Random walks on random planar triangulations via the circle packing theorem

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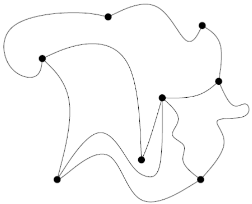
Mini-school on Random Maps and the Gaussian Free Field,
ENS Lyon, May 15th 2017

Basic terminology

- Planar map: a graph embedded in $\mathbb{R}^2$ so that vertices are mapped to points and edges to non-intersecting curves.



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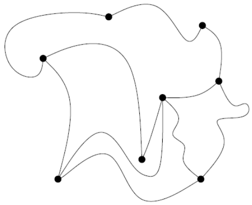

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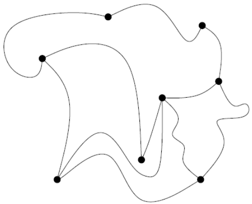

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- The simple random walk on a graph starts at an arbitrary vertex and in each step moves to a uniformly chosen neighbor.

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Lastly, can we analyze models of statistical physics on such graphs? (percolation, self-avoiding walk, Ising model)

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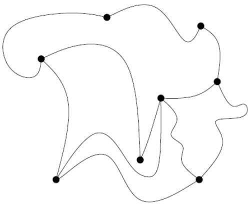

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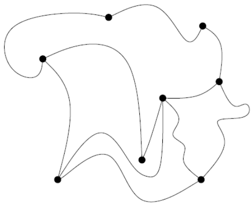

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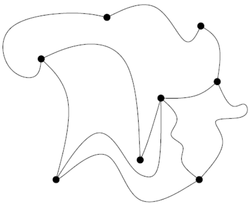

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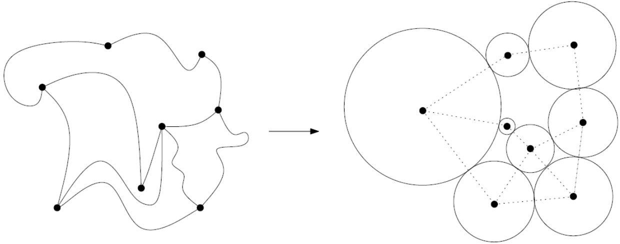

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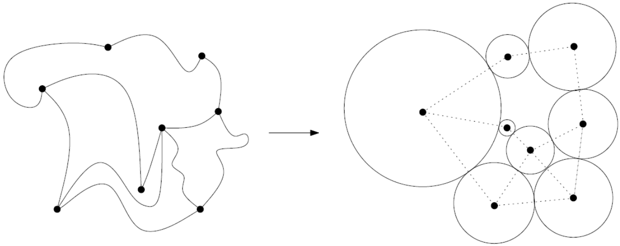

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If G is a triangulation, then the drawing is unique up to Möbius transformations and reflections.

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- No rigidity: different limits may lead to very different looking packings (that are **not** Möbius equivalent). However, under some natural conditions, the **type** of the limiting packing is determined.
- The main point: the type of the packing encapsulates probabilistic information: recurrence/transience of the random walk, existence of non-trivial bounded harmonic functions, speed of the random walk, etc.

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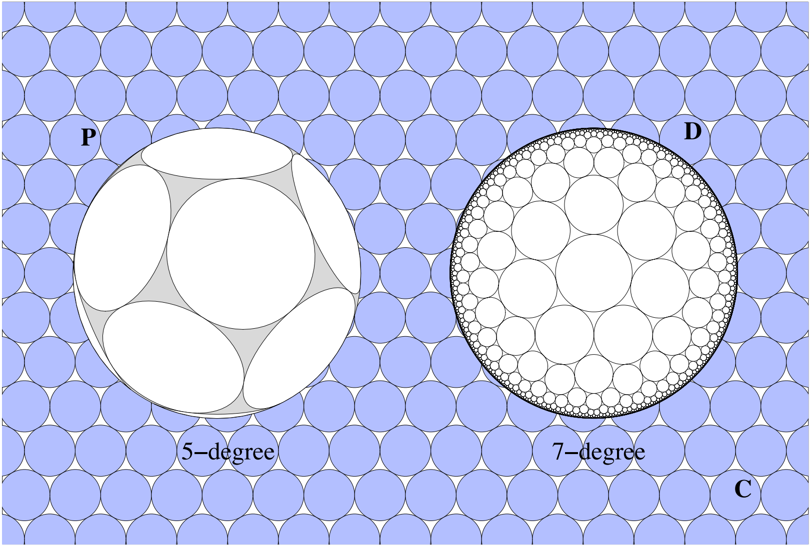
- The set of accumulation points $A(P)$ is the boundary of the carrier $\mathrm{carr}(P)$.



