The decoupling has completely removed soft-collinear interactions from the Lagrangian

$$\mathcal{L}_{SCET} = \mathcal{L}_c^{(0)} + \mathcal{L}_{\bar{c}}^{(0)} + \mathcal{L}_s$$

Since there are no longer any interactions, we are dealing with independent theories of soft and collinear particles. Also the states separate

$$|X\rangle = |X_c\rangle \otimes |X_{\bar{c}}\rangle \otimes |X_s\rangle$$

and the soft physics manifests itself as Wilson lines along the directions of the energetic particles, e.g. for $J_\mu^{(0)}$ we have

$$\bar{\chi}_c(t\bar{n})\,\gamma_\perp^\mu\,\chi(sn) = \bar{\chi}_c^{(0)}(t\bar{n})\,S_n^\dagger(0)\,\gamma_\perp^\mu\,S_{\bar{n}}(0)\,\chi_{\bar{c}}^{(0)}(sn)$$

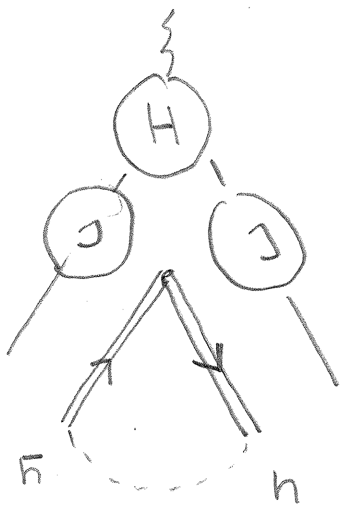
Applied to the Sudakov form factor F, we have

coll. matrix element
↓

$$F(Q^2, P^2, Q^2) = \tilde{C}_V(Q^2,\mu^2)\, \tilde{J}(P^2,\mu^2)\, \tilde{J}(L^2,\mu^2) \times \tilde{S}(\Lambda_s^2,\mu^2)$$

↖ soft matrix element

$\Lambda_s \sim \frac{P^2 L^2}{Q^2}$



← Wilson line with cusp

$$\frac{d}{d\ln\mu} \ln F = \Gamma_H + 2\Gamma_J + \Gamma_S \stackrel{!}{=} 0$$

$$= C_F\, \gamma_{cusp} \ln\frac{Q^2}{\mu^2} + \gamma_V$$

$$- C_F\, \gamma_{cusp} \left(\ln\frac{P^2}{\mu^2} + \ln\frac{L^2}{\mu^2}\right) + 2\gamma_J$$

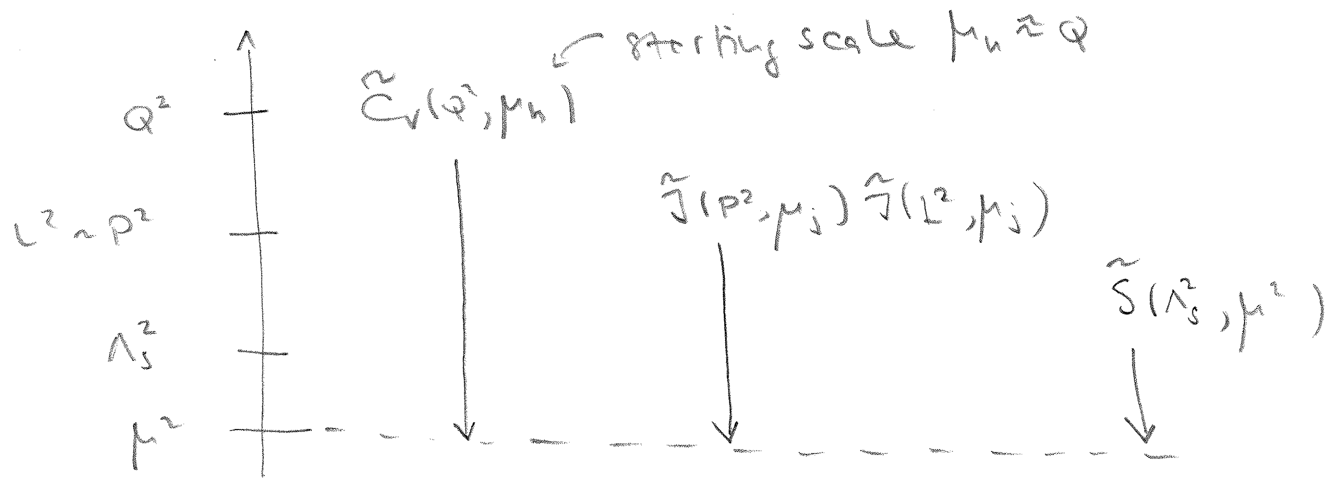
$$+ C_F\, \gamma_{cusp} \ln\frac{\Lambda_s^2}{\mu^2} + 2\gamma_S$$

