3. High-Energy Dynamics

- CCWZ Formalism
- Heavy Fields
- Low-Energy Constants
- Large–N_C Limit
- Asymptotic Behaviour

Energy Scale	Fields	Effective Theory
M_W	t, b, c $s, d, u \; ; \; G^a$	$\text{QCD}^{N_f=6}$
$\lesssim m_c$	$s, d, u \; ; \; G^a$	$\text{QCD}^{N_f=3}$
Λ_χ	V, A, S, P π, K, η	$\text{R}\chi\text{T}$
$\lesssim M_K$	π, K, η	$\chi\text{PT}^{N_f=3}$
$\lesssim M_\pi$	π	$\chi\text{PT}^{N_f=2}$

Goldstones and Coset-Space Coordinates: $G \xrightarrow{SSB} H$

Goldstone fields: $\vec{\phi} \equiv (\phi_1, \ldots, \phi_N) \quad \longrightarrow \quad \vec{\phi}' = \vec{\mathcal{F}}(g, \vec{\phi}) \quad , \quad g \in G$

$$N = \dim(G) - \dim(H) \qquad , \qquad \vec{\mathcal{F}}(e, \vec{\phi}) = \vec{\phi} \qquad , \qquad \vec{\mathcal{F}}(g_1 g_2, \vec{\phi}) = \vec{\mathcal{F}}\left(g_1, \vec{\mathcal{F}}(g_2, \vec{\phi})\right)$$

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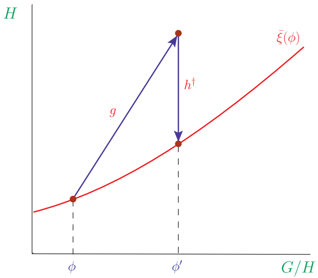
$\tilde{\mathcal{F}}$: invertible mapping between Goldstone fields and G/H



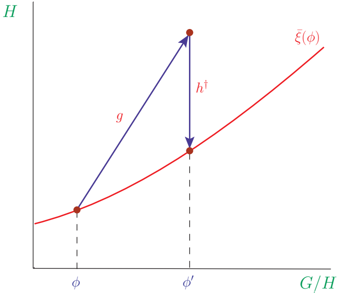
$$\vec{\mathcal{F}}(gh, \vec{0}) = \vec{\mathcal{F}}(g, \vec{0}) \qquad \forall g \in G \,, \forall h \in H$$

$$\vec{\mathcal{F}}(h, \vec{0}) = \vec{0} \quad , \quad h \in H \qquad \text{(vacuum invariant)}$$

$$\vec{\mathcal{F}}(g_i, \vec{0}) = \vec{\mathcal{F}}(g_j, \vec{0}) \quad \longrightarrow \quad g_i^{-1} g_j \in H$$

Coset representative: $\bar{\xi}(\phi) \in G$

Coset Space Coordinates: $G \equiv SU(3)_L \otimes SU(3)_R \xrightarrow{SCSB} H \equiv SU(3)_V$



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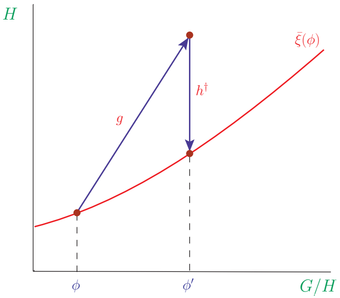

$$\bar{\xi}(\phi) \equiv (\xi_L(\phi), \xi_R(\phi)) \in G$$

$$\xi_L(\phi) \xrightarrow{G} g_L\, \xi_L(\phi)\, h^\dagger(\phi, g)$$

$$\xi_R(\phi) \xrightarrow{G} g_R\, \xi_R(\phi)\, h^\dagger(\phi, g)$$

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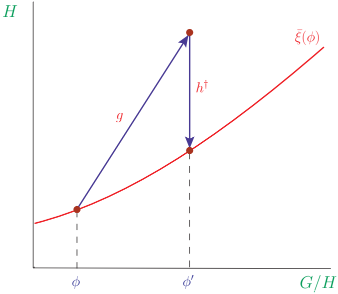

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$$\xi_R(\phi) \xrightarrow{G} g_R\, \xi_R(\phi)\, h^\dagger(\phi, g)$$

$$\mathbf{U}(\phi) \equiv \xi_R(\phi)\, \xi_L^\dagger(\phi) \xrightarrow{G} g_R\, \mathbf{U}(\phi)\, g_L^\dagger$$

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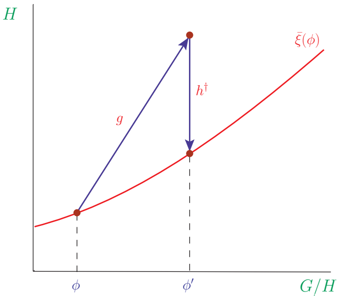

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$$\mathbf{U}(\phi) \equiv \xi_R(\phi)\, \xi_L^\dagger(\phi) \xrightarrow{G} g_R\, \mathbf{U}(\phi)\, g_L^\dagger$$

Canonical choice:

$$\xi_R(\phi) = \xi_L(\phi)^\dagger \equiv \mathbf{u}(\phi) \xrightarrow{G} g_R\, \mathbf{u}(\phi)\, h^\dagger(\phi, g) = h(\phi, g)\, \mathbf{u}(\phi)\, g_L^\dagger$$

$$\mathbf{U}(\phi) = \mathbf{u}(\phi)^2 = \exp\left\{ i\, \frac{\sqrt{2}}{f}\, \mathbf{\Phi} \right\}$$

