

The decoupling has completely removed soft-collinear interactions from the Lagrangian

$$\mathcal{L}_{\text{SCET}} = \mathcal{L}_c^{(0)} + \mathcal{L}_{\bar{c}}^{(0)} + \mathcal{L}_s$$

Since there are no longer any interactions, we are dealing with independent theories of soft and collinear particles. Also the states separate

$$|X\rangle = |X_c\rangle \otimes |X_{\bar{c}}\rangle \otimes |X_s\rangle$$

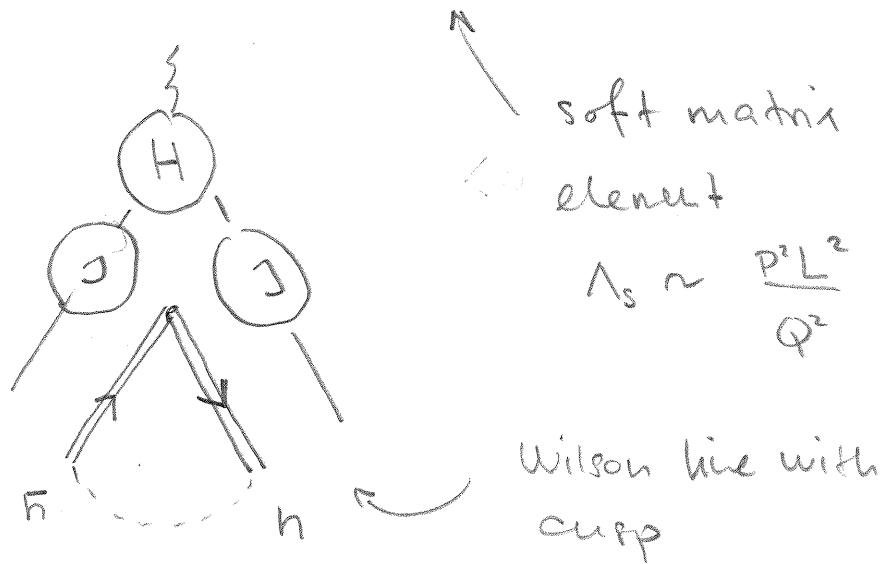
and the soft physics manifests itself as Wilson lines along the directions of the energetic particles, e.g. for $J_{\mu}^{(0)}$ we have

$$\bar{\chi}_c(t\bar{u}) \gamma_{\pm}^{\mu} \chi(su) = \bar{\chi}_c^{(0)}(t\bar{u}) S_n^{\pm(0)} \gamma_{\pm}^{\mu} S_{\bar{u}}^{(0)} \chi_{\bar{c}}^{(0)}(su)$$

Applied to the Sudakov form factor F , we have

coll. matrix element
↓

$$F(Q^2, P^2, L^2) = \tilde{C}_V(Q^2, \mu^2) \tilde{J}(P^2, \mu^2) \tilde{J}(L^2, \mu^2) \times \tilde{S}(\Lambda_s^2, \mu^2)$$



$$\frac{d}{d \ln \mu} \ln F = \Gamma_H + 2 \Gamma_J + \Gamma_S \stackrel{!}{=} 0$$

$$= C_F \gamma_{\text{cusp}} \ln \frac{Q^2}{\mu^2} + \gamma_V$$

$$- C_F \gamma_{\text{cusp}} \left(\ln \frac{P^2}{\mu^2} + \ln \frac{L^2}{\mu^2} \right) + 2 \gamma_J$$

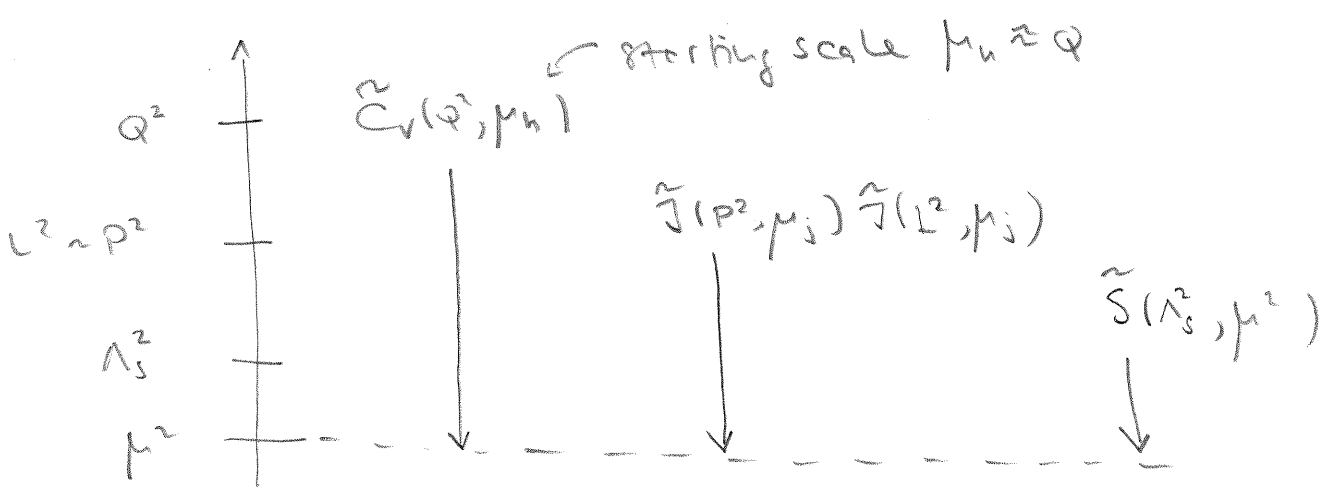
$$+ C_F \gamma_{\text{cusp}} \ln \frac{\Lambda_s^2}{\mu^2} + 2 \gamma_S$$