This cancellation only works because the anomalous dim.'s are <u>linear</u> in $\ln(\mu^2)$ with a universal coefficient.

(Exercise: Show that it cannot work with higher logs.)

The name "cusp anomalous dimension" is appropriate because the soft matrix element is given by a Wilson line with a cusp.

To resum logs in the form factor, we can solve the RG equations and evolve all ingredients to a common scale



It is easy to solve the RG equation. Let us first neglect the running coupling. In this case, the solution would simply be

$$\tilde{C}_V(Q^2,\mu^2) \cong \exp\left[-\frac{1}{2}\gamma_{cusp}(\alpha_s)\ln^2\left(\frac{Q^2}{\mu^2}\right) - \gamma_V(\alpha_s)\ln\left(\frac{Q^2}{\mu^2}\right)\right] \cdot \tilde{C}_V(Q^2,\mu^2=Q^2)$$

This makes it obvious, that the leading logarithms in C_V have the form $\alpha_s^n \ln^{2n}\left(\frac{Q^2}{\mu^2}\right)$. These are the so-called Sudakov logarithms.

Including the running coupling, one gets

$$\tilde{C}_V(Q^2,\mu^2) = \exp\left\{\int_{\mu_h}^{\mu}\frac{d\mu'}{\mu'}\left[C_F\,\gamma_{cusp}(\alpha_s)\ln\left(\frac{Q^2}{\mu'^2}\right)+\gamma_V\right]\right\} \cdot \tilde{C}_V(Q^2,\mu_h^2)$$

$$= U(\mu_h,\mu)\,\tilde{C}_V(Q^2,\mu_h)$$

Using $\frac{d\alpha_s(\mu)}{d\ln\mu} = \beta(\alpha_s(\mu))$, one can rewrite $\int_{\mu_h}^{\mu} \frac{d\mu}{\mu} = \int_{\alpha(\mu_h)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)}$. Expanding the exponent in α_s, one then gets the result for $\mathcal{U}(\mu_h, \mu)$ in RG-improved PT (see Matthias Neubert's lecture).

One finds

$$\mathcal{U}(\mu_h, \mu) = \exp\left[2C_F \, S(\mu_h, \mu) - A_{\gamma_V}(\mu_h, \mu)\right] \cdot \left(\frac{Q^2}{\mu_h^2}\right)^{-C_F A_{\gamma_{cusp}}(\mu_h, \mu)}$$

where

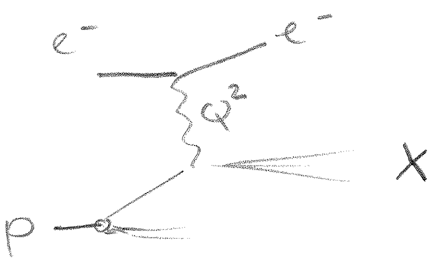
$$S(\nu, \mu) = - \int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\gamma_{cusp}(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\nu)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}$$

$$A_\gamma(\nu, \mu) = - \int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\gamma(\alpha)}{\beta(\alpha)}$$

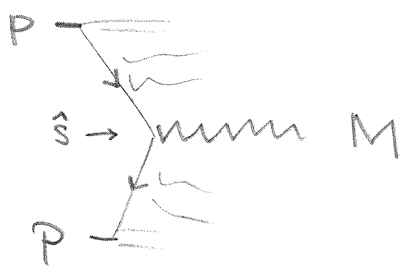
To get explicit results, one then expands $\gamma^i(\alpha_s)$, $\beta(\alpha_s)$ in α_s and integrates.

