References

- EFT: `hep-ph/9606222`
- Power Counting: `arXiv:1601.07551`
- Matching in HQET and field redefinitions: `hep-ph/9701294`
- Invariants: `arXiv:0907.4763, 1010.3161, 1503.07537, 1512.03433, 1706.08520`
- SMEFT holomorphy: `1409.0868`

Problems v7

1. Show that for a *connected* graph, $V - I + L = 1$. What is the formula if the graph has n connected components?

2. Work out the properties of fermion bilinears $\overline{\psi}(\mathbf{x},t)\Gamma P_L \chi(\mathbf{x},t)$ under C, P, T, where $\Gamma = 1, \gamma^\mu, \sigma^{\mu\nu}$. The results for $P_L \to P_R$ can be obtained by using $L \leftrightarrow R$.

3. Fierz identities are relations of the form

$$(\overline{A}\Gamma B)(\overline{C}\Gamma D) = \sum_i (\overline{C}\Gamma_i B)(\overline{A}\Gamma_i D)$$

   where A, B, C, D are fermion fields. They are much simpler if written in terms of chiral fields, where $\Gamma_i = 1, \gamma^\mu, \sigma^{\mu\nu}$. Work out the Fierz relations for

$$(\overline{A}P_L B)(\overline{C}P_L D),\ (\overline{A}P_L B)(\overline{C}P_R D),\ (\overline{A}\gamma^\mu P_L B)(\overline{C}\gamma_\mu P_L D), (\overline{A}\gamma^\mu P_L B)(\overline{C}\gamma_\mu P_R D),$$
$$(\overline{A}\sigma^{\mu\nu} P_L B)(\overline{C}\sigma_{\mu\nu} P_L D),\ (\overline{A}\sigma^{\mu\nu} P_L B)(\overline{C}\sigma_{\mu\nu} P_R D)$$

   The $P_R\ P_R$ identities are given by using $L \leftrightarrow R$ on the $L\ L$ identities. Do not forget the Fermi minus sign.

4. In $d = 4$ spacetime dimensions, work out the field content of Lorentz-invariant operators with dimension n for $n = 1, \ldots, 6$. At this point, do not try and work out which operators are independent, just the possible structure of allowed operators. Use the notation ϕ for a scalar, ψ for a fermion, $X_{\mu\nu}$ for a field strength, and D for a derivative. For example, an operator of type $\phi^2 D$ such as $\phi D_\mu \phi$ is not allowed because it is not Lorentz-invariant. An operator of type $\phi^2 D^2$ could be either $D_\mu \phi D^\mu \phi$ or $\phi D^2 \phi$, so a $\phi^2 D^2$ operator is allowed, and we will worry later about how many independent operators $\phi^2 D^2$ we can construct.

5. For $d = 2, 3, 4, 5, 6$ dimensions, work out the field content of operators with dimension $n \leq d$, i.e. the "renormalizable" operators.

6. Show that if $\alpha_s(\mu)$ is fixed at some high scale, say $\mu = 1\,\text{TeV}$, then $m_p \propto m_t^{2/27}$, where m_p is the proton mass and m_t is the top quark mass.

7. (a) Compute in dimensional regularization in $d = 4 - 2\epsilon$ dimensions

$$I_F = -i\mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)(k^2 - M^2)} + \text{c.t.}$$
$$I_{\text{EFT}} = -i\mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)} \left[-\frac{1}{M^2} - \frac{k^2}{M^4} + \ldots \right] + \text{c.t.}.$$

Both integrals only have UV divergences, so the $1/\epsilon$ pieces are canceled by the counterterms. Determine the counterterm contributions $I_{F,ct}$, $I_{\text{EFT},ct}$.

(b) Compute $I_M \equiv (I_F + I_{F,ct}) - (I_{\text{EFT}} + I_{\text{EFT},ct})$ and show that it is analytic in m.

(c) Compute $I_F^{(\text{exp})}$, i.e. I_F with the IR m scale expanded out

$$I_F^{(\text{exp})} = -i\mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - M^2)} \left[\frac{1}{k^2} + \frac{m^2}{k^4} + \ldots \right]$$

Note that the first term in the expansion has a $1/\epsilon$ UV divergence, and the remaining terms have $1/\epsilon$ IR divergences.

