A(f)$, for studying the polarization contributions to it and its behavior frequency-by-frequency;
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Known results from the literature have been recovered adding only later further assumptions and simplifications to this much general analysis algorithm.

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How can we claim a detection of an SGWB signal?

We want to verify if the measured data set supports better the hypothesis H_0 of absence of an SGWB signal or that H_1 of its presence:

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H_1, presence of an SGWB signal:

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To do that we need an **hypothesis test**, that is, a decision rule that, given some measures, select for us one of the two previous hypotheses.

Every rule of this kind will be imperfect for the possible occurrence of some error:

Type I, or false alarm error	wrongly reject H_0
Type II error, or false dismissal error	wrongly reject H_1

Neyman-Pearson criterion: Among the various possible tests, we chose the one that at fixed false alarm probability α gives the smallest false dismissal probability β, or, conversely, the highest detection probability $\gamma \equiv 1-\beta$.

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Type II error, or false dismissal error	wrongly reject H_1

Neyman-Pearson criterion: Among the various possible tests, we chose the one that at fixed false alarm probability α gives the smallest false dismissal probability β, or, conversely, the highest detection probability $\gamma \equiv 1-\beta$.

Neyman-Pearson test

How can we construct an optimal test statistic?

Neyman-Pearson Lemma: Given the statistic $Y(\tilde{s})$ constructed by means of the ratio of the likelihood functions (Likelihood Ratio Test, LRT) of the alternative and the null hypotheses, the test the chooses

$$Y(\tilde{s}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta, \qquad \text{where} \quad P\big(Y(\tilde{s}) > \eta | H_0\big) = \alpha$$

for a fixed false alarm probability α is the one that maximizes the detection probability γ. [Neyman, 1937]

Given the cross-correlations for the hypotheses H_1 and H_0, the resulting test statistic is:

$$Y(\tilde{s}) = \int df\, \tilde{s}_i^*(f) \frac{\sum_A \Gamma_{ij}^A(f)\, S_h^A(f)}{P_i(f)\, P_j(f)} \tilde{s}_j(f),$$

which equals the standard test statistic obtained maximizing the SNR, but which includes also non-standard polarizations.

The is not yet an actual test statistic because it contains the unknown functions $S_h^A(f)$.

Generalized Likelihood Ratio Test (GLRT): Perform the standard Neyman-Pearson, LRT, where to the unknown parameters are substituted their Maximum Likelihood Estimators. Optimality, in the sense of the NP criterion, is guaranteed by the asymptotic properties of MLEs.[Kay, 1998]

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Parameter estimation

From the SGWB cross-correlation, we can find the unbiased estimators for the SGWB components, $\widehat{S_h^A}(f)$ minimizing the distance:

$$\left\|\Sigma_{ij}(f) - \tfrac{T}{2}\sum_A \Gamma^A_{ij}(f)\,\widehat{S_h^A}(f)\right\|^2, \qquad \text{where} \quad \Sigma_{ij}(f) \equiv \frac{1}{n}\sum_{I=1}^{n} \tilde{s}^*_{I,i}(f)\,\tilde{s}_{I,j}(f), \quad T_{\text{tot}} = nT.$$

It can be shown that these estimators are also asymptotically MLEs.

An estimator of this kind can be constructed if the number of the available detector pairs $\binom{N}{2}$ is greater than the number polarization we want to reconstruct P. In this case:

$$\widehat{S_h^A}(f) = \frac{2}{T}\sum_B \left[\left(\Gamma^T\Gamma\right)^{-1}\right]^{AB} \sum_{i,j} \Gamma^B_{ij}\Sigma_{ij},$$

where the term $\left(\Gamma^T\Gamma\right)^{-1}\Gamma^T$ is the left pseudo-inverse of the ORF matrix $\Gamma^A_{ij}(f)$, and it reduces to the usual inverse when $\binom{N}{2} = P$:

$$\widehat{S_h^A}(f) = \frac{2}{T}\sum_{i,j} \left(\Gamma^{-1}\right)^A_{ij}\Sigma_{ij}.$$

This is the most interesting case as it describes the situation when we want to reconstruct 3 modes of polarizations with the 3 currently operating kilometer-size interferometric detectors: LIGO H and L and Virgo.