CCWZ Formalism

Callan–Coleman–Wess–Zumino

$$\mathbf{u}(\varphi) \quad \xrightarrow{G} \quad g_R\, \mathbf{u}(\varphi)\, h^\dagger(\phi, g) \;=\; h(\phi, g)\, \mathbf{u}(\phi)\, g_L^\dagger$$

$SU(3)_V$ octets: $$\mathbf{X} \quad \xrightarrow{\mathbf{G}} \quad \mathbf{h}(\phi, \mathbf{g})\;\mathbf{X}\;\mathbf{h}(\phi, \mathbf{g})^\dagger$$

$$\mathbf{R} \equiv \tfrac{1}{2}\lambda^a R^a \quad , \quad \nabla_\mu R = \partial_\mu R + [\Gamma_\mu, R]$$

$$u_\mu \equiv i\, u^\dagger D_\mu U\, u^\dagger = u_\mu^\dagger \quad , \quad h^{\mu\nu} = \nabla^\mu u^\nu + \nabla^\nu u^\mu$$

$$f_\pm^{\mu\nu} = u\, F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu}\, u \quad , \quad \chi_\pm \equiv u^\dagger \chi\, u^\dagger \pm u\, \chi^\dagger u$$

$$\Gamma_\mu = \frac{1}{2}\left\{ u^\dagger(\partial_\mu - i\, r_\mu)\, u + u\,(\partial_\mu - i\,\ell_\mu) u^\dagger \right\}$$

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$$\mathcal{L}_2 = \frac{f^2}{4}\, \langle u^\mu u_\mu + \chi_+ \rangle$$

RχT

Resonance Nonet Multiplets: $\mathbf{V}(1^{--})\,,\ \mathbf{A}(1^{++})\,,\ \mathbf{S}(0^{++})\,,\ \mathbf{P}(0^{-+})$

$$\mathcal{L}_2^V = \frac{F_V}{2\sqrt{2}}\,\langle V_{\mu\nu}\, f_+^{\mu\nu}\rangle + \frac{i\,G_V}{\sqrt{2}}\,\langle V_{\mu\nu}\, u^\mu\, u^\nu\rangle$$

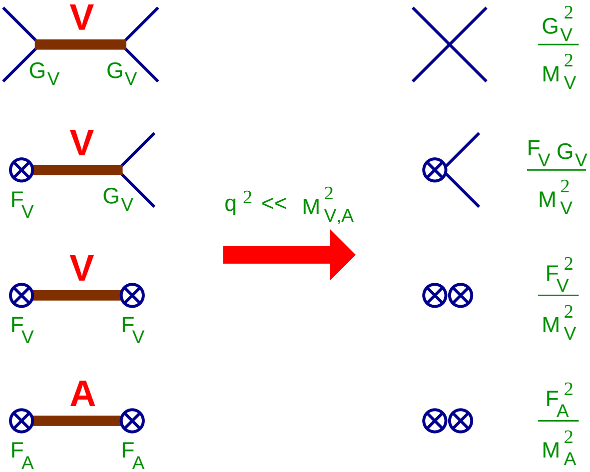
$$\mathcal{L}_2^A = \frac{F_A}{2\sqrt{2}}\,\langle A_{\mu\nu}\, f_-^{\mu\nu}\rangle$$

$$\mathcal{L}_2^S = c_d\,\langle S\, u^\mu\, u_\mu\rangle + c_m\,\langle S\,\chi_+\rangle$$

$$\mathcal{L}_2^P = i\, d_m\,\langle P\,\chi_-\rangle$$

$$u_\mu = i\, u^\dagger\, D_\mu U\, u^\dagger = u_\mu^\dagger \qquad ; \qquad U = u^2$$

$$f_\pm^{\mu\nu} = u\, F_L^{\mu\nu}\, u^\dagger \pm u^\dagger\, F_R^{\mu\nu}\, u \qquad ; \qquad \chi_\pm = u^\dagger\,\chi\, u^\dagger \pm u\,\chi^\dagger\, u$$



$$\left\{ \mathbf{V}\,[M_V, G_V, F_V]\;,\; \mathbf{A}\,[M_A, F_A] \right\} \longleftrightarrow \left\{ \mathbf{S}\,[M_S, c_d, c_m]\;,\; \mathbf{P}\,[M_P, d_m] \right\}$$

$O(N_C)$:

$$2\,L_1 \;=\; L_2 \;=\; \sum_i \frac{G_{V_i}^2}{4\,M_{V_i}^2} \qquad ; \qquad L_3 \;=\; \sum_i \left\{ -\frac{3\,G_{V_i}^2}{4\,M_{V_i}^2} + \frac{c_{d_i}^2}{2\,M_{S_i}^2} \right\}$$

$$L_5 \;=\; \sum_i \frac{c_{d_i}\, c_{m_i}}{M_{S_i}^2} \qquad ; \qquad L_8 \;=\; \sum_i \left\{ \frac{c_{m_i}^2}{2\,M_{S_i}^2} - \frac{d_{m_i}^2}{2\,M_{P_i}^2} \right\}$$

$$L_9 \;=\; \sum_i \frac{F_{V_i}\, G_{V_i}}{2\,M_{V_i}^2} \qquad ; \qquad L_{10} \;=\; \frac{1}{4} \sum_i \left\{ \frac{F_{A_i}^2}{M_{A_i}^2} - \frac{F_{V_i}^2}{M_{V_i}^2} \right\}$$