(d) Compute $I_F^{(\text{exp})} + I_{F,ct}$. Show that the UV divergence cancels, and the remaining $1/\epsilon$ IR divergence is the same as the UV divergent counterterm $I_{\text{EFT},ct}$ in the EFT.

(e) Compute $I_{\text{EFT}}^{(\text{exp})}$, i.e. I_{EFT} with the IR m scale expanded out. Show that it is a scaleless integral which vanishes. Using the known UV divergence from (a), write it in the form

$$I_{\text{EFT}}^{(\text{exp})} = \frac{1}{16\pi^2} \left[\frac{C}{\epsilon_{\text{UV}}} - \frac{C}{\epsilon_{\text{IR}}} \right]$$

and that the IR divergence agrees with that in $I_F^{(\text{exp})} + I_{F,ct}$.

(f) Compute $\left(I_F^{(\text{exp})} + I_{F,ct} \right) - \left(I_{\text{EFT}}^{(\text{exp})} + I_{\text{EFT},ct} \right)$ and show that all the $1/\epsilon$ divergences (both UV and IR) cancel, and the result is equal to I_M found in (b).

(g) Make sure you understand why you can compute I_M simply by taking $I_F^{(\text{exp})}$ and dropping all $1/\epsilon$ terms (both UV and IR).

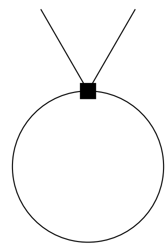
8. Show that for $SU(N)$,

$$[T^A]^\alpha{}_\beta [T^A]^\lambda{}_\sigma = \frac{1}{2} \delta^\alpha_\sigma \delta^\lambda_\beta - \frac{1}{2N} \delta^\alpha_\beta \delta^\lambda_\sigma$$

and the color Fierz identities

$$\delta^\alpha{}_\beta \delta^\lambda{}_\sigma = \frac{1}{N} \delta^\alpha_\sigma \delta^\lambda_\beta + 2 [T^A]^\alpha{}_\sigma [T^A]^\lambda{}_\beta$$
$$[T^A]^\alpha{}_\beta [T^A]^\lambda{}_\sigma = \frac{N^2 - 1}{2N^2} \delta^\alpha_\sigma \delta^\lambda_\beta - \frac{1}{N} [T^A]^\alpha{}_\sigma [T^A]^\lambda{}_\beta$$

9. Compute the one-loop scalar graph with a scalar of mass m and interaction $-\lambda\phi^4/4!$ in the $\overline{\text{MS}}$ scheme.



10. $*$ Compute the decay rate $\Gamma(b \to c e^- \overline{\nu}_e)$ with the interaction Lagrangian

$$L = -\frac{4G_F}{\sqrt{2}} V_{cb} (\overline{c}\gamma^\mu P_L b)(\overline{\nu}_e \gamma_\mu P_L e)$$

with $m_e \to 0$, $m_\nu \to 0$, but retaining the dependence on $\rho = m_c^2/m_b^2$. It is convenient to write the three-body phase space in terms of the variables $x_1 = 2E_e/m_b$ and $x_2 = 2E_\nu/m_b$.

11. Compute the anomalous dimension of $\overline{q}q$ in QCD. Start with massless QCD, and treat $L = -m\overline{q}q$ as an operator insertion.

12. $*$ Compute the anomalous dimension mixing matrix of

$$O_1 = (\overline{b}^\alpha \gamma^\mu P_L c_\alpha)(\overline{u}^\alpha \gamma^\mu P_L d_\alpha) \qquad\qquad O_2 = (\overline{b}^\alpha \gamma^\mu P_L c_\beta)(\overline{u}^\beta \gamma^\mu P_L d_\alpha)$$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Two other often used bases are

$$Q_1 = (\overline{b}\gamma^\mu P_L c)(\overline{u}\gamma^\mu P_L d) \qquad\qquad Q_2 = (\overline{b}\gamma^\mu P_L T^A c)(\overline{u}\gamma^\mu P_L T^A d)$$

and

$$O_\pm = O_1 \pm O_2$$

So let

$$\mathcal{L} = c_1 O_1 + c_2 O_2 = d_1 Q_1 + d_2 Q_2 = c_+ O_+ + c_- O_-$$

and work out the transformation between the anomalous dimensions for $d_{1,2}$ and $c_{+,-}$ in terms of those for $c_{1,2}$,