More often, we are in an under-determined situation: $\binom{N}{2} < P$. We have to recur to a generalized, power-law template approach:

$$\text{Assuming:} \quad S_h^A(f) = S_\nu^A\left(f/f_0\right)^\nu, \qquad \widehat{S_\nu^A} = \frac{2}{T}\sum_B M^{AB}\int df\,\left(f/f_0\right)^\nu \Gamma^A_{ij}\Sigma_{ij},$$

where $M^{AB} = \left[\int df\,\left(f/f_0\right)^{2\nu}\Gamma^A_{ij}(f)\,\Gamma^B_{ij}(f)\right]^{-1}$ is a normalization factor.

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Generalized test statistic

We can define the generalized test statistic substituting the previous estimators into the expression for $Y(\tilde{s})$ (as prescribed by the GLRT).

- If $\binom{N}{2} > P$:

$$Y_G \equiv \frac{1}{(nT)^2} \sum_{I,J}^{n} \sum_{\substack{i,j \\ (i\neq j)}} \sum_{\substack{k,l \\ (k\neq l)}} \int df \, \frac{\tilde{s}^*_{I,i}(f)\,\tilde{s}_{I,j}(f)\,\tilde{s}^*_{J,k}(f)\,\tilde{s}_{J,l}(f)}{P_i(f)\,P_j(f)} \sum_{A,B} \underbrace{\Gamma^A_{ij}\left(\left(\Gamma^T\Gamma\right)^{-1}\right)^{AB}\Gamma^B_{kl}}_{\text{projector on the null space of } \Gamma^A_{ij}} .$$

- If $\binom{N}{2} \leqslant P$:

$$Y_G \equiv \frac{2}{(nT)^2} \sum_{I,J}^{n} \sum_{\substack{i,j \\ (i\neq j)}} \int df \, \frac{\tilde{s}^*_{I,i}(f)\,\tilde{s}_{I,j}(f)\,\tilde{s}^*_{J,i}(f)\,\tilde{s}_{J,j}(f)}{P_i(f)\,P_j(f)} .$$

- Power-law template reconstruction:

$$Y_G \equiv \frac{2}{(nT)^2} \sum_{I,J} \sum_{A,B} M^{AB} \iint df\, df' \, \frac{\tilde{s}^*_{I,i}(f)\,\tilde{s}_{I,j}(f)\,\tilde{s}^*_{J,i}(f')\,\tilde{s}_{J,j}(f')}{P_i(f)\,P_j(f)} \Gamma^A_{ij}(f)\,\Gamma^B_{ij}(f') \left(\frac{ff'}{f_0^2}\right)^{\nu} .$$

Disposing of a set of data $\tilde{s}$, we can compute the corresponding value of the previous generalized statistics and perform the GLRT hypothesis test comparing their values with the threshold η obtained from the fixed false alarm probability α and the statistical properties of Y_G (thought as a random variable itself).

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Data analysis on LIGO S5 data

We tested the LIGO S5 data with our detection algorithm.

Currently NO DETECTION was possible with the LIGO S5 data. The Bayesian upper limit (based on previous measures by LIGO S4 and S3) on the value of its energy density,

$$\Omega^M_{\text{gw}}(f) \equiv \left(\frac{2\pi^2}{3H_0^2}\right) f^3 S_h^M(f), \qquad \text{for every mode } M = T, V, S$$

published in [Abbott et al., 2009] is:

$$\Omega_0^T < 6.9 \times 10^{-6} \qquad \text{at } 95\% \text{ confidence level.}$$

Our algorithm yielded that:

- NO detection was possible with our frequentist approach based on the NP criterion, for reasonable fixed values of the false alarm probability α (1, 5 or 10%). The minimum α that can support a detection is 23%; $\leftarrow$ **too high!**
- we computed then the sensitivity level, that is, the maximum value of the SGWB that LIGO missed to detect:

$$\Omega_\nu^A \geqslant \frac{1}{\sqrt{T_{\text{tot}}}} \frac{3H_0^2}{2\pi^2} \left(\int df \left(\frac{f}{f_0}\right)^{2\nu} \frac{\left(\Gamma^A_{ij}(f)\right)^2}{f^6 P_i(f) P_j(f)} \right)^{-1/2} P^{1/4} \left(\text{erfc}^{-1}(2\alpha) - \text{erfc}^{-1}(2\gamma)\right)^{1/2}$$

Assuming: $\alpha = 5\%$, $\gamma = 95\%$, $\Rightarrow$
$T_{\text{tot}} = 10^7$ sec, $f_0 = 100\,\text{Hz}$
$P = 5$ polarizations