VI Applications

The Sudakov form factor was unphysical, but the Lagrangian and current operator J^μ can be used to study a variety of physical processes. These include



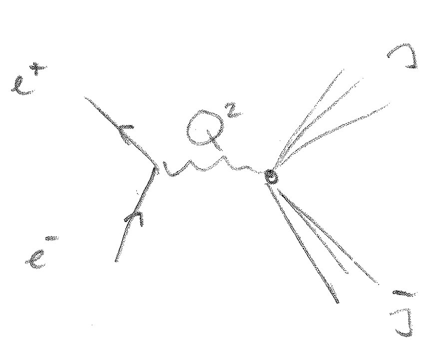
for $M_X^2 \ll Q^2$
i.e. $x \to 1$



for $\hat{s} \to M^2$

$$d\sigma \sim |C_V(Q^2)|^2 J \otimes \phi_q \qquad d\sigma \sim |C_V(-M^2 - i\varepsilon)|^2 S \otimes \phi_q \otimes \phi_{\bar{q}}$$

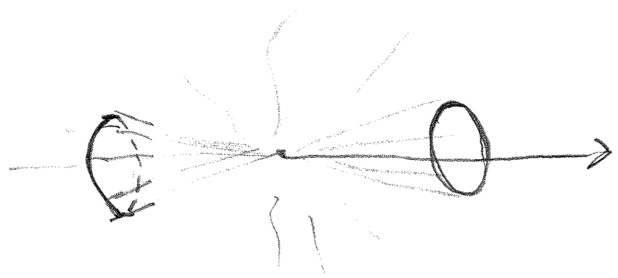
We will study 2-jet production for $M_J^2 \ll Q^2$



There are different ways to define 2-jet events. One possibility is to demand that all energy, up to a small fraction, is inside two cones. This definition was originally proposed by Sterman and Weinberg '77. The basic idea has evolved into a variety of modern jet definitions (see e.g. "Towards jetography" by G. Salam 0906.1833)

A simpler set of observables are event shapes, which characterize the geometry of collider events. The prototype event shape is called thrust Fahri '77. In the following, we will analyze thrust and, time permitting, briefly discuss resummation for jets.

Factorization for thrust



$$T = \frac{1}{Q} \max_{\{\vec{n}_T\}} \sum_i |\vec{n}_T \cdot \vec{p}_i|$$ (Farhi '85)

$$Q = \sum_i |\vec{p}_i|$$ (for massless particles)

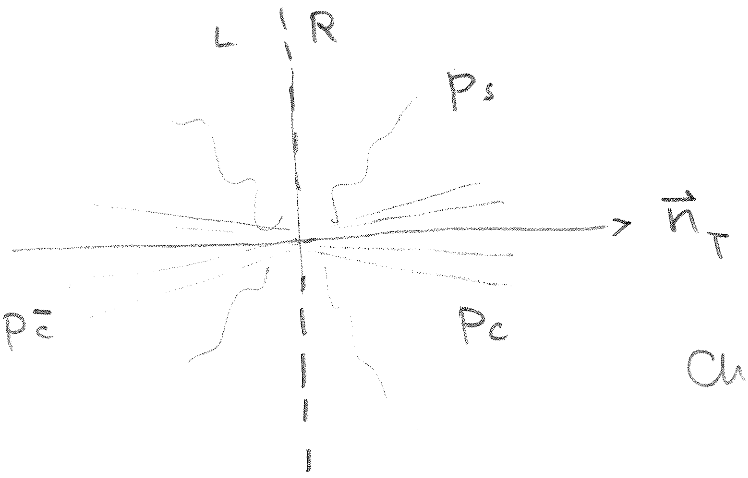
$$T = \frac{\text{momentum flow along } \vec{n}}{\text{total mom flow}} = 1 - \tau$$

$T = 1$ $\leftrightarrow$ all momenta parallel to $\vec{n}$
($\tau = 0$) pencil-like event.