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$$2\,L_1 - L_2 \;=\; L_4 \;=\; L_6 \;=\; 0 \qquad ; \qquad L_7 \;=\; -\frac{\tilde{d}_m^{\,2}}{2\,M_{\eta_1}^2}$$

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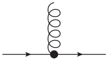
BUT $\qquad M_{\eta_1}^2 \;\sim\; O\left(\frac{1}{N_C},\, \mathcal{M}\right)$

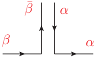
Large-N_C QCD:

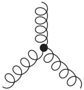
$$G^\mu \equiv \sum_{a=1}^{N_C^2-1} \frac{\lambda^a}{2} G_a^\mu \;\sim\; N_C^2 - 1 \approx N_C^2 \qquad , \qquad q_\alpha \;\sim\; N_C$$

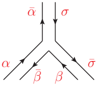
$$N_C \to \infty \qquad , \qquad N_C\, g^2 \sim 1 \qquad , \qquad \alpha \sim 1/\beta_1 \sim 1/N_C$$

$\bar{q}_\alpha \,(G_\mu)^\alpha{}_\beta\, q^\beta$









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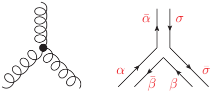
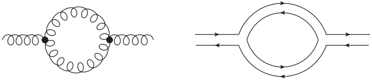
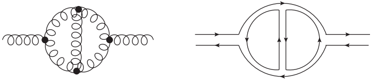




$$\left(\frac{1}{\sqrt{N_C}}\right)^2 N_C = 1$$



$$\left(\frac{1}{\sqrt{N_C}}\right)^4 N_C^2 = 1$$

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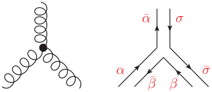

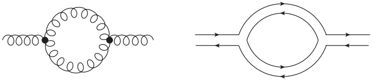

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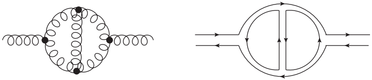

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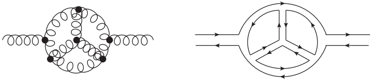
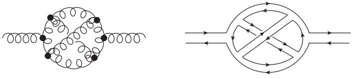


$$\left(\frac{1}{\sqrt{N_C}}\right)^6 N_C^3 = 1$$

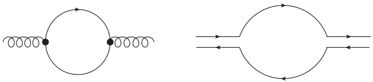
Planar topology



$$\left(\frac{1}{\sqrt{N_C}}\right)^6 N_C = \frac{1}{N_C^2}$$

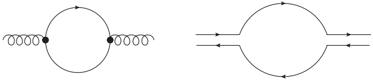
Non-planar topology

- **Quark loops suppressed:**



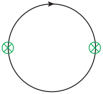
$$\left(\frac{1}{\sqrt{N_C}}\right)^2 = \frac{1}{N_C}$$

- **Quark loops suppressed:**

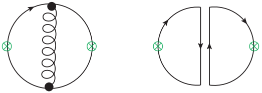


$$\left(\frac{1}{\sqrt{N_C}}\right)^2 = \frac{1}{N_C}$$

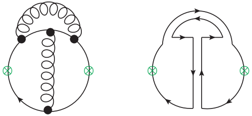
- **Quark correlation functions:** $\otimes \;=\; \bar{q}\,\Gamma q$



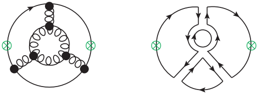
N_C



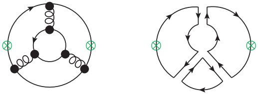
$$\left(\frac{1}{\sqrt{N_C}}\right)^2 N_C^2 = N_C$$



$$\left(\frac{1}{\sqrt{N_C}}\right)^4 N_C = \frac{1}{N_C}$$



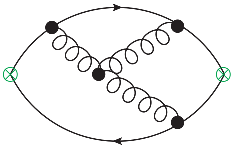
$$\left(\frac{1}{\sqrt{N_C}}\right)^6 N_C^4 = N_C$$

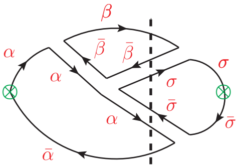


$$\left(\frac{1}{\sqrt{N_C}}\right)^6 N_C^3 = 1$$

Planar diagrams with only a single quark loop at the edge

Assume that large–N_C QCD confines





$\bar{q}_\alpha \, G^\alpha{}_\sigma \, G^\sigma{}_\beta \, q^\beta$

All diagram cuts correspond to one-meson colour-singlet states

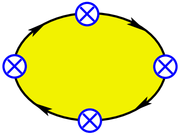
$$\langle J(k)\, J(-k)\rangle \;=\; \sum_n \frac{|a_n|^2}{k^2 - M_n^2} \;\sim\; \mathcal{O}(N_C)$$

$$a_n \;=\; \langle 0|J|n\rangle \;\sim\; \mathcal{O}\left(\sqrt{N_C}\right) \qquad , \qquad M_N \;\sim\; \mathcal{O}(1)$$