Picture due to Ken Stephenson.

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- We call a circle packing of an infinite triangulation a packing **in the disc** if its carrier is the unit disc $\mathbb{D}$, and **in the plane** if its carrier is $\mathbb{C}$.

Theorem (He-Schramm '95)

*Any simple triangulation of the plane can be circle packed in the plane $\mathbb{C}$ or the unit disc $\mathbb{D}$, but not both (**CP parabolic** vs. **CP Hyperbolic**).*

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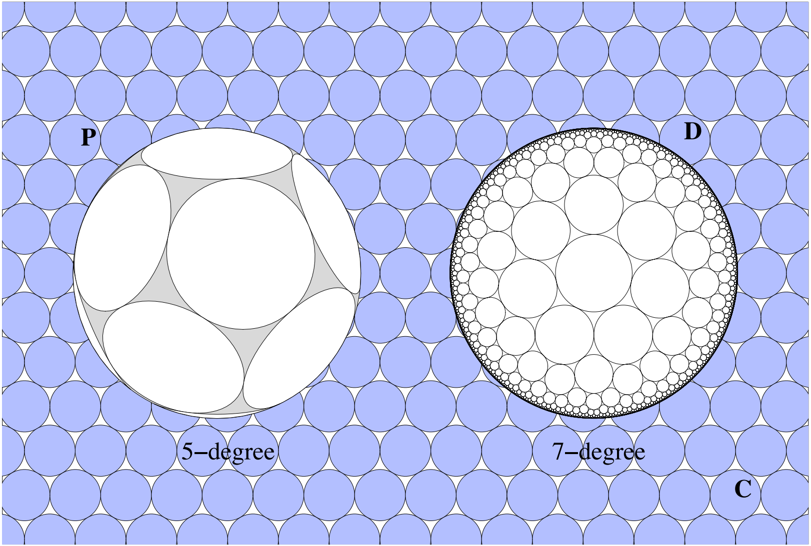
Theorem (Schramm's rigidity '91)

The above circle packing is unique up to Möbius transformations of the plane or the sphere as appropriate.



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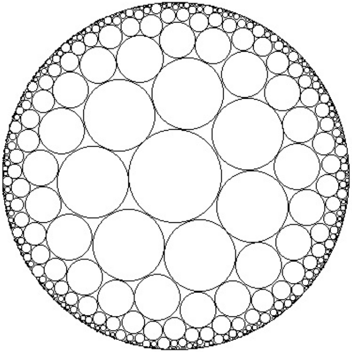
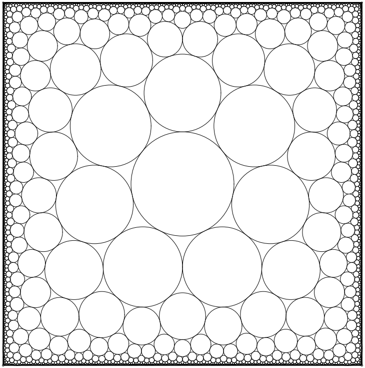
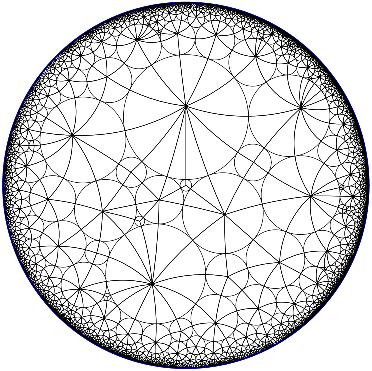
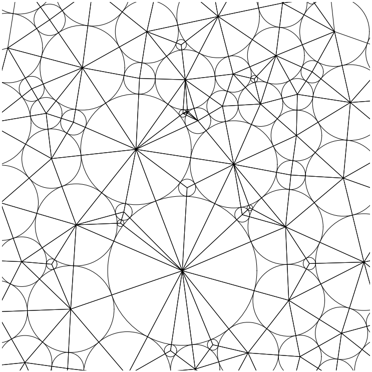
7-regular hyperbolic tessellation





Circle packing also gives us a drawing of the graph with either straight lines or hyperbolic geodesics depending on the type.