$T = 1/2$ $\leftrightarrow$ completely spherical event

Thrust is a two jet event shape. Generalization to N jet & hadron colliders is N-jettiness (Stewart et al. '10)

Thrust is soft and collinear safe, i.e. its value does not change under collinear splittings or soft emissions. This property makes it possible to compute it perturbatively. However, for small $\tau \ll 1$, large logarithms arise.



Choose $n^\mu = (1, \vec{n}_T)$

$$\tau Q = \sum_i |\vec{p}_i| - |\vec{n} \cdot \vec{p}_i|$$

$$= \sum_{i \in c} n \cdot p_{c_i} + \sum_{i \in \bar{c}} \bar{n} \cdot p_{\bar{c}_i} +$$

$$\sum_i n \cdot p^R_{s_i} + \sum_i \bar{n} \cdot p^L_{s_i}$$

This result has a simple physical interpretation.

By definition $p^{\perp}_{X_c} = p^{\perp}_{X_{\bar{c}}} = 0 + O(\lambda^2)$

$$M_R^2 = \left(p_{X_c} + p^R_{X_s} \right)^2 = \bar{n}\cdot p_{X_c}\, n p_{X_c}$$

$$+ \bar{n} p_{X_c} \cdot n p^R_{X_s} + O(\lambda^4)$$

$$= Q \left(n\cdot p_{X_c} + n\cdot p^R_{X_s} \right)$$

$$= M_c^2 + Q\, n\cdot p^R_{X_s}$$

So $\tau Q^2 = M_L^2 + M_R^2$

$$= M_c^2 + M_{\bar{c}}^2 + Q \left(n\cdot p_s^R + n p_s^L \right)$$

for $\tau \ll 1$

The fact that the soft and collinear contributions are additive will be important to establish factorization.

So let's now compute the cross section

$$\frac{d\sigma}{d\tau} = \frac{1}{2Q^2} \sum_X |m(e^+e^- \to \gamma^* \to X)|^2 (2\pi)^4 \delta^{(4)}(q-p_X) \cdot \delta(\tau - \tau(X))$$

$|m|^2 = L_{\mu\nu} H^{\mu\nu}$, and the QCD part is given by

$$H_{\mu\nu} = \langle 0 | J^\dagger_\mu(0) | X \rangle \langle X | J_\nu(0) | 0 \rangle$$

and $L_{\mu\nu}$ encodes the trivial leptonic part

Introduce

$$1 = \int d^3\vec{n}\, \delta^{(3)}(\vec{n} - \vec{n}_T)$$

$$= 2\pi \int d\cos\theta \left(\frac{Q}{2}\right)^2 \delta^{(2)}(p^\perp_{X_c})$$

(angle to beam)

↑

$\vec{n}_T = \vec{p}_{X_c} / |\vec{p}_{X_c}|$; $|\vec{p}_{X_c}| = Q/2$, up to higher orders.

and expand

$$\delta^{(4)}(q - p_{X_c} - p_{X_{\bar c}} - p_{X_s})\, \delta^{(2)}(p_{X_c}^{\perp})$$

$$= 2\, \delta(\bar n \cdot p_{X_c} - Q)\, \delta(n \cdot p_{X_{\bar c}} - Q)$$

$$\delta^{(2)}(p_{X_{c_1}}^{\perp})\, \delta^{(2)}(p_{X_{c_2}}^{\perp})$$

After all this preparation, we can now plug in our SCET result for the current:

$$\frac{d\sigma}{d\tau\, d\cos\theta} = \frac{\pi}{2} L_{\mu\nu} \left| \tilde C_V(-q^2 - i\varepsilon, \mu) \right|^2$$

$$* \int dM_c^2 \int dM_{\bar c}^2 \int d\omega \; \delta\!\left(\tau - \frac{M_c^2 + M_{\bar c}^2 + Q\omega}{Q^2}\right)$$

$$\sum_{X_c} \langle 0 | \bar\chi^a_{c,\alpha}(0) | X_c \rangle \langle X_c | \chi^b_{c,\beta} | 0 \rangle$$