13. The equation of motion for $\lambda\phi^4$ theory,

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \qquad\qquad E = (-\partial^2 - m^2)\phi - \frac{\lambda}{3!}\phi^3$$

The EOM Ward identity for $\theta = F(\phi)E$ is

$$\langle 0|T\left\{\phi(x_1)\ldots\phi(x_n)\theta(z)\right\}|0\rangle = i\sum_{r=1}^{n}\delta(z-x_r)\langle 0|T\left\{\phi(x_1)\ldots\cancel{\phi(x_r)}\ldots\phi(x_n)F(z)\right\}|0\rangle$$

In momentum space, integrate both sides with

$$\int \mathrm{d}z e^{-iq\cdot z}\prod_i \int \mathrm{d}x_i e^{-ip_i\cdot x_i}$$

to give the momentum space version

$$\langle 0|T\left\{\widetilde{\phi}(p_1)\ldots\widetilde{\phi}(p_n)\widetilde{\theta}(q)\right\}|0\rangle = i\sum_{r=1}^{n}\delta(z-x_r)\langle 0|T\left\{\widetilde{\phi}(p_1)\ldots\cancel{\widetilde{\phi}(p_r)}\ldots\widetilde{\phi}(x_n)\widetilde{F}(q+p_r)\right\}|0\rangle$$

(a) Consider the equation of motion operator

$$\theta_1 = \phi\, E = \phi(-\partial^2 - m^2)\phi - \frac{\lambda}{3!}\phi^4$$

and verify the Ward identity by explicit calculation at order λ (i.e. tree level) for $\phi\phi$ scattering, i.e. a graph with four ϕ fields, $n = 4$.

(b) Take the on-shell limit $p_r^2 \to m^2$ at fixed $q \neq 0$ of

$$\prod_r (-i)(p_r^2 - m^2) \times \text{Ward Identity}$$

and verify that both sides of the Ward identity vanish. Note that both sides do not vanish if one first takes $q = 0$ and then takes the on-shell limit.

(c) * Repeat the above calculation to order λ^2, i.e. one loop.

(d) * Repeat (to one loop) for the equation of motion operator

$$\theta_2 = \phi^3\, E = \phi^3(-\partial^2 - m^2)\phi - \frac{\lambda}{3!}\phi^6$$

14. Take the heavy quark Lagrangian

$$\begin{aligned}\mathcal{L}_v &= \bar{Q}_v\left\{iv\cdot D + i\not{D}_\perp \frac{1}{2m + iv\cdot D} i\not{D}_\perp\right\}Q_v \\ &= \bar{Q}_v\left\{iv\cdot D - \frac{1}{2m}\not{D}_\perp\not{D}_\perp + \frac{1}{4m^2}\not{D}_\perp (iv\cdot D)\not{D}_\perp + \ldots\right\}Q_v\end{aligned}$$

and use a sequence of field redefinitions to eliminate the $1/m^2$ suppressed $v \cdot D$ term. $(iv \cdot D)Q_v = 0$ is the equation of motion for the heavy quark field, so this example shows how you eliminate equation of motion operators. Here v^μ is a velocity vector with $v \cdot v = 1$, and for a four-vector A,

$$D_\perp^\mu \equiv D^\mu - (v \cdot D) v^\mu$$

If you prefer, you can work in the rest frame of the heavy quark, where $v^\mu = (1, 0, 0, 0)$, $v \cdot D = D^0$ and $D_\perp^\mu = (0, \mathbf{D})$.

15. $*$ Compute the on-shell electron form factors $F_1(q^2)$ and $F_2(q^2)$ expanded to first order in q^2/m^2 using dimensional regularization to regulate the IR and UV divergences. This gives the one-loop matching to heavy-electron EFT. The non-Abelian version (in `hep-ph/9701294`) gives the one-loop matching to the HQET Lagrangian. Note that it is much simpler to *first* expand and then do the Feynman parameter integrals. $F_{1,2}(q^2)$ are given in many field theory textbooks, but usually not in pure dim reg.