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Theorem (Benjamini-Schramm '96)

Assume G is a bounded degree, CP hyperbolic triangulation of the plane circle packed in the unit disc $\mathbb{D}$. Then the random walk converges to $\partial\mathbb{D}$, the exit measure is non-atomic and has full measure.

A dichotomy for bounded degree plane triangulations

If G is bounded degrees triangulation of the plane, then either:

Random walk on G is recurrent, G is CP parabolic and all bounded harmonic functions are constant,

or

Random walk on G is transient, G is CP hyperbolic and any bounded Borel $g : \partial\mathbb{D} \to \mathbf{R}_+$ extends to G.

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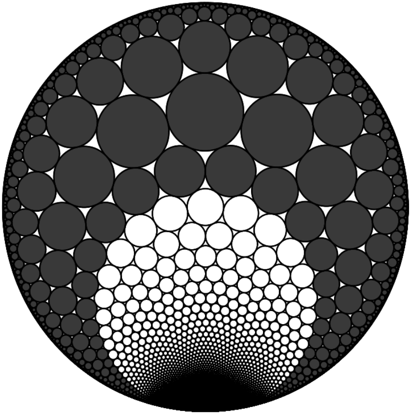
Theorem: There are no other bounded harmonic functions (Angel, Barlow, Gurel-Gurevich, N. '13).

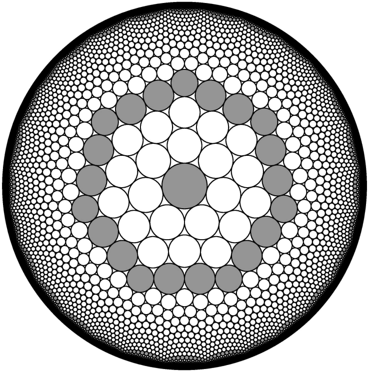
More examples

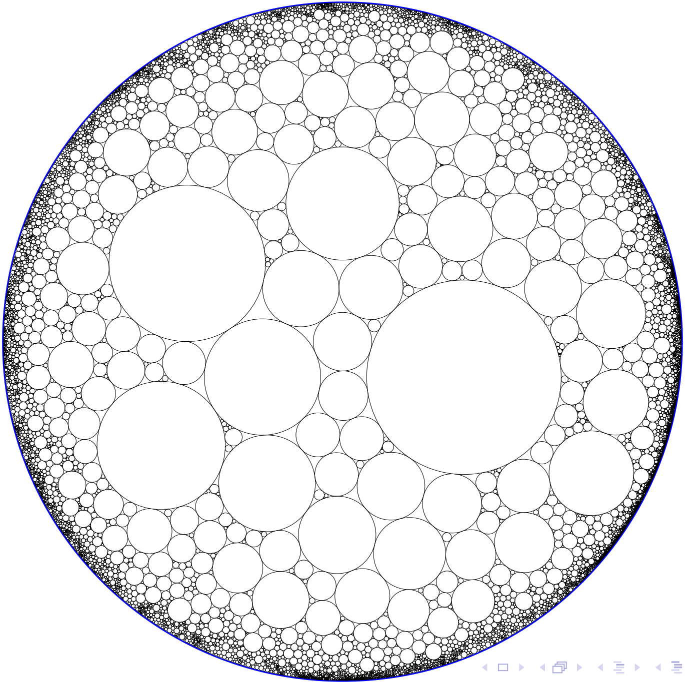
The 7-degree hyperbolic half-space glued with the 6-degree triangular half-space.