This cancellation only works because the anomalous dim.'s are linear in $\ln(\mu^2)$ with a universal coefficient.

(Exercise: Show that it cannot work with higher logs.)

The name "cusp anomalous dimension" is appropriate because the soft matrix element is given by a Wilson line with a cusp.

To resum logs in the form factor, we can solve the RG equations and evolve all ingredients to a common scale



It is easy to solve the RG equation. Let us first neglect the running coupling. In this case, the solution would simply be

$$\tilde{C}_V(Q^2, \mu^2) \approx \exp \left[-\frac{1}{2} \gamma_{\text{cusp}}(\alpha_s) \ln^2 \left(\frac{Q^2}{\mu^2} \right) - \gamma_V(\alpha_s) \ln \left(\frac{Q^2}{\mu^2} \right) \right] \cdot \tilde{C}_V(Q^2, \mu^2 = Q^2)$$

This makes it obvious, that the leading logarithms in C_V have the form $\alpha_s^n \ln^{2n} \left(\frac{Q^2}{\mu^2} \right)$. These are the so-called Sudakov logarithms.

Including the running coupling, one gets

$$\tilde{C}_V(Q^2, \mu^2) = \exp \left\{ \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[C_F \gamma_{\text{cusp}}(\alpha_s) \ln \left(\frac{Q^2}{\mu'^2} \right) + \gamma_V \right] \right\} \cdot \tilde{C}_V(Q^2, \mu_0^2)$$

$$= U(\mu_0, \mu) \tilde{C}_V(Q^2, \mu_0)$$

Using $\frac{d\alpha_s(\mu)}{d \ln \mu} = \beta(\alpha_s(\mu))$, one can

rewrite $\int_{\mu_0}^{\mu} \frac{d\mu}{\mu} = \int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)}$. Expanding the

exponent in α_s , one then gets the result for

$\mathcal{U}(\mu_0, \mu)$ in RG-improved PT (see Matthias

Neuberger's lecture).

one finds

$$\mathcal{U}(\mu_0, \mu) = \exp \left[2C_F S(\mu_0, \mu) - A_{g_s}(\mu_0, \mu) \right] \\ \cdot \left(\frac{Q^2}{\mu_0^2} \right)^{-C_F A_{\text{loop}}(\mu_0, \mu)}$$

where

$$S(\mu_0, \mu) = - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha \frac{\gamma_{\text{loop}}(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\mu)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}$$

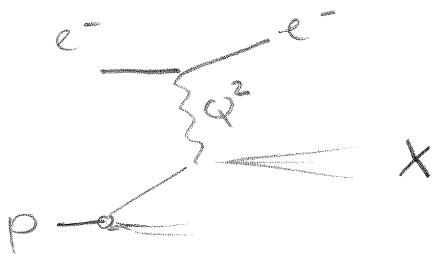
$$A_{g_s}(\mu_0, \mu) = - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha \frac{\gamma(\alpha)}{\beta(\alpha)}$$

To get explicit results, one then expands $\gamma(\alpha)$

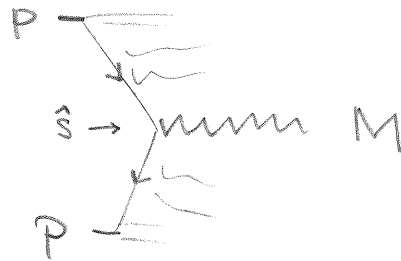
$\beta(\alpha)$ in α_s and integrates.