16. The SCET matching for the vector current $\overline{\psi}\gamma^\mu\psi$ for the Sudakov form factor can be done by repeating the previous problem with external masses $m \to 0$ and $p^2 \to 0$, and doing the integral in pure dim reg with $Q^2 = -q^2 \neq 0$. Here Q^2 is the big scale, whereas in the previous problem q^2 was the small scale. The spacelike calculation $Q^2 > 0$ avoids having to deal with the $+i0^+$ terms in the Feynman propagator which lead to imaginary parts. The timelike result can then be obtained by analytic continuation.

17. Compute the SCET matching for thrust by analytically continuing the result in the previous problem to timelike q^2. Be careful about the sign of the imaginary parts.

18. Show that the power counting formula

$$L \sim \frac{\Lambda^4}{16\pi^2} \left[\frac{\partial}{\Lambda}\right]^{N_p} \left[\frac{4\pi\,\phi}{\Lambda}\right]^{N_\phi} \left[\frac{4\pi\,A}{\Lambda}\right]^{N_A} \left[\frac{4\pi\,\psi}{\Lambda^{3/2}}\right]^{N_\psi} \left[\frac{g}{4\pi}\right]^{N_g} \left[\frac{y}{4\pi}\right]^{N_y} \left[\frac{\lambda}{16\pi^2}\right]^{N_\lambda},$$

for an EFT Lagrangian is self-consistent, i.e. an arbitrary graph with insertions of vertices of this form generates an interaction which maintains the same form. [See `1601.07551` and NPB 234 (1984) 189]

19. Show (by explicit calculation) for a general 2×2 matrix A that

$$0 = \frac{1}{6}\langle A\rangle^3 - \frac{1}{2}\langle A\rangle\langle A^2\rangle + \frac{1}{3}\langle A^3\rangle, \qquad 0 = \frac{1}{2}\langle A\rangle^2 - \frac{1}{2}\langle A^2\rangle - \langle A\rangle A + A^2$$

and for general 2×2 matrices A, B, C that

$$0 = \langle A\rangle\langle B\rangle\langle C\rangle - \langle A\rangle\langle BC\rangle - \langle B\rangle\langle AC\rangle - \langle C\rangle\langle AB\rangle + \langle ABC\rangle + \langle ACB\rangle\,.$$

Identities analogous to this for 3×3 matrices are used to remove L_0 and replaced it by $L_{1,2,3}$ in χPT, as discussed by Pich in his lectures.

20. Compute the Hilbert series for the ring of invariants generated by

(a) x, y (each of dimension 1), and invariant under the transformation $(x, y) \to (-x, -y)$.

(b) x, y, z (each of dimension 1), and invariant under the transformation $(x, y, z) \to (-x, -y, -z)$.

21. Show that the Jarlskog invariant

$$J = \left\langle X_u^2 X_d^2 X_u X_d \right\rangle - \left\langle X_d^2 X_u^2 X_d X_u \right\rangle$$

is the lowest order CP-odd invariant made of the quark mass matrices. Here

$$X_u \equiv M_u M_u^\dagger, \qquad X_d = M_d M_d^\dagger \qquad M_u \to L M_u R_u^\dagger \qquad M_d \to L M_d R_d^\dagger \tag{1}$$

Show that J can also be written in the form

$$J = \frac{1}{3} \left\langle [X_u, X_d]^3 \right\rangle$$

and explicitly work out J in the SM using the CKM matrix convention of the PDG.

22. Show that $(\psi_{Lr}^T C \psi_{Ls})$ is symmetric in rs and $(\psi_{Lr}^T C \sigma^{\mu\nu} \psi_{Ls})$ is antisymmetric in rs.

23. (a) Show that the unique dimension 5 operator in the SMEFT for $n_g = 1$ generation is

$$L = c_5 \; (l_i^T C l_j) H_k H_l \epsilon^{ik} \epsilon^{jl}$$

(b) How many such operators are there for n_g generations?

(c) Show that this operator generates a Majorana neutrino mass when H gets a VEV, and find M_ν in terms of c_5 and v.

24. $*$ In the SMEFT for n_g generations, how many operators are there of the following kind:

(a) Q_{He}

(b) Q_{ledq}

(c) $Q_{lq}^{(1)}$

(d) $Q_{qq}^{(1)}$

(e) Q_{ll}

(f) Q_{uu}

(g) $*$ Q_{ee}

(h) $*$ Show that there are a total of 2499 dimension-six $\Delta B = \Delta L = 0$ operators.