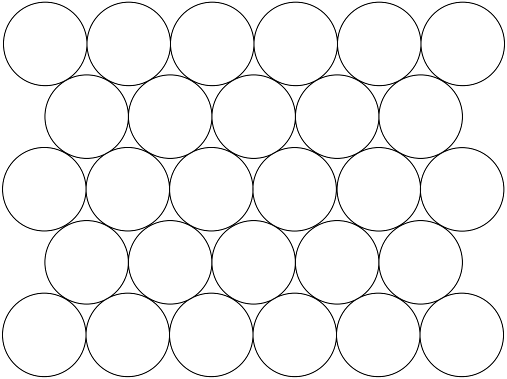
More examples

Rings of degree 7 (grey) are separated by growing bands of degree six vertices (white), causing the hyperbolic radii of circles to decay. The bands of degree six vertices can grow surprisingly quickly without the triangulation becoming recurrent.





Without bounded degree, there is no theory!



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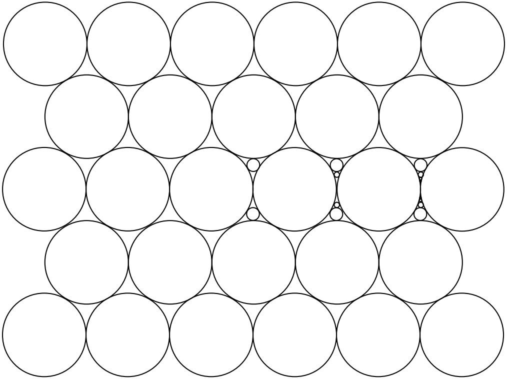

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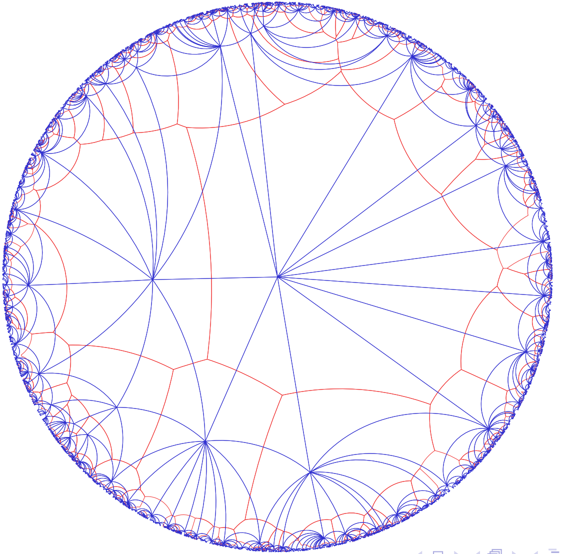
And, in the hyperbolic case,

- Question 3: Does the walker converge to a point in the boundary of the disc? Does the law of the limit have full support and no atoms almost surely?
- Question 4: Is the unit circle a realisation of the Poisson boundary?

Example: Poisson-Voronoi triangulation



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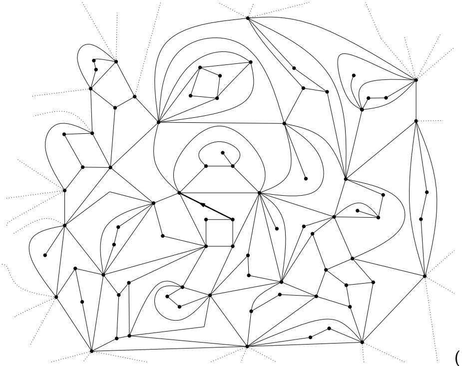

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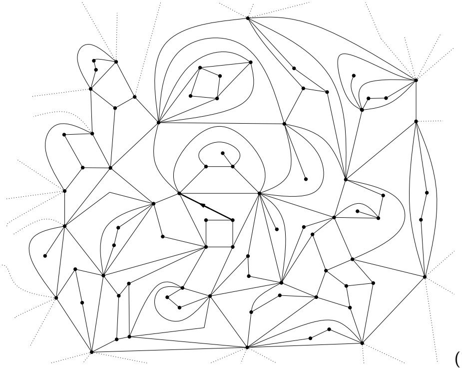

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When $n \to \infty$, what does it look like?