$$* \; \delta(M_c^2 - p_{X_c}^2)\, \delta^{(2)}(p_X^{\perp})\, \delta(\bar n \cdot p_{X_c} - Q)$$

$$* \sum_{X_{\bar c}} \langle 0 | \chi^c_{\bar c,\delta}(0) | X_{\bar c} \rangle \langle X_{\bar c} | \bar\chi^d_{\bar c,\gamma}(0) | 0 \rangle$$

$$\delta(M_{\bar c}^2 - p_{X_{\bar c}}^2)\, \delta^{(2)}(p_X^{\perp})\, \delta(n \cdot p_{X_{\bar c}} - Q)$$

$$* \sum_{X_s} \langle 0 | \left[S_n^{+}(0) \, S_{\bar{n}}(0) \right]_{ac} | X_s \rangle$$

$$* \langle X_s | \left[S_{\bar{n}}^{+}(0) \, S_n(0) \right]_{db} | 0 \rangle$$

$$\delta(\omega - n \cdot p_s^R - \bar{n} \, p_s^L)$$

$$* \, (\gamma_\perp^\mu)_{\alpha\beta} \, (\gamma_\perp^\nu)_{\gamma\delta}$$

At this point, some work is needed to perform the Dirac algebra and to analyze the different (collinear, soft) matrix elements. An important point is that the collinear matrix elements are color-diagonal $\propto \delta^{ab} \delta^{cd}$. The soft matrix elements take the form

$$S(\omega) = \sum_{X_s} \frac{1}{N_c} \langle 0 | \left[S_n^{+}(0) \, S_{\bar{n}}(0) \right]_{ab} | X_s \rangle \langle X_s | \left[S_{\bar{n}}^{+}(0) \, S_n(0) \right]_{ba} | 0 \rangle$$

$$\delta(\omega - n \cdot p_s^R - \bar{n} \, p_s^L)$$

The final result for the cross section is then obtained as

$$\frac{d\sigma}{d\tau\, d\cos\theta} = \frac{\pi N_c Q_f^2 \alpha^2}{2Q^2}(1+\cos^2\theta)$$

$$* \left| C_V(-Q^2-i\varepsilon, \mu^2) \right|^2 \int_0^\infty dM_c^2 \int_0^\infty dM_{\bar c}^2$$

$$\int_0^\infty d\omega \;\delta\left(\tau - \frac{M_c^2 + M_{\bar c}^2 + \omega Q}{Q^2}\right)$$

$$J_c(M_c^2, \mu^2)\, J_{\bar c}(M_{\bar c}^2, \mu^2)\, S(\omega, \mu^2)$$

The soft function was given above and the jet function is related to the matrix element through

$$\frac{\delta^{ab}}{2(2\pi)^3}\left(\frac{\not{n}}{2}\right)_{\beta\alpha} J_c(M^2) = \sum_{X_c} \langle 0 | \bar{\chi}^a_{c,\alpha}(0) | X_c \rangle$$

$$\langle X_c | \chi^b_{c,\beta}(0) | 0 \rangle$$

$$\delta(M^2 - p_{X_c}^2)\, \delta(Q - \bar{n}\cdot p_{X_c})\, \delta^{(2)}(p_{X_c}^\perp)$$

To resum the large logarithms in this cross section, one solves the RG equations to evaluate C_v, J, S at their natural scales

$$\mu_h^2 \sim Q^2 \quad \text{for } |C_v|^2,$$

$$\mu_j^2 \sim \tau Q^2 \quad \text{for } J,$$

$$\mu_s^2 \sim \tau^2 Q^2 \quad \text{for } S.$$

See the supplemental slides, how this improves convergence!