Ω_0^T	Ω_0^V	Ω_0^S
7.93×10^{-5}	2.43×10^{-5}	2.45×10^{-5}

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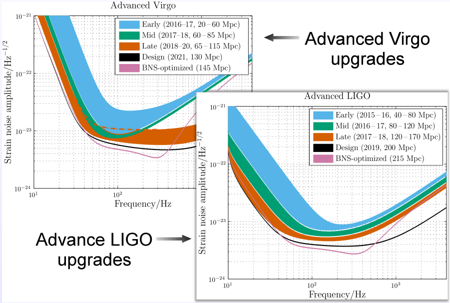
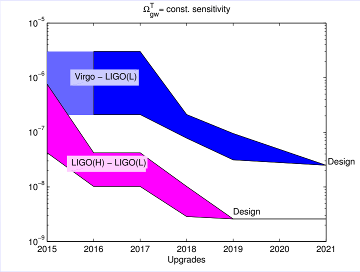
Projected strain sensitivities

We made use of the projections on the strain sensitivities of the Advanced detectors during the scheduled upgrade phases in order to compute what will be their sensitivity to an SGWB with non-standard polarizations.

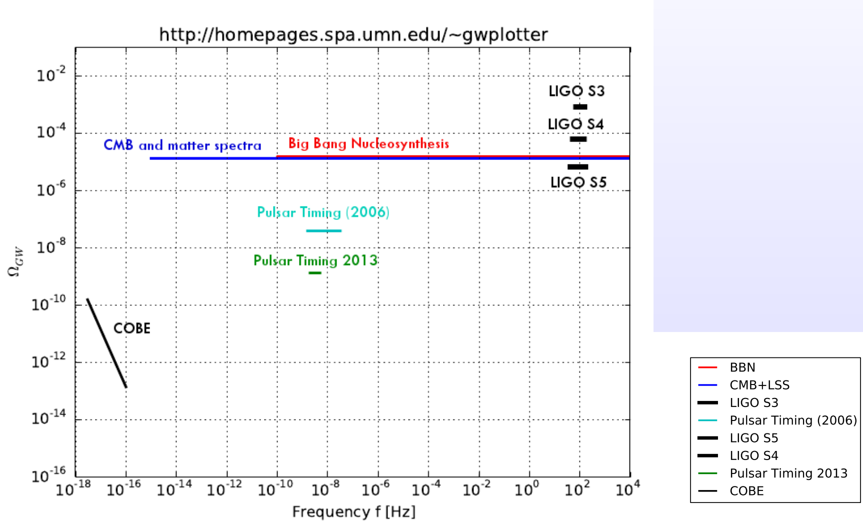
P.-law ν	Pol. comp.	Detector pair	Design sens. (2019-2021)
0	Ω_0^T	AdV - AdLIGO(L)	2.49×10^{-8}
		AdLIGO(L) - (H)	2.59×10^{-9}
	Ω_0^V	AdV - AdLIGO(L)	1.99×10^{-8}
		AdLIGO(L) - (H)	3.47×10^{-9}
	Ω_0^S	AdV - AdLIGO(L)	1.47×10^{-8}
		AdLIGO(L) - (H)	3.53×10^{-9}
3	Ω_3^T	AdV - AdvLIGO(L)	1.27×10^{-7}
		AdvLIGO(L) - (H)	5.58×10^{-8}
	Ω_3^V	AdV - AdLIGO(L)	1.45×10^{-7}
		AdLIGO(L) - (H)	2.69×10^{-8}
	Ω_3^S	AdV - AdLIGO(L)	1.40×10^{-7}
		AdLIGO(L) - (H)	2.17×10^{-8}





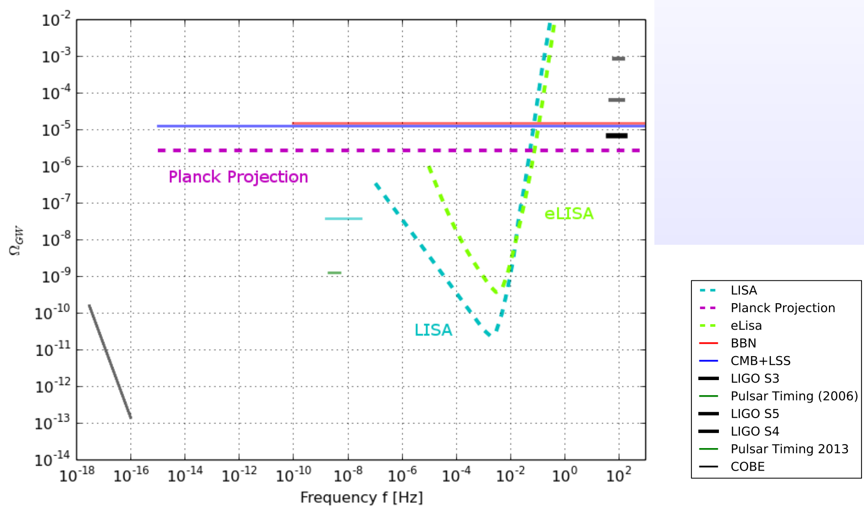
Some results and prospects for the future