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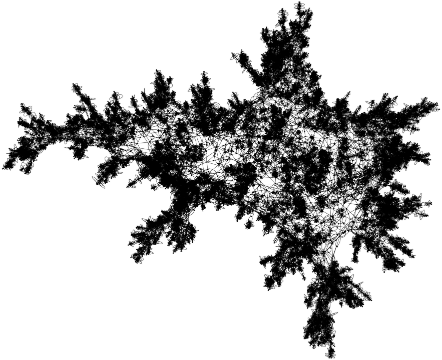

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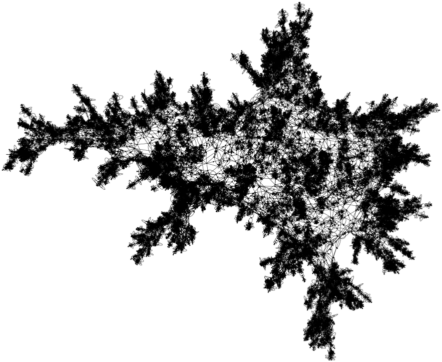

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This is what you get when you scale graph distances by a factor of $n^{-1/4}$ so that the diameter is roughly a constant, then take a Gromov-Hausdorff limit. The resulting limit is called the *Brownian map* and is a random compact metric space homeomorphic to the sphere (Le-Gall 2011, Miermont 2011).

Local limits

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Definition. Given a sequence of *finite* graphs G_n let ρ_n be a uniform random vertex of G_n. We say that the *random rooted graph* (G, ρ) is the **distributional limit** of G_n if for any $r > 0$

$$B_{G_n}(\rho_n, r) \stackrel{(d)}{\Longrightarrow} B_G(\rho, r)\,,$$

where $B_G(\rho, r)$ is the ball of radius r in graph distance around ρ.

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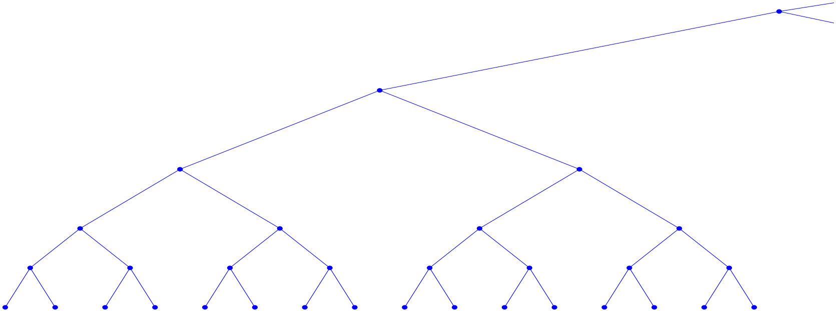

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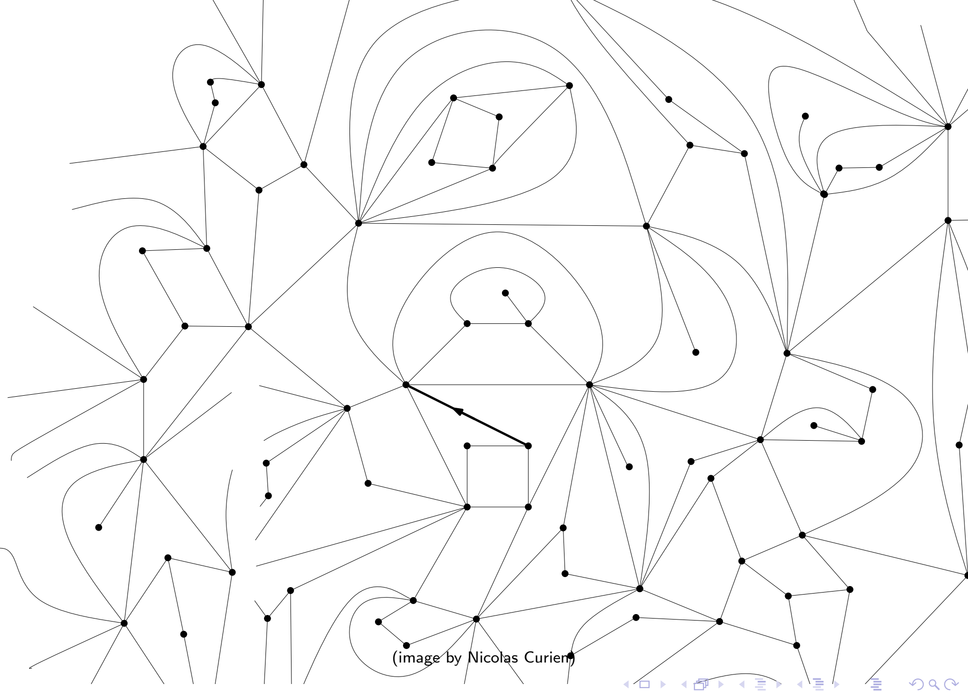
Universality: Other models of random planar graphs are expected to have a distributional limit with similar properties.