Large–N_C Counting Rules

't Hooft '74, Witten '79, Manohar hep-ph/9802419

$$g_s \sim 1/\sqrt{N_C} \qquad ; \qquad \alpha_s \sim 1/N_C \qquad ; \qquad \langle T\,(J_1 \cdots J_n)\rangle \;\sim\; N_C$$

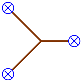
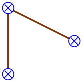


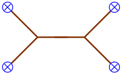
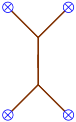
- Dominance of planar gluonic exchanges
- Non-planar diagrams suppressed by $1/N_C^2$
- Internal quark loops suppressed by $1/N_C$

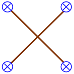
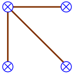
Colour Confinement ⟶ $J\,|0\rangle \;\sim\; |1\;\text{Meson}\rangle$

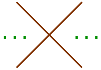
$$\langle J(k)\,J(-k)\rangle \;=\; \sum_n \frac{f_n^2}{k^2 - M_n^2}$$

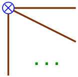
- **Infinite** number of mesons $(\sim \ln k^2)$
- $f_n = \langle 0|J|n\rangle \sim \sqrt{N_C}$; $M_n \sim \mathcal{O}(1)$
- Mesons are **free**, **stable** and **non-interacting**

$\langle J\,J\,J \rangle \;=\; \sum$  $+\;\sum$ 

$\langle J\,J\,J\,J \rangle \;=\; \sum$  $+\;\sum$ 

$+\;\sum$  $+\;\sum$ 

 $\sim\; N_C^{1-\frac{n}{2}}$

 $\sim\; N_C^{1-\frac{n}{2}}$

Crossing + Unitarity ➡ **Tree Approximation to some Local Effective Meson Lagrangian**

$O(N_C)$:

$$2\,L_1 \;=\; L_2 \;=\; \sum_i \frac{G_{V_i}^2}{4\,M_{V_i}^2} \qquad ; \qquad L_3 \;=\; \sum_i \left\{ -\frac{3\,G_{V_i}^2}{4\,M_{V_i}^2} + \frac{c_{d_i}^2}{2\,M_{S_i}^2} \right\}$$

$$L_5 \;=\; \sum_i \frac{c_{d_i}\,c_{m_i}}{M_{S_i}^2} \qquad ; \qquad L_8 \;=\; \sum_i \left\{ \frac{c_{m_i}^2}{2\,M_{S_i}^2} - \frac{d_{m_i}^2}{2\,M_{P_i}^2} \right\}$$

$$L_9 \;=\; \sum_i \frac{F_{V_i}\,G_{V_i}}{2\,M_{V_i}^2} \qquad ; \qquad L_{10} \;=\; \frac{1}{4} \sum_i \left\{ \frac{F_{A_i}^2}{M_{A_i}^2} - \frac{F_{V_i}^2}{M_{V_i}^2} \right\}$$

$\mathbf{O(N_C)}$:

$$2\, L_1 \,=\, L_2 \,=\, \sum_i \frac{G_{V_i}^2}{4\, M_{V_i}^2} \qquad ; \qquad L_3 \,=\, \sum_i \left\{ -\frac{3\, G_{V_i}^2}{4\, M_{V_i}^2} + \frac{c_{d_i}^2}{2\, M_{S_i}^2} \right\}$$

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$\mathbf{O(1)}$: $$2\, L_1 - L_2 \,=\, L_4 \,=\, L_6 \,=\, 0 \qquad ; \qquad L_7 \,=\, -\frac{\tilde{d}_m^{\,2}}{2\, M_{\eta_1}^2}$$

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$$2\,L_1 \;=\; L_2 \;=\; \sum_i \frac{G_{V_i}^2}{4\,M_{V_i}^2} \qquad ; \qquad L_3 \;=\; \sum_i \left\{ -\frac{3\,G_{V_i}^2}{4\,M_{V_i}^2} + \frac{c_{d_i}^2}{2\,M_{S_i}^2} \right\}$$

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$$L_9 \;=\; \sum_i \frac{F_{V_i}\, G_{V_i}}{2\,M_{V_i}^2} \qquad ; \qquad L_{10} \;=\; \frac{1}{4} \sum_i \left\{ \frac{F_{A_i}^2}{M_{A_i}^2} - \frac{F_{V_i}^2}{M_{V_i}^2} \right\}$$

$\mathbf{O(1)}$:

$$2\,L_1 - L_2 \;=\; L_4 \;=\; L_6 \;=\; 0 \qquad ; \qquad L_7 \;=\; -\frac{\tilde{d}_m^{\,2}}{2\,M_{\eta_1}^2}$$

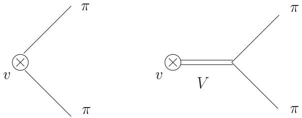
BUT $\qquad M_{\eta_1}^2 \;\sim\; O\left(\frac{1}{N_C},\, \mathcal{M}\right)$

L_i's from Resonance Exchange

i	$10^3 \cdot L_i^r(M_\rho)$	V	A	S	η_1	Total
1	0.7 ± 0.3	0.6	0	0	0	0.6
2	1.3 ± 0.3	1.2	0	0	0	1.2
3	-3.5 ± 1.1	-3.6	0	0.6	0	-3.0
4	-0.3 ± 0.5	0	0	0	0	0.0
5	1.4 ± 0.5	0	0	$1.4^{a)}$	0	1.4
6	-0.2 ± 0.3	0	0	0	0	0.0
7	-0.4 ± 0.2	0	0	0	-0.3	-0.3
8	0.9 ± 0.3	0	0	$0.9^{a)}$	0	0.9
9	6.9 ± 0.7	$6.9^{a)}$	0	0	0	6.9
10	-5.5 ± 0.7	-10.0	4.0	0	0	-6.0