Current experimental limits



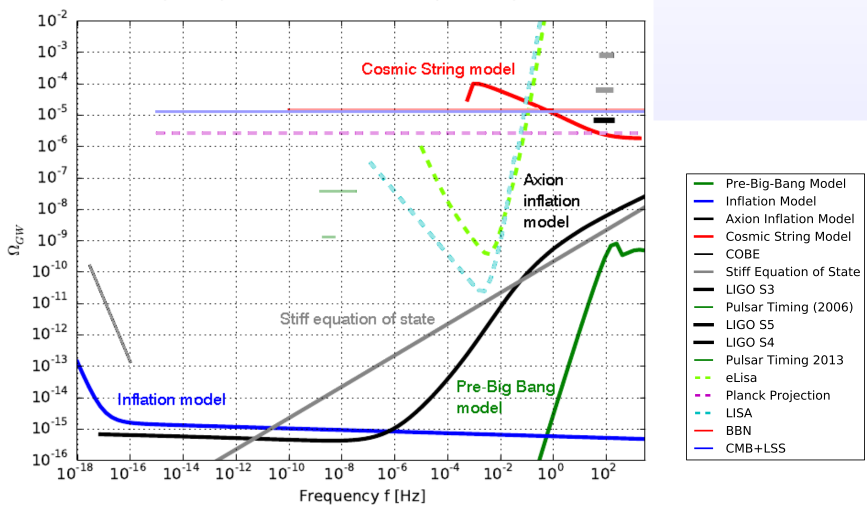
Some results and prospects for the future

Projected sensitivities



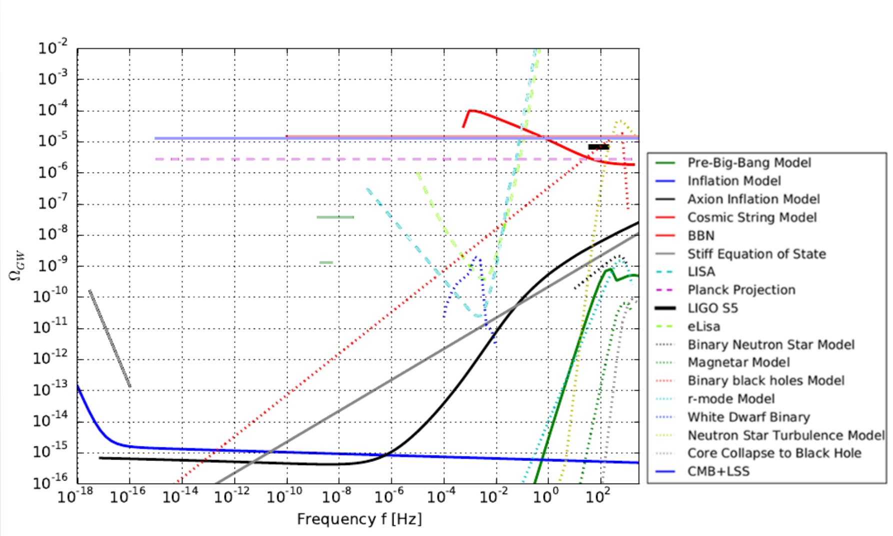
Some results and prospects for the future