VI Applications

The Sudakov form factor was unphysical, but the Lagrangian and current operator J^μ can be used to study a variety of physical processes. These include



for $M_X^2 \ll Q^2$
i.e. $x \rightarrow 1$

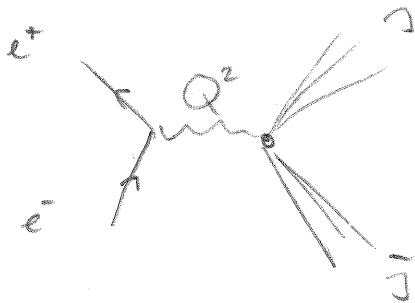


for $\hat{s} \rightarrow M^2$

$$d\sigma \sim |C_V(Q^2)|^2 J \otimes \phi_q$$

$$d\sigma \sim |C_V(-M^2 - i\epsilon)|^2 S \otimes \phi_q \otimes \phi_{\bar{q}}$$

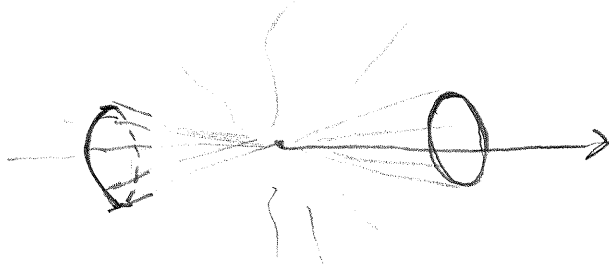
We will study 2-jet production for $M_0^2 \ll Q^2$



There are different ways to define 2-jet events. One possibility is to demand that all energy, up to a small fraction, is inside two cones. This definition was originally proposed by Stroman and Weinberg '77. The basic idea has evolved into a variety of modern jet definitions (see e.g. "Towards Jetography" by G. Salam 0906.1833)

A simpler set of observables are event shapes, which characterize the geometry of collision events. The prototype event shape is called thrust Fabri '77. In the following, we will analyze thrust and, time permitting, briefly discuss recommendations for jets.

Factorization for Thrust



$$T = \frac{T}{Q} \max_{\vec{n}} \sum_i |\vec{n} \cdot \vec{p}_i| \quad (\text{Farchi '85})$$

$$Q = \sum_i |\vec{p}_i| \quad (\text{for massless particles})$$

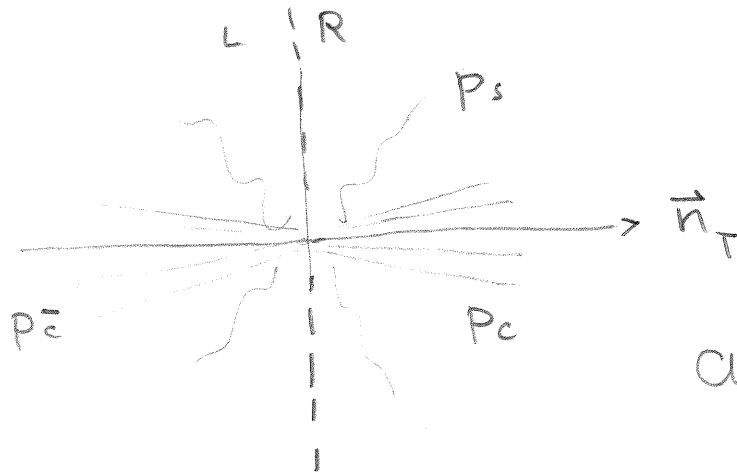
$$T \equiv \frac{\text{momentum flow along } \vec{n}}{\text{total mom flow}} = 1 - \tau$$

$T = 1 \iff$ all momenta parallel to \vec{n}
 ($\tau = 0$) pencil-like event,

$T = 1/2 \iff$ completely spherical event

Thrust is a two jet event shape. Generalization to N jet & hadron colliders is N -jettiness
 (Stewart et al. 10)

Thrust is soft and collinear safe, i.e. its value does not change under collinear splittings or soft emissions. This property makes it possible to compute it perturbatively. However, for small α_s , large logarithms arise.



Choose $n^\mu = (1, \vec{n}_T)$

$$\begin{aligned}
 \ln Q &= \sum_i |\vec{p}_i| - |\vec{n} \cdot \vec{p}_i| \\
 &= \sum_{i \in c} n \cdot p_{c_i} + \sum_{i \in \bar{c}} \bar{n} \cdot p_{\bar{c}_i} + \\
 &\quad \sum_i n \cdot p_{s_i}^R + \sum_i \bar{n} \cdot p_{s_i}^L
 \end{aligned}$$

This result has a simple physical interpretation.

