II. Renormalization-group eqns. and running couplings

Consider a QED observable $\mathcal{O}$ ($\overset{\text{e.g. a}}{\text{scattering}}$ amplitude or cross section) computed in the on-shell renormalization scheme and in the $\overline{\text{MS}}$ scheme:

$$\mathcal{O} = \mathcal{O}\Big(\underset{\substack{\uparrow \\ \text{Thomson limit} \\ (q^2=0)}}{\alpha}\,,\ \underset{\substack{\uparrow \\ \text{physical (pole)} \\ \text{mass}}}{m}\,,\ \overbrace{\ln\frac{s}{m^2}\,,\ \ldots}^{\text{dependence on kin. variables}}\Big) \qquad \text{on-shell scheme}$$

$$= \mathcal{O}\Big(\underbrace{\alpha(\mu),\ m(\mu)}_{\substack{\overline{\text{MS}}\text{ parameters} \\ \text{are } \mu\text{-dependent}}}\,,\ \ln\frac{s}{\mu^2}\,,\ \ldots\Big) \qquad \overline{\text{MS}} \text{ scheme}$$

[if more than one scales appear, use EFTs to separate them]

Comments: ◦ both results are equivalent and μ independent!

◦ sometimes on-shell renormalization is inconvenient, since it leaves large logs $\ln s/m^2$ (for $s \gg m^2$), which can spoil the perturbative expansion (e.g. if $\alpha \ln s/m^2 = O(1)$)

◦ choosing $\mu^2 \approx s$ can fix this problem, giving a well-behaved perturbative expansion in terms of parameters $\alpha(\mu^2)$ and $m(\mu^2)$, which are however different from the "physical" parameters α and m:

$$\alpha(\mu) = \alpha\,\frac{Z_3^{\overline{\text{MS}}}}{Z_3^{\text{on-shell}}} = \alpha\left(1 + \frac{\alpha}{3\pi}\ln\frac{\mu^2}{m^2}\right)$$

large logs (need to be resummed)

$$m(\mu) = m\,\frac{(Z_m/Z_2)^{\text{on-shell}}}{(Z_m/Z_2)^{\overline{\text{MS}}}} = m\left[1 - \frac{3\alpha}{4\pi}\left(\ln\frac{\mu^2}{m^2} + \frac{4}{3}\right)\right]$$

- μ independence of $\mathcal{O}$ in the $\overline{MS}$ scheme can be expressed in terms of the partial differential equation:

$$\mu \frac{d}{d\mu} \mathcal{O} = \mu \frac{d\alpha(\mu)}{d\mu} \frac{\partial \mathcal{O}}{\partial \alpha} + \mu \frac{dm(\mu)}{d\mu} \frac{\partial \mathcal{O}}{\partial m} + \frac{\partial \mathcal{O}}{\partial \ln\mu} = 0$$

$\rightarrow$ Callan - Symanzik equation (usually written for Green's fns., but equivalent)

$\rightarrow$ RGE (same thing)

one defines:

$$\mu \frac{d\alpha(\mu)}{d\mu} = \beta(\alpha(\mu)) \qquad \text{"}\beta\text{-function"}$$

$$\mu \frac{dm(\mu)}{d\mu} = \gamma_m(\alpha(\mu)) \cdot m(\mu)$$

$\uparrow$ mass "anomalous dimension"

with these definitions:

$$\beta(\alpha) \frac{\partial \mathcal{O}}{\partial \alpha} + \gamma_m(\alpha)\, m \frac{\partial \mathcal{O}}{\partial m} + \mu \frac{\partial \mathcal{O}}{\partial \mu} = 0$$

- in QCD, using the $\overline{MS}$ scheme is a must, and setting the light quark masses to zero one obtains:

$$\beta(\alpha_s) \frac{\partial \mathcal{O}}{\partial \alpha_s(\mu)} + \mu \frac{\partial \mathcal{O}}{\partial \mu} = 0$$

Calculation of β-functions and anomalous dimensions:

Recall the general relations (now for QCD):

$$\alpha_{s,0} = \mu^{2\epsilon} \left(Z_1 Z_2^{-1} Z_3^{-1/2}\right)^2_\mu \alpha_s(\mu) \equiv \mu^{2\epsilon} Z_\alpha(\mu)\, \alpha_s(\mu)$$

$$\uparrow$$

$$\frac{g_{s,0}^2}{4\pi}$$

$$m_{q,0} = \left(Z_m Z_2^{-1}\right)_\mu m_q(\mu) \equiv Z_m'(\mu)\, m_q(\mu)$$

where in the $\overline{MS}$ scheme:

$$Z_\alpha(\mu) = 1 - \frac{\alpha_s(\mu)}{4\pi\hat{\varepsilon}} \beta_0 + O(\alpha_s^2) \; ; \quad \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f$$

$$Z_m'(\mu) = 1 - 3C_F \frac{\alpha_s(\mu)}{4\pi\hat{\varepsilon}} + O(\alpha_s^2)$$

From the fact that the bare parameters are μ-independent, it follows that:

$$\frac{d}{d\ln\mu} \alpha_{s,0} = 0 = \mu^{2\epsilon} Z_\alpha(\mu)\, \alpha_s(\mu) \left[2\epsilon + Z_\alpha^{-1} \frac{dZ_\alpha}{d\ln\mu} + \frac{1}{\alpha_s} \frac{d\alpha_s}{d\ln\mu} \right]$$

$$\mu \frac{d}{d\mu} \qquad \Rightarrow \quad \frac{d\alpha_s(\mu)}{d\ln\mu} = \alpha_s \left[-2\epsilon - Z_\alpha^{-1} \frac{dZ_\alpha}{d\ln\mu} \right] \equiv \beta(\alpha_s(\mu), \varepsilon)$$

and:

$$\frac{d}{d\ln\mu} m_{q,0} = 0 = Z_m'(\mu)\, m_q(\mu) \left[Z_m'^{-1} \frac{dZ_m'}{d\ln\mu} + \frac{1}{m_q} \frac{dm_q}{d\ln\mu} \right]$$

$$\Rightarrow \quad \frac{1}{m_q(\mu)} \frac{dm_q(\mu)}{d\ln\mu} = - Z_m'^{-1} \frac{dZ_m'}{d\ln\mu} = \gamma_m(\alpha_s(\mu))$$

In the $\overline{MS}$ scheme the Z-factors only contain $1/\hat{\varepsilon}^n$ poles and therefore depend on μ only via the coupling $\alpha_s(\mu)$. It follows that:

$$\beta(\alpha_s(\mu), \varepsilon) = \alpha_s(\mu) \left[-2\varepsilon - \beta(\alpha_s(\mu), \varepsilon)\, Z_\alpha^{-1} \frac{dZ_\alpha}{d\alpha_s} \right] \qquad (*)$$

$$\gamma_m(\alpha_s(\mu)) = -\beta(\alpha_s(\mu), \varepsilon)\, Z_m'^{-1} \frac{dZ_m'}{d\alpha_s}$$

To solve the first equation, one expands: (use MS rather than $\overline{\text{MS}}$)

results are identical

$$\beta(\alpha_s, \varepsilon) = \beta(\alpha_s) + \sum_{k=1}^{\infty} \varepsilon^k \beta^{(k)}(\alpha_s)$$

$$Z_\alpha = 1 + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} Z_\alpha^{(k)}(\alpha_s)$$

start at $O(\alpha_s^k)$

From the fact that all pole terms $\sim 1/\varepsilon^n$ must cancel on the RHS of (*) one can derive an infinite set of relations between $\beta^{(k)}(\alpha_s)$ and $Z_\alpha^{(k)}$. The solution is:

$$\beta^{(1)}(\alpha_s) = -2\alpha_s$$

$$\beta^{(k)}(\alpha_s) = 0 \qquad \text{for all } k \geq 2$$

$$\beta(\alpha_s) = 2\alpha_s^2 \frac{dZ_\alpha^{(1)}}{d\alpha_s}$$

This yields: (exercise)

coefficient of $1/\varepsilon$ pole

$$\beta(\alpha_s, \varepsilon) = \beta(\alpha_s) - 2\varepsilon\alpha_s = 2\alpha_s^2 \frac{dZ_\alpha^{(1)}}{d\alpha_s} - 2\varepsilon\alpha_s$$

(exact relation!)