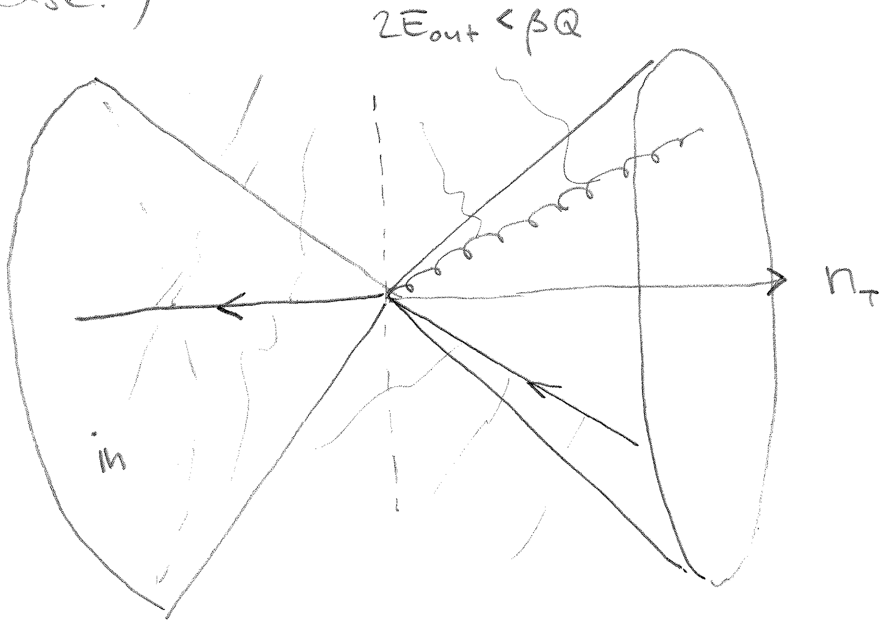
Factorization for cone jet cross sections

TB, Neubert, Rothen, Shao '15, '16

Interestingly, the pattern of logarithms for jet cross sections is much more complicated than for thrust.

Extra "nonglobal" logs were discovered by Dasgupta & Salam '01. They were able to resum the leading logs at large N_c, but general pattern and factorization formula for such logs was unknown. This problem was solved in the papers above and we now know how to resum logs for cone-jet processes, in principle to arbitrary accuracy, in practice at leading log accuracy in the large-N_c limit. Interestingly, the RG equations are equivalent to a parton shower, the traditional method for leading-log resummations. SCET then shows what ingredients are needed ~~to~~ construct a parton shower which also resums subleading logs.

For simplicity, let's choose $\vec{n}_T$ as the jet axis and choose a large jet angle δ. (In the papers, we also treat the more complicated small angle case.)



Momentum regions: soft: $p_s^\mu \sim \beta Q$ (in or out)

hard: $p_h^\mu \sim Q$ in only!

Factorization (cf. first lecture!)

$$S_1(n_1)\, S_2(n_2) \cdots S_m(n_m)\, | \mathcal{M}(\{p_1, p_2, \ldots, p_m\}) \rangle$$

↑ Wilson lines for each hard parton. S_i is matrix acting on color index of parton i.

↑ m hard partons inside

$|\mathcal{M}\rangle$ is amplitude in "color space".

Cross section takes the form

$$\sigma(\beta, \delta) = \sum_{m=2}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, Q, \mu) \otimes \mathcal{S}_m(\{\underline{n}\}, Q\beta, \mu) \rangle$$

↑ color trace.

↑ integral over directions $\{\underline{n}\}$: $\prod_i \int \frac{d\Omega_i}{4\pi}$

where

$$\mathcal{S}_m(\{\underline{n}\}, Q\beta) = \sum_{X_s} \langle 0 | S_1^\dagger(n_1) \cdots S_m^\dagger(n_m) | X_s \rangle \langle X_s | S_1(n_1) \cdots S_m(n_m) | 0 \rangle \, \Theta(Q\beta - 2E_{in})$$

$$\mathcal{H}_m(\{\underline{n}\}, Q) = \frac{1}{2Q^2} \prod_{i=1}^{m} \int \frac{dE_i \, E_i^{d-3}}{(2\pi)^{d-2}}$$

$$\cdot |m(\{p\})\rangle\langle m(\{p\})| \; \delta(Q - E_x)\, \delta^{(d-1)}(\vec{p}_x)$$

$$\prod_i \Theta_{in}(\vec{p}_i)$$

To resum logs $\alpha_s^n \ln^n\left(\frac{E_{in}}{E_{out}}\right) = \alpha_s^n \ln^n\left(\frac{1}{\beta}\right)$, one solves the RG for the hard functions $\mathcal{H}_m$ and evolves from $\mu_h \approx Q$ to $\mu_s \approx \beta Q$.