By definition $p_{x_c}^\perp = p_{x_c}^\perp = 0 + O(\lambda^2)$

$$M_R^2 = (p_{x_c} + p_{x_s}^R)^2 = \bar{n} \cdot p_{x_c} n p_{x_c} + \bar{n} p_{x_c} \cdot n p_{x_s}^R + O(\lambda^4)$$

$$= Q (n \cdot p_{x_c} + n \cdot p_{x_s}^R)$$

$$= M_c^2 + Q n \cdot p_{x_s}^R$$

So $\tau Q^2 = M_L^2 + M_R^2$

$$= M_c^2 + M_c^2 + Q (n \cdot p_s^R + n p_s^L)$$

for $\tau \ll 1$

The fact that the soft and collinear contributions are additive will be important to establish factorization.

So let's now compute the cross section

$$\frac{d\sigma}{d\tau} = \frac{1}{2Q^2} \int_X |m(e^- \rightarrow \gamma^* \rightarrow X)|^2 (2\pi)^4 \delta^{(4)}(q - P_X) \cdot \delta(\tau - \tau(X))$$

$|m|^2 = L_{\mu\nu} H^{\mu\nu}$, and the QCD part is given by

$$H_{\mu\nu} = \langle 0 | J_\mu^\dagger(0) | X \rangle \langle X | J_\nu(0) | 0 \rangle$$

and $L_{\mu\nu}$ encodes the trivial leptonic part

Introduce

$$1 = \int d^3 \vec{n} \delta^{(3)}(\vec{n} - \vec{n}_\tau)$$

$$= 2\pi \int d\cos\theta \left(\frac{Q}{2}\right)^2 \delta^{(2)}(p_{X_c}^\perp)$$

↙ angle to beam

↑

$$\vec{n}_\tau = \frac{\vec{p}_{X_c}}{|\vec{p}_{X_c}|} \quad ; \quad |\vec{p}_{X_c}| = Q/2, \text{ up to}$$

higher orders.

and expand

$$\begin{aligned} & \delta^{(4)}(q - p_{X_c} - p_{X_{\bar{c}}} - p_{X_S}) \delta^{(2)}(p_{X_c}^\perp) \\ &= 2 \delta(\vec{n} \cdot p_{X_c} - Q) \delta(\vec{n} \cdot p_{X_{\bar{c}}} - Q) \\ & \quad \delta^{(2)}(p_{X_{c_1}}^\perp) \delta^{(2)}(p_{X_{c_2}}^\perp) \end{aligned}$$

After all this preparation, we can now plug in our SCET result for the current:

$$\frac{d\sigma}{d\tau d\cos\theta} = \frac{\pi}{2} L_{\mu\nu} |\tilde{C}_V(-Q^2 - i\epsilon, \mu)|^2$$

$$* \int dM_c^2 \int dM_{\bar{c}}^2 \int d\omega \delta\left(\tau - \frac{M_c^2 + M_{\bar{c}}^2 + Q\omega}{Q^2}\right)$$

$$\int_{X_c} \langle 0 | \bar{\chi}_{c,\alpha}^a(0) | X_c \rangle \langle X_c | \chi_{c,\beta}^b | 0 \rangle$$

$$* \delta(M_c^2 - p_{X_c}^2) \delta^{(2)}(p_{X_c}^\perp) \delta(\vec{n} \cdot p_{X_c} - Q)$$

$$* \int_{X_{\bar{c}}} \langle 0 | \bar{\chi}_{\bar{c},\delta}^c(0) | X_{\bar{c}} \rangle \langle X_{\bar{c}} | \bar{\chi}_{\bar{c},\gamma}^d(0) | 0 \rangle$$

$$\delta(M_{\bar{c}}^2 - p_{X_{\bar{c}}}^2) \delta^{(2)}(p_{X_{\bar{c}}}^\perp) \delta(\vec{n} \cdot p_{X_{\bar{c}}} - Q)$$

$$* \int_{X_s} \langle 0 | [S_n^+(0) S_{\bar{n}}(0)]_{ac} | X_s \rangle$$

$$+ \langle X_s | [S_{\bar{n}}^+(0) S_n(0)]_{db} | 0 \rangle$$

$$\delta(\omega - n \cdot p_s^R - \bar{n} \cdot p_s^L)$$

$$* (\gamma_{\perp}^{\mu})_{\alpha\beta} (\gamma_{\perp}^{\nu})_{\gamma\delta}$$

At this point, some work is needed to perform the Dirac algebra and to analyze the different (collinear, soft) matrix elements. An important point is that the collinear matrix elements are color-diagonal $\propto \delta^{ab} \delta^{cd}$. The soft matrix elements take the form

$$S(\omega) = \int_{X_s} \frac{1}{N_c} \langle 0 | [S_n^+(0) S_{\bar{n}}(0)]_{ab} | X_s \rangle \langle X_s | [S_{\bar{n}}^+(0) S_n(0)]_{ba} | 0 \rangle$$

$$\delta(\omega - n \cdot p_s^R - \bar{n} \cdot p_s^L)$$