Likewise, one can show that: (exercise)

$$\gamma_m(\alpha_s) = 2\alpha_s \frac{dZ_m'^{(1)}}{d\alpha_s} \qquad \text{(exact relation)}$$

At one-loop order, one finds:

$$\beta(\alpha_s) = -2\alpha_s \left(\beta_0 \frac{\alpha_s}{4\pi} + \dots \right), \qquad \gamma_m(\alpha_s) = -6C_F \frac{\alpha_s}{4\pi} + \dots$$

(valid in $\overline{\text{MS}}$ scheme)

Side remark:

The function $\beta(\alpha_s, \varepsilon)$ governs the scale dependence of the renormalized coupling in the regularized theory with $d = 4 - 2\varepsilon$ (relevant for the Z_i factors), while $\beta(\alpha_s)$ controls the scale dependence of the renormalized coupling in 4 dimensions (relevant for observables).

We can use the above results to show that the QED coupling and electron mass defined in the on-shell scheme are μ independent despite appearance. Consider first the relation for the pole mass:

$$m_0 = \underset{\substack{\uparrow \\ \text{pole mass}}}{m} \left[1 - \frac{3\alpha(\mu)}{4\pi} \left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln \frac{\mu^2}{m^2} + \frac{4}{3} + O(\varepsilon) \right) + O(\alpha^2) \right]$$

We find:

$$\frac{d}{d\ln\mu}\left[1 - \dots\right] = -\frac{3}{4\pi} \overbrace{\frac{d\alpha(\mu)}{4\pi}}^{-2\varepsilon\alpha + \overset{=0 \text{ (see below)}}{\beta(\alpha)} = -2\varepsilon\alpha} \left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{m^2} + \frac{4}{3} + O(\varepsilon) \right)$$

$$- \frac{3\alpha}{4\pi} \cdot 2 + O(\alpha^2)$$

$$= O(\alpha^2, \varepsilon\alpha) \quad \text{would be cancelled by higher-order terms}$$

$$= 0 + \text{higher-order terms}$$

$$\Rightarrow \frac{d m_{\text{pole}}}{d\ln\mu} = 0$$

Similarly, we find for the coupling constant:

$$\alpha_0 = \mu^{2\varepsilon} \alpha(\mu) \underbrace{\left[1 - \frac{\alpha(\mu)}{3\pi} \left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln \frac{\mu^2}{m^2} + O(\varepsilon) \right) + O(\alpha^2) \right]}_{\mu \text{ independent (see above)}}$$

$$\Rightarrow 0 = \frac{d}{d\ln\mu}\left[\mu^{2\varepsilon}\alpha(\mu)\right] = \mu^{2\varepsilon}\Big[2\varepsilon\alpha(\mu) + \underbrace{\frac{d\alpha(\mu)}{d\ln\mu}}_{-2\varepsilon\alpha + \beta(\alpha)}\Big] \Rightarrow \beta(\alpha) = 0$$

Leading-order solution to the evolution equations:

Running coupling: $(\varepsilon \to 0)$

$$\frac{d\alpha_s(\mu)}{d\ln\mu} = -2\beta_0 \frac{\alpha_s^2(\mu)}{4\pi} \quad \Leftrightarrow \quad -\frac{d\alpha_s}{\alpha_s^2} = \frac{2\beta_0}{4\pi} d\ln\mu = \frac{\beta_0}{4\pi} d\ln\mu^2$$

Integrate:

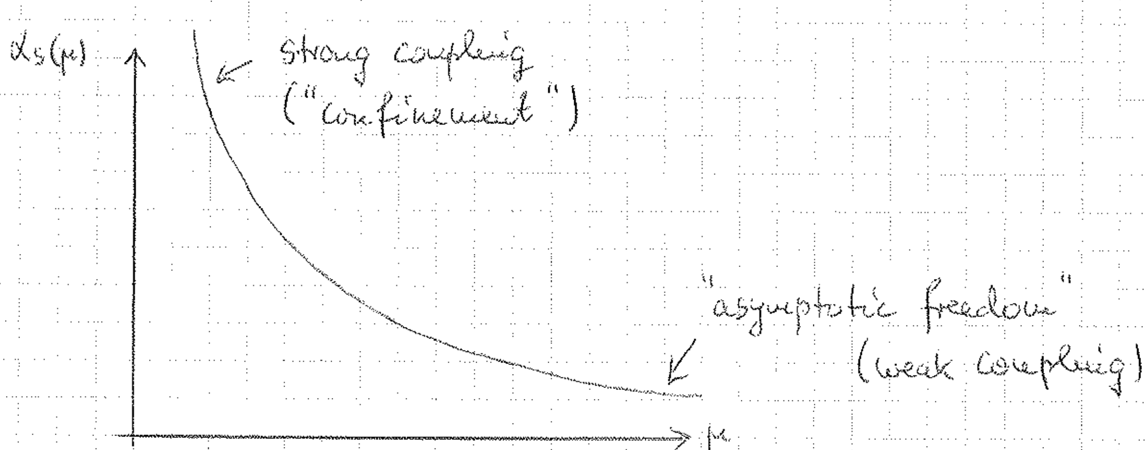
$$-\int_{\alpha_s(Q)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\alpha_s^2} = \frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(Q)} = \frac{\beta_0}{4\pi} \ln\frac{\mu^2}{Q^2}$$

$$\Rightarrow \quad \alpha_s(\mu) = \frac{\alpha_s(Q)}{1 + \frac{\beta_0}{4\pi}\underbrace{}_{\alpha_s(Q)} \ln \mu^2/Q^2} \quad ; \quad Q: \text{ some reference scale}$$

One usually chooses:

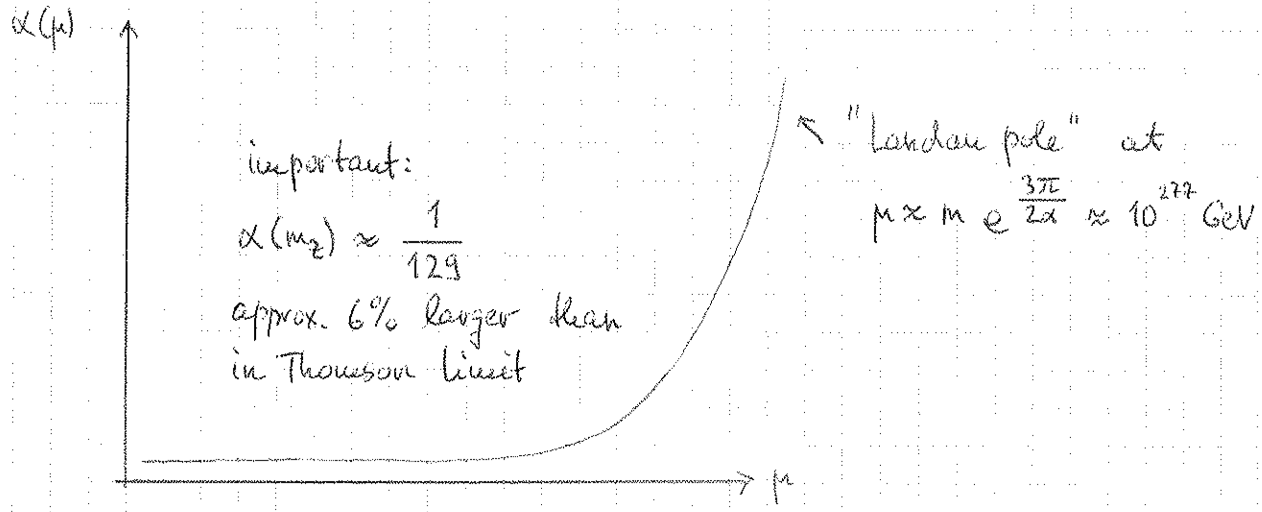
$$Q = m_Z \, , \quad \alpha_s(m_Z) \simeq 0.118$$

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_q = 11 - \frac{2}{3} n_q > 0 \quad \text{for } n_q \leq 16$$



Apply the same formalism to QED, where $\underset{\overline{MS}}{\underset{\uparrow}{\alpha(\mu=m)}} = \underset{\text{on-shell}}{\underset{\uparrow}{\alpha}} \simeq \frac{1}{137.036}$, so that:

$$\alpha(\mu) = \frac{\alpha}{1 + \frac{\beta_0}{4\pi}\,\alpha \ln \mu^2/m^2} \quad ; \quad \beta_0 = -\frac{4}{3} < 0$$



It is not difficult to include higher-order corrections (two-loop and beyond) in the calculation of running couplings.