$^{a)}$ Input

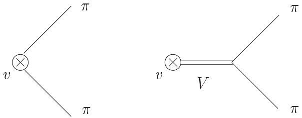
Vector Form Factor



$\langle \pi | \mathbf{v}_\mu | \pi \rangle$ **:**

$$F_V(t) \, = \, 1 \, + \, \sum_i \, \frac{F_{V_i} \, G_{V_i}}{f^2} \, \frac{t}{M_{V_i}^2 - t}$$

Vector Form Factor



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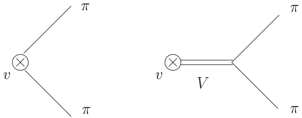
Short-distance QCD constraint: $$\lim_{t \to \infty} F_V(t) = 0$$

Vector Form Factor



$\langle \pi | \mathbf{v}_\mu | \pi \rangle$ **:** $$F_V(t) \, = \, 1 \, + \, \sum_i \, \frac{F_{V_i} \, G_{V_i}}{f^2} \, \frac{t}{M_{V_i}^2 - t}$$

Short-distance QCD constraint: $$\lim_{t\to\infty} F_V(t) = 0$$

$\Rightarrow$ $$\boxed{\sum_i \, F_{V_i} \, G_{V_i} \, = \, f^2}$$

Weinberg Sum Rules

Chiral Symmetry:

$$\Pi_{LR}^{\mu\nu}(q) \equiv \int d^4x \, \mathrm{e}^{iqx} \, \langle 0 | T(J_L^{\mu}(x) \, J_R^{\nu}(0)^{\dagger}) | 0 \rangle = (-g^{\mu\nu} q^2 + q^{\mu} q^{\nu}) \, \Pi_{LR}(q^2) = 0$$

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OPE: $\Pi_{LR}^{\mu\nu}(q) \neq 0$ only through **order parameters of EWSB**
(operators invariant under **H** but not under **G**)

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Asymptotically-Free Theories: $\lim_{s\to\infty} s^2 \, \Pi_{LR}(s) = 0$ **(Bernard et al.)**

Weinberg Sum Rules

Chiral Symmetry:

$$\Pi^{\mu\nu}_{LR}(q) \;\equiv\; \int d^4x\; e^{iqx}\; \langle 0| T(J^\mu_L(x)\, J^\nu_R(0)^\dagger)|0\rangle \;=\; (-g^{\mu\nu} q^2 + q^\mu q^\nu)\; \Pi_{LR}(q^2) \;=\; 0$$

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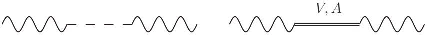
$\Rightarrow$

$$\frac{1}{\pi}\int_0^\infty ds\; [\mathrm{Im}\Pi_{VV}(s) - \mathrm{Im}\Pi_{AA}(s)] \;=\; f^2 \qquad (1^{\mathrm{st}}\ \mathrm{WSR})$$

$$\frac{1}{\pi}\int_0^\infty ds\; s\; [\mathrm{Im}\Pi_{VV}(s) - \mathrm{Im}\Pi_{AA}(s)] \;=\; 0 \qquad (2^{\mathrm{nd}}\ \mathrm{WSR})$$

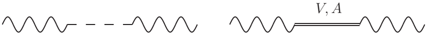
WSRs @ LO



$$\Pi_{LR}(s) \; = \; \frac{f^2}{s} + \sum_i \frac{F_{V_i}^2}{M_{V_i}^2 - s} - \sum_i \frac{F_{A_i}^2}{M_{A_i}^2 - s}$$

- **1st WSR:** $\sum_i F_{V_i}^2 - \sum_i F_{A_i}^2 \; = \; f^2$
- **2nd WSR:** $\sum_i F_{V_i}^2 M_{V_i}^2 - \sum_i F_{A_i}^2 M_{A_i}^2 \; = \; 0$

WSRs @ LO

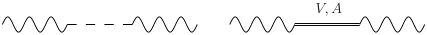


$$\Pi_{LR}(s) = \frac{f^2}{s} + \sum_i \frac{F_{V_i}^2}{M_{V_i}^2 - s} - \sum_i \frac{F_{A_i}^2}{M_{A_i}^2 - s}$$

- **1st WSR:** $\sum_i F_{V_i}^2 - \sum_i F_{A_i}^2 = f^2$
- **2nd WSR:** $\sum_i F_{V_i}^2 M_{V_i}^2 - \sum_i F_{A_i}^2 M_{A_i}^2 = 0$

SRA → $F_V^2 = f^2 \frac{M_A^2}{M_A^2 - M_V^2} \quad , \quad F_A^2 = f^2 \frac{M_V^2}{M_A^2 - M_V^2} \quad , \quad M_A > M_V$

WSRs @ LO



$$\Pi_{LR}(s) \;=\; \frac{f^2}{s} + \sum_i \frac{F_{V_i}^2}{M_{V_i}^2 - s} - \sum_i \frac{F_{A_i}^2}{M_{A_i}^2 - s}$$