This RG takes the form

$$\frac{d}{d\ln\mu} \mathcal{H}_m = -\sum_\ell \mathcal{H}_\ell \, \Gamma_{\ell m}$$

↑ operator mixing!

At one loop:

$$\Gamma = \frac{\alpha_s}{4\pi} \begin{pmatrix} V_2 & R_2 & & 0 \\ & V_3 & R_3 & \\ & & V_4 & \\ 0 & & & \ddots \end{pmatrix} + O(\alpha_s^2)$$

One-loop RG

$$\frac{d}{d\ln\mu}\mathcal{H}_m = \frac{\alpha_s}{4\pi}\left[V_m\,\mathcal{H}_m + R_{m-1}\,\mathcal{H}_{m-1}\right]$$

or

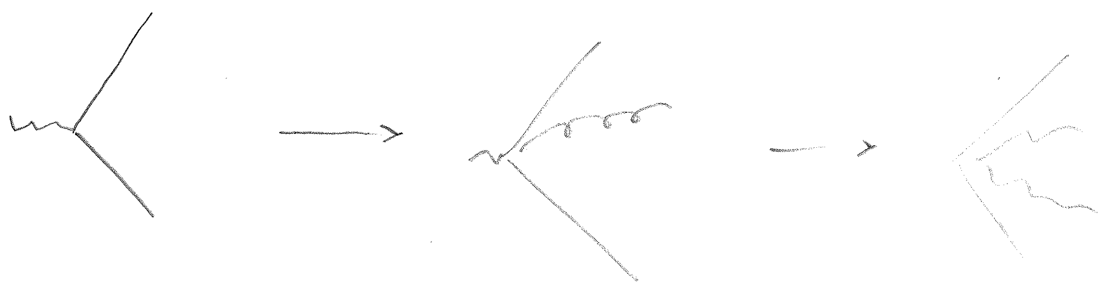
"no branch"

$$\mathcal{H}_m(t) = \mathcal{H}_m(t)\, e^{(t-t_0)V_m} + \int_{t_0}^{t} dt'\, \mathcal{H}_{m-1}(t')\, R_{m-1}\, e^{(t-t')V_m}$$

"branch"

$$t = \int_{\alpha(\mu)}^{\alpha(\mu_h)} \frac{d\alpha}{\beta(\alpha)}\frac{\alpha}{4\pi}$$

Parton shower equation!



$\mu = \mu_h$

$t = 0$ | $t = t_1$ | $t = t_2$

H_2 | H_3 | H_4

To get the cross section, we first generate $\mathcal{H}_m(\mu_s)$ from running the shower. Then one uses $\mathcal{S}_m(\{\underline{n}\}, Q\beta, \mu_s) = \mathbb{1} + \mathcal{O}(\alpha_s)$

↑ no large logs!

and obtains the cross section as

$$\sigma(\beta, \delta) = \sum_m \langle \mathcal{H}_m(\mu_s) \otimes \mathbb{1} \rangle ,$$

i.e. by summing over m and integrating over the directions of the additional particles.

(This integral over directions is "automatically" done when generating the extra partons using MC.)

It is remarkable that we make direct contact to PS resummation using our EFT.

Since the resummation is based on the RG, we also know how to extend it to subleading logs. For NLL, we need

$S_m^{r(1)}$, $\mathcal{H}_2^{(1)}$, $\mathcal{H}_3^{(1)}$: one-loop matching.

$\Gamma_{em}^{(2)}$: two-loop anomalous dimension.

Of course, these then need to be implemented into an MC framework.

Automating higher-log resummations and making them available for a wider class of observables is indeed one of the main goals for the future. (Such automation has been achieved for NLO fixed order over the past few years.)