<u>Running quark mass:</u>

$$\frac{dm_q(\mu)}{d\ln\mu} = m_q(\mu)\, \overbrace{\gamma_m^0 \frac{\alpha_s}{4\pi}}^{\gamma_m(\alpha_s)} \quad ; \quad \gamma_m^0 = -6C_F = -8$$

$$= \beta(\alpha_s)\, \frac{dm_q(\mu)}{d\alpha_s(\mu)}$$

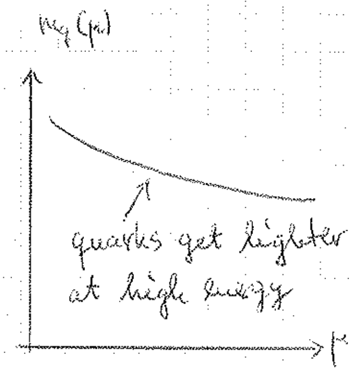
$$\Leftrightarrow \quad \frac{dm_q}{m_q} = \frac{\gamma_m(\alpha_s)}{\beta(\alpha_s)}\, d\alpha_s = -\frac{\gamma_m^0}{2\beta_0}\, \frac{d\alpha_s}{\alpha_s}$$

Integrate:

$$\ln\frac{m_q(\mu)}{m_q(Q)} = -\frac{\gamma_m^0}{2\beta_0} \ln\frac{\alpha_s(\mu)}{\alpha_s(Q)}$$

$$\Rightarrow \quad m_q(\mu) = m_q(Q) \left(\frac{\alpha_s(\mu)}{\alpha_s(Q)}\right)^{-\frac{\gamma_m^0}{2\beta_0}}$$

> 0



Again it is not difficult to include higher-order corrections (exercise). A typical reference scale for running quark masses is $Q = 2\,\text{GeV}$ $(q = u,d,s)$ or $Q = m_Q$ $(Q = c,b,t)$. Consider for example the b-quark mass:

$$m_b(m_b) \simeq 4.18 \text{ GeV}$$

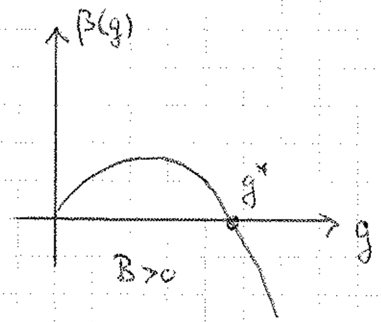
$$\Rightarrow \quad m_b(m_h) = m_b(m_b) \left(\frac{\alpha_s(m_h)}{\alpha_s(m_b)} \right)^{12/23} \simeq 2.79 \text{ GeV}$$

relevant for Higgs physics

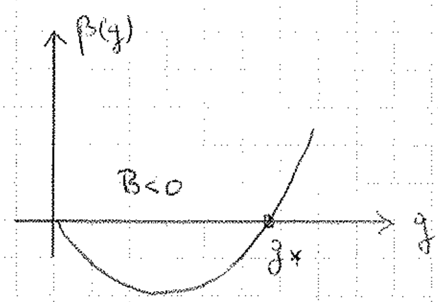
↳ large effect!

<u>Fixed points of running couplings:</u>

An interesting possibility is that $\beta(g)$ in some QFT has a zero at some value $g_* \neq 0$ of the coupling, e.g.:



or:



Near the fixed point we have:

$$\beta(g) \simeq -B\,(g - g_*) = \frac{dg}{d\ln\mu}$$

$$\Rightarrow \quad g(\mu) = g_* + [g(Q) - g_*] \left(\frac{\mu}{Q} \right)^{-B}$$

Two cases:

$B > 0$: $g(\mu) \longrightarrow g_*$ for $\mu \to \infty$ "UV stable FP"

$B < 0$: $g(\mu) \longrightarrow g_*$ for $\mu \to 0$ "IR stable FP"

Green's functions obey power-like scaling laws at the fixed point, with <u>critical exponents</u> given in terms of anomalous dimensions $\gamma(g_*) = \text{const.}$

Critical phenomena in condensed-matter physics (e.g. phase transitions) are described by anomalous dimensions in simple QFTs, such as ϕ^4 theory (see e.g. Chapters 12+13 in Peskin & Schroeder).