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SRA ➡ $F_V^2 \;=\; f^2\,\frac{M_A^2}{M_A^2 - M_V^2} \quad , \quad F_A^2 \;=\; f^2\,\frac{M_V^2}{M_A^2 - M_V^2} \quad , \quad M_A > M_V$

1st WSR likely valid also in gauge theories with non-trivial UV fixed points

2nd WSR questionable (not valid) in walking (conformal) TC scenarios

Appelquist–Sannino, Orgogozo–Rychkov

Short-Distance Constraints

Vector Form Factor $\langle\pi|v_\mu|\pi\rangle$:

$$F_V(t) \, = \, 1 \, + \, \sum_i \frac{F_{V_i}\, G_{V_i}}{f^2} \, \frac{t}{M_{V_i}^2 - t}$$

$$\lim_{t\to\infty} F_V(t) = 0 \quad \longrightarrow \quad \boxed{\sum_i \, F_{V_i}\, G_{V_i} \, = \, f^2}$$

Axial Form Factor $\langle\gamma|a_\mu|\pi\rangle$:

$$G_A(t) = \sum_i \left\{ \frac{2\, F_{V_i}\, G_{V_i} - F_{V_i}^2}{M_{V_i}^2} + \frac{F_{A_i}^2}{M_{A_i}^2 - t} \right\}$$

$$\lim_{t\to\infty} G_A(t) = 0 \quad \longrightarrow \quad \boxed{\sum_i \, \left(2\, F_{V_i}\, G_{V_i} - F_{V_i}^2\right) / M_{V_i}^2 \, = \, 0}$$

Weinberg Sum Rules:

$$\Pi_{LR}(t) \, = \, -\frac{f^2}{t} \, + \, \sum_i \frac{F_{V_i}^2}{M_{V_i}^2 + t} \, - \, \sum_i \frac{F_{A_i}^2}{M_{A_i}^2 + t}$$

$$\lim_{t\to\infty} t \; \Pi_{LR}(t) = 0$$

$$\lim_{t\to\infty} t^2 \, \Pi_{LR}(t) = 0$$

$$\longrightarrow \quad \boxed{\begin{array}{c} \sum_i \, \left(F_{V_i}^2 - F_{A_i}^2\right) = f^2 \\ \sum_i \, \left(M_{V_i}^2\, F_{V_i}^2 - M_{A_i}^2\, F_{A_i}^2\right) = 0 \end{array}}$$

Scalar FF: $$F^S_{K\pi}(s) = 1 + \sum_i \frac{4c_{m_i}}{f^2}\left[c_{d_i} + (c_{m_i} - c_{d_i})\,\frac{M_K^2 + M_\pi^2}{M^2_{S_i}}\right]\frac{s}{M^2_{S_i} - s}$$

$$\lim_{s\to\infty} F^S_{K\pi}(s) = 0 \quad \longrightarrow \quad \boxed{4\sum_i c_{d_i} c_{m_i} = f^2 \quad ; \quad \sum_i \frac{c_{m_i}}{M^2_{S_i}}\,(c_{m_i} - c_{d_i}) = 0}$$

S − P Sum Rules: $$\Pi_{SS-PP}(t) = 16 B_0^2 \left\{\sum_i \frac{c^2_{m_i}}{M^2_{S_i} + t} - \sum_i \frac{d^2_{m_i}}{M^2_{P_i} + t} - \frac{f^2}{8\,t}\right\}$$

$$\lim_{t\to\infty} t\ \Pi_{SS-PP}(t) = 0 \quad \longrightarrow \quad \boxed{8\sum_i (c^2_{m_i} - d^2_{m_i}) = f^2}$$

Pseudoscalar Nonet: $U(3)_L \otimes U(3)_R$ symmetry at $N_C \to \infty$

$$\mathcal{L}_2 \doteq \frac{f^2}{4}\,\langle \chi\, U^\dagger + U\,\chi^\dagger\rangle \sim -i\,\frac{f}{\sqrt{24}}\,\eta_1\,\langle\chi_-\rangle \quad \longrightarrow \quad \boxed{\tilde{d}_m = -\frac{f}{\sqrt{24}}}$$

1–Resonance Approximation: Ecker, Gasser, Leutwyler, Pich, de Rafael

$$F_V = 2\,G_V = \sqrt{2}\,F_A = \sqrt{2}\,f \quad ; \quad M_A = \sqrt{2}\,M_V \quad ; \quad d_m = \frac{1}{2\sqrt{2}}\,f$$

$$c_m = c_d = \frac{1}{2}\,f \qquad \text{(Jamin, Oller, Pich)} \qquad ; \qquad M_P \approx \sqrt{2}\,M_S$$

$$\longrightarrow \quad \boxed{\begin{array}{c} 2\,L_1 = L_2 = \frac{1}{4}\,L_9 = -\frac{1}{3}\,L_{10} = \frac{f^2}{8\,M_V^2} \\ L_3 = -\frac{3\,f^2}{8\,M_V^2} + \frac{f^2}{8\,M_S^2} \quad ; \quad L_5 = \frac{f^2}{4\,M_S^2} \\ L_8 = \frac{f^2}{8\,M_S^2} - \frac{f^2}{16\,M_P^2} \quad ; \quad L_7 = -\frac{f^2}{48\,M_{\eta_1}^2} \end{array}}$$

1–Resonance Approximation:

Ecker, Gasser, Leutwyler, Pich, de Rafael

$$F_V = 2\,G_V = \sqrt{2}\,F_A = \sqrt{2}\,f \quad ; \quad M_A = \sqrt{2}\,M_V \quad ; \quad d_m = \frac{1}{2\sqrt{2}}\,f$$

$$c_m = c_d = \frac{1}{2}\,f$$ (Jamin, Oller, Pich) ; $$M_P \approx \sqrt{2}\,M_S$$

→

$$\mathbf{2\,L_1 = L_2 = \frac{1}{4}\,L_9 = -\frac{1}{3}\,L_{10} = \frac{f^2}{8\,M_V^2}}$$

$$\mathbf{L_3 = -\frac{3\,f^2}{8\,M_V^2} + \frac{f^2}{8\,M_S^2} \quad ; \quad L_5 = \frac{f^2}{4\,M_S^2}}$$

$$\mathbf{L_8 = \frac{f^2}{8\,M_S^2} - \frac{f^2}{16\,M_P^2} \quad ; \quad L_7 = -\frac{f^2}{48\,M_{\eta_1}^2}}$$

Additional SD input:

Pich *(hep-ph/0205030)*; Ledwig, Nieves, Pich, Ruiz-Arriola, Ruiz de Elvira

$$\sqrt{2}\,M_S = \sqrt{2}\,M_V = M_P = M_A = \sqrt{\frac{48}{N_C}}\,\pi\,f$$

→

$$\mathbf{2\,L_1 = L_2 = -\frac{1}{2}\,L_3 = \frac{1}{2}\,L_5 = \frac{4}{3}\,L_8 = \frac{1}{4}\,L_9 = -\frac{1}{3}\,L_{10} = \frac{N_C}{192\,\pi^2}}$$

L_i's from Resonance Exchange

i	$10^3 \cdot L_i^r(M_\rho)$	V	A	S	η_1	Total		
1	0.7 ± 0.3	0.6	0	0	0	0.6		
2	1.3 ± 0.3	1.2	0	0	0	1.2		
3	-3.5 ± 1.1	-3.6	0	0.6	0	-3.0		
4	-0.3 ± 0.5	0	0	0	0	0.0		
5	1.4 ± 0.5	0	0	$1.4^{a)}$	0	1.4		
6	-0.2 ± 0.3	0	0	0	0	0.0		
7	-0.4 ± 0.2	0	0	0	-0.3	-0.3		
8	0.9 ± 0.3	0	0	$0.9^{a)}$	0	0.9		
9	6.9 ± 0.7	$6.9^{a)}$	0	0	0	6.9		
10	-5.5 ± 0.7	-10.0	4.0	0	0	-6.0		

a : Input

L_i's from Resonance Exchange

i	$10^3 \cdot L_i^r(M_\rho)$	V	A	S	η_1	Total	SD1	
1	0.7 ± 0.3	0.6	0	0	0	0.6	0.9	
2	1.3 ± 0.3	1.2	0	0	0	1.2	1.8	
3	-3.5 ± 1.1	-3.6	0	0.6	0	-3.0	-4.3	
4	-0.3 ± 0.5	0	0	0	0	0.0	0.0	
5	1.4 ± 0.5	0	0	$1.4^{a)}$	0	1.4	2.1	
6	-0.2 ± 0.3	0	0	0	0	0.0	0.0	
7	-0.4 ± 0.2	0	0	0	-0.3	-0.3	-0.3	
8	0.9 ± 0.3	0	0	$0.9^{a)}$	0	0.9	0.8	
9	6.9 ± 0.7	$6.9^{a)}$	0	0	0	6.9	7.2	
10	-5.5 ± 0.7	-10.0	4.0	0	0	-6.0	-5.4	

a : Input   SD1: Short-Distance Constraints

L_i's from Resonance Exchange

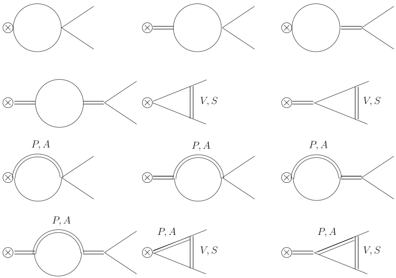
i	$10^3 \cdot L_i^r(M_\rho)$	V	A	S	η_1	Total	SD1	SD2
1	0.7 ± 0.3	0.6	0	0	0	0.6	0.9	0.8
2	1.3 ± 0.3	1.2	0	0	0	1.2	1.8	1.6
3	-3.5 ± 1.1	-3.6	0	0.6	0	-3.0	-4.3	-3.2
4	-0.3 ± 0.5	0	0	0	0	0.0	0.0	0.0
5	1.4 ± 0.5	0	0	$1.4^{a)}$	0	1.4	2.1	3.2
6	-0.2 ± 0.3	0	0	0	0	0.0	0.0	0.0
7	-0.4 ± 0.2	0	0	0	-0.3	-0.3	-0.3	-0.3
8	0.9 ± 0.3	0	0	$0.9^{a)}$	0	0.9	0.8	1.2
9	6.9 ± 0.7	$6.9^{a)}$	0	0	0	6.9	7.2	6.3
10	-5.5 ± 0.7	-10.0	4.0	0	0	-6.0	-5.4	-4.7

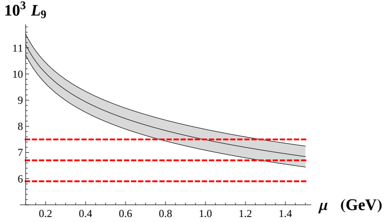
a : Input SD1: Short-Distance Constraints SD2: $L_i \sim \# N_C$