Cosmological Models



Some results and prospects for the future

Astrophysical Models



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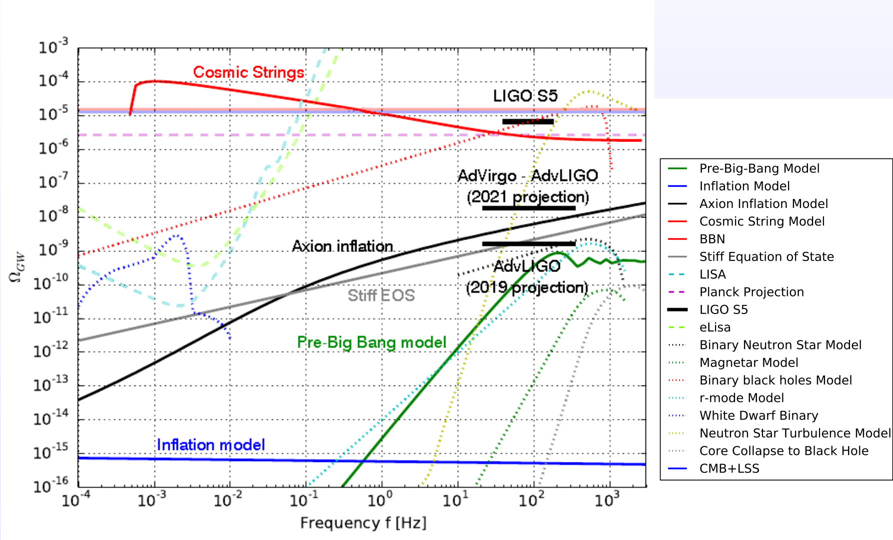
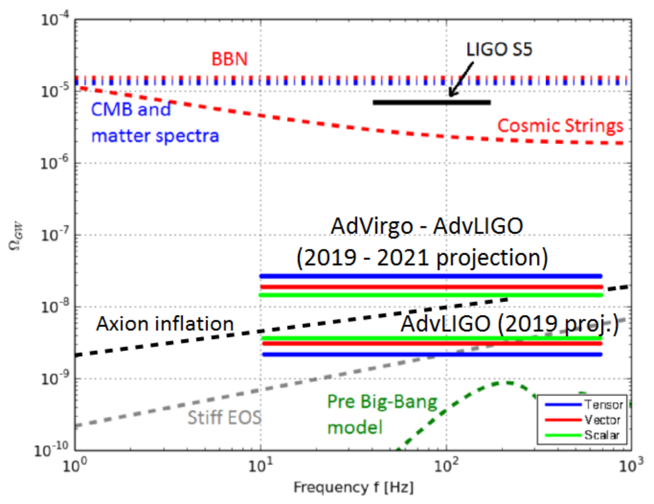


Some results and prospects for the future

Sensitivity to the various polarization modes



Summary and Conclusions

- Many theoretical models, both astrophysical and cosmological, predict a stochastic background of gravitational waves. This background is of great interest in the study of the early Universe cosmology and very-high energy physics;
- for the study of the SGWB, we must include the possibilities of Alternative Theories of Gravity. The SGWB signal can also be used to test these theories;
- the most general SGWB can allow at most 6 modes of polarization: detecting these modes can be a valuable, "model-independent" test for alternative theories of gravity;
- In the present work, we generalized the standard cross-correlation analysis, developed by [Allen and Romano, 1999], including the possibility of non-standard polarizations and of a frequency-by-frequency reconstruction of the SGWB spectrum;
- we made use of the LIGO S5 data to test the possible presence of an SGWB signal. No evidence for an SGWB signal in these data;
- with the predicted strain sensitivities of the advanced GW detectors, we evaluated the projections on their sensitivities to the SGWB;
- these projections lie under the energy densities expected by many cosmological and astrophysical models;
- if these models are predictive, we will be able in the next decade to detect an SGWB signal. Otherwise, we will improve current upper limits and bounds on them.

Summary and Conclusions

- Many theoretical models, both astrophysical and cosmological, predict a stochastic background of gravitational waves. This background is of great interest in the study of the early Universe cosmology and very-high energy physics;
- for the study of the SGWB, we must include the possibilities of Alternative Theories of Gravity. The SGWB signal can also be used to test these theories;
- the most general SGWB can allow at most 6 modes of polarization: detecting these modes can be a valuable, "model-independent" test for alternative theories of gravity;

- In the present work, we generalized the standard cross-correlation analysis, developed by [Allen and Romano, 1999], including the possibility of non-standard polarizations and of a frequency-by-frequency reconstruction of the SGWB spectrum;
- we made use of the LIGO S5 data to test the possible presence of an SGWB signal. No evidence for an SGWB signal in these data;
- with the predicted strain sensitivities of the advanced GW detectors, we evaluated the projections on their sensitivities to the SGWB;

- these projections lie under the energy densities expected by many cosmological and astrophysical models;
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Backup material

Notes on the assumptions about the SGWB and its characterization

We make some assumptions, and "first-order" approximations, in order to study a very general SGWB, produced by any mechanism within the paradigm of any generic theory of gravity:

- Stationarity: it means that its statistical properties must not change for all the duration of our experiments (usually several orders of magnitude shorter than the SGWB time scales);
- Gaussianity: justified by the central limit theorem if the number of independent sources that contribute to the SGWB is large enough;
- Isotropy: that is, no preferred directions, as it is, in first approximation, for the CMB.