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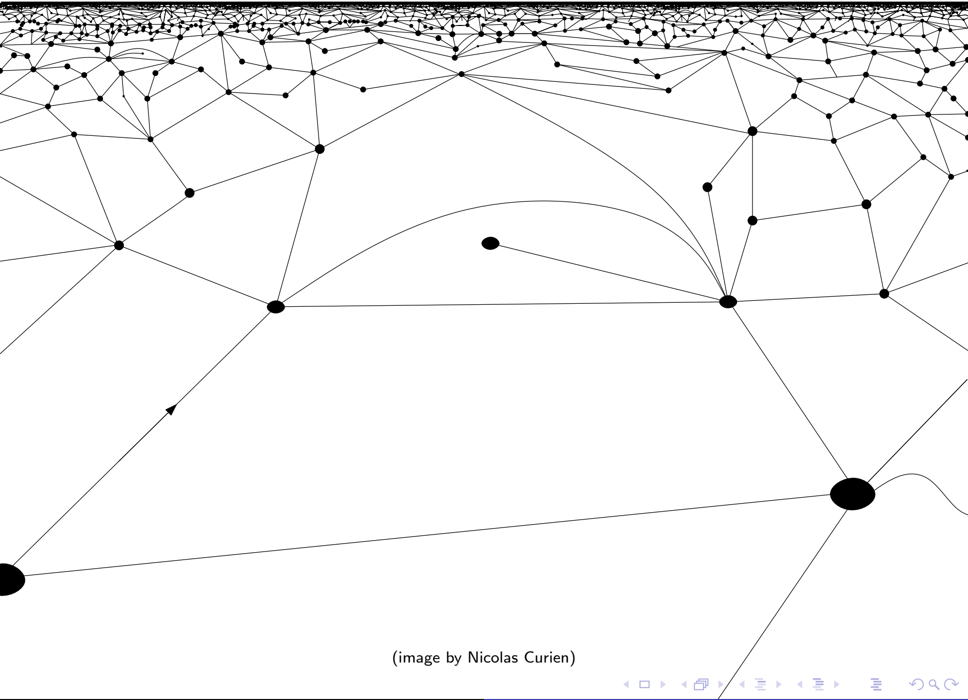

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- Balls of radius r have size roughly r^4 (Angel 2003, Chassaing-Schaeffer 2004).

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- Balls of radius r have size roughly r^4 (Angel 2003, Chassaing-Schaeffer 2004).
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Properties of the UIPT/UIPQ vs. Euclidean geometry

The UIPT/UIPQ behaves like $\mathbb{Z}^2$ in some aspects:

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Question: is it recurrent or transient a.s.?

Recurrence?

Conjecture (Benjamini-Schramm 2001, Angel-Schramm 2003)

The UIPT/UIPQ is almost surely recurrent.

If only the UIPT had bounded degrees...

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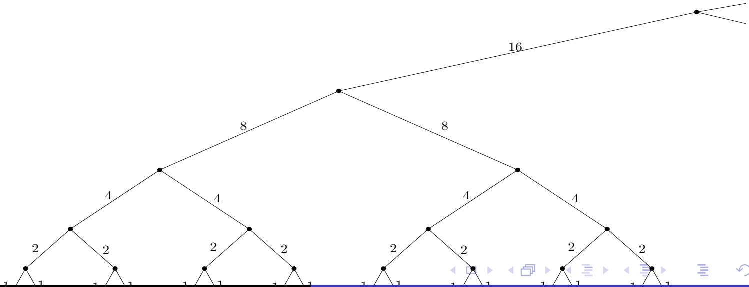

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Sharpness: slightly fatter tail than exponential is not enough.

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