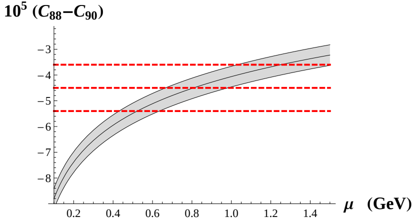
LECs at NLO in $1/N_C$

A.P., I. Rosell, J.J. Sanz-Cillero, hep-ph/0407240, hep-ph/0610290, 0803.1567, 1011.5771

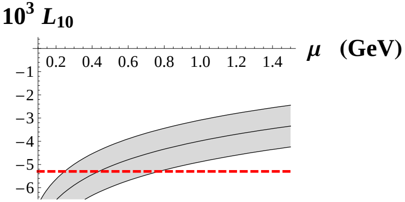
- **Vector Form Factor:**

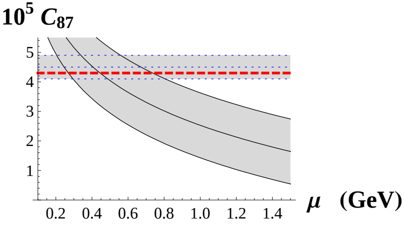






- $\Pi_{LR}(t)$ **(Weinberg Sum Rules):**



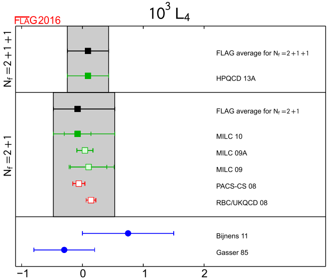


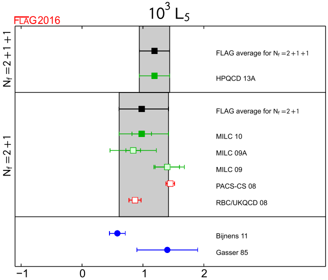
- $\Pi_{SS-PP}(t)$ **:**

$$L_8^r(M_\rho^2) = (0.6 \pm 0.4) \cdot 10^{-3} \qquad , \qquad C_{38}^r(M_\rho^2) = (2 \pm 6) \cdot 10^{-6}$$

Full μ dependence at NLO in $1/N_C$

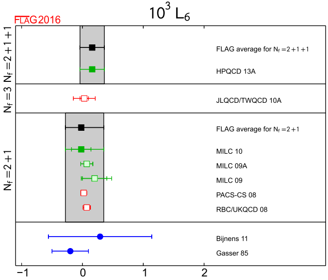
Lattice Determination of SU(3) LECs

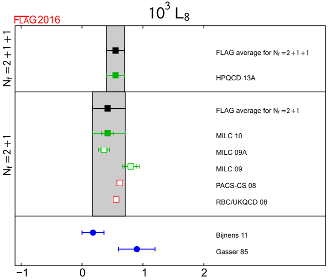




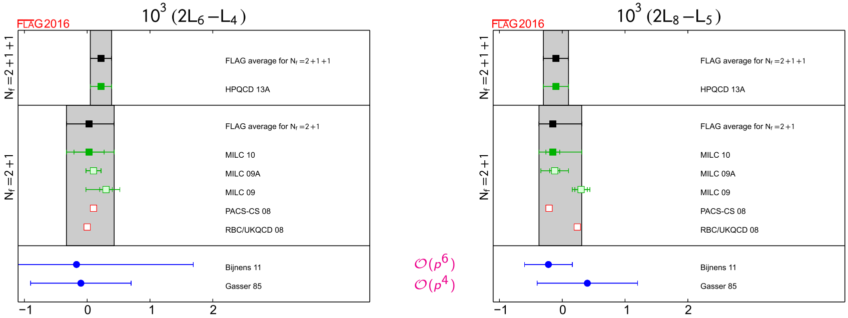
$\mathcal{O}(p^6)$

$\mathcal{O}(p^4)$

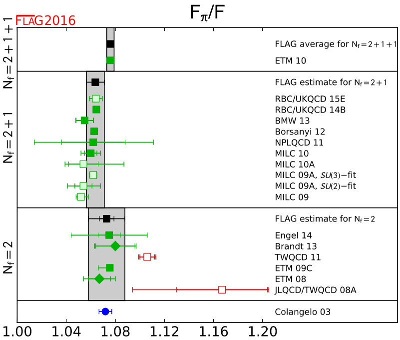


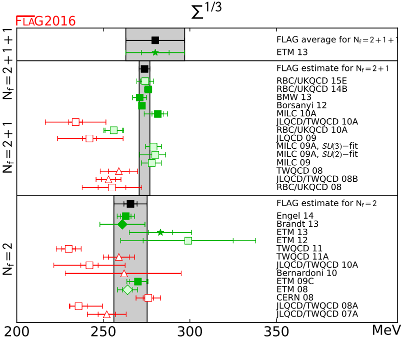


Lattice Determination of SU(3) LECs



Lattice Determination of SU(2) LECs





$$F = F_\pi\big|_{m_u=m_d=0} \quad , \quad \Sigma = -\langle 0|\bar{u}u|0\rangle\big|_{m_u=m_d=0}$$

Lattice Determination of SU(2) LECs