All these assumptions are well justified for a background of cosmological origin. On the other hand, in increasing order of approximation, they may not hold for an SGWB of astrophysical origin if the number of sources is small and they are distributed mostly in our galaxy.

If we take these assumption as true, the most general SGWB we are looking for:

- can be described at most by six modes of polarization: two tensor circular polarizations (± 2), two vector circular polarizations (± 1) and two scalar modes (b and ℓ);
- it can be fully characterized by the two point correlator of the signal outputs of a sufficient number of detector pairs (ij):

$$h_i(t) = h_{ab}(t)\, D_i^{ab}: \qquad \langle \tilde{h}_i^*(f)\, \tilde{h}_j(f') \rangle = \delta(f - f') \sum_A \frac{1}{2} S_h^A(f) \Gamma_{ij}^A(f)$$

$A = \pm 2, \pm 1, 0$ and $i, j = 1, 2, \ldots, N$ for a network of N GW detectors.

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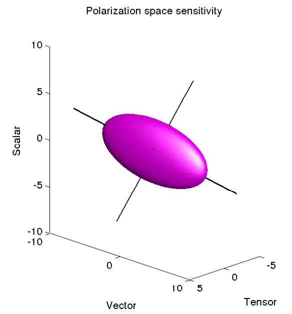
Notes on the proposed algorithm for studying the SGWB

- The MLEs for the components of the power spectrum density $S_h^A(f)$ are equal to those obtained by Seto and Taruya and Nishizawa et al. in the particular cases of circular tensor polarization and tensor, vector and scalar modes with no circular polarizations, respectively;
- we can resolve all the five components of the power spectrum density if we have a large enough number N of detectors ($\binom{N}{2} > 5$, or 3 if we exclude circular polarizations) or if we assume some power-low model for these spectra: $S_h^A(f) = S_h^A \cdot f^\nu$. With less detectors we can only find some directions, in the polarization space, where the detectability is most favorable, that is, our apparatus is most sensitive;
- we also recovered the algorithm described by Allen and Romano, where they considered only unpolarized tensor modes, as a special case. Respect to their algorithm, the introduction of other degrees of freedom, in the form of non-standard polarizations, reduce the statistics and hence the sensitivity we can reach;



SGWB energy density sensitivity

What is the minimum value of the SGWB spectral density required so that our detectors and our decision rule are able to correctly identify its presence with a (fixed) detection probability γ?
We find those values of $S_h^A(f)$ for which the detection probability $\gamma(\mu) = 1 - \beta(\mu)$ is equal to or greater than the desired rate γ:

$$\gamma(\mu) \equiv 1 - \beta \equiv \text{detection rate} = P\left(\tilde{s} \in R_1 | H_1\right) = \int_{R_1} d\tilde{s}\, p\left(\tilde{s}|H_1\right)$$

$$= \tfrac{1}{2}\text{erfc}\left(\text{erfc}^{-1}(2\alpha) - \mu/\sqrt{2\sigma^2}\right) \geqslant \gamma$$

$$\text{erfc}^{-1}(2\alpha) - \text{erfc}^{-1}(2\gamma) \leqslant \mu/\sqrt{2\sigma^2},$$

assuming a power-law template

$$S_h^A(f) = S_\nu^A \left(\frac{f}{f_0}\right)^\nu,$$

and substituting the expectation value and the variance of the stochastic variable Y_G, we obtain:

$$S_\nu^A \geqslant \frac{1}{\sqrt{T_{\text{tot}}}} \left(\int df \left(\frac{f}{f_0}\right)^{2\nu} \frac{\left(\Gamma_{ij}^A(f)\right)^2}{P_i(f)P_j(f)}\right)^{-1/2} P^{1/4}\left(\text{erfc}^{-1}(2\alpha) - \text{erfc}^{-1}(2\gamma)\right)^{1/2}.$$

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