III. EFTs, composite operators and Wilson's approach to RG

Recall the basic ideas of EFTs:

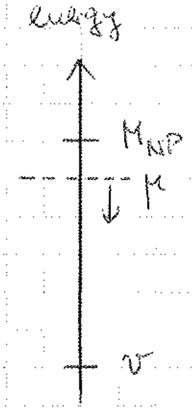
- consider a QFT with two very different scales
- obtain a simpler EFT by constructing a systematic expansion in the ratio of these scales

Two examples:

i) $\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_{NP}$

$\mathcal{L}_{SM}$: scale v $\quad$ $\mathcal{L}_{NP}$: scale $M_{NP} \gg v$

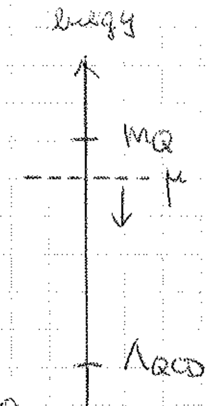
EFT = SMEFT



$\rightarrow$ expansion in powers of v/M_{NP} (modulo logs)

ii) $\mathcal{L} = \mathcal{L}_{QCD}$ (scale Λ_{QCD}) containing a heavy quark Q (scale $m_Q \gg \Lambda_{QCD}$)

EFT = HQET, NRQCD, QCD' (without Q)



$\rightarrow$ expansion in powers of Λ_{QCD}/m_Q (mod. logs)

[These EFTs will be discussed by Manohar and Hamel during this school!]

"Integrating out" the heavy degrees of freedom associated with the high scale M ($M_{NP} \hat{=}$ masses of new heavy particles, m_Q = mass of a heavy quark) from the generating functional one obtains a <u>non-local</u> action functional, which can be expanded in an infinite tower of local operators:

$$\mathcal{L}_{EFT} = \sum_{n=0}^{\infty} \sum_i \frac{C_i^{(n)}}{M^n} O_i^{(n)}$$

Comments:

- For fixed $n \geq 0$, $\{O_i^{(n)}\}$ is a complete set (basis) of local, $D = 4+n$ composite operators built out of the fields of the low-energy theory:

    i) all SM fields
    ii) all QCD fields without the heavy quark Q

- These operators are only constrained by the symmetries of the low-energy theory (Lorentz invariance, gauge invariance, C, P or T invariance, flavor symmetries, ...).

- The Wilson coefficients $C_i^{(n)}$ are dimensionless and contain information about the short-distance physics which has been integrated out.

- The above result is useful only because the $\sum_{n\geq 0}$ can be truncated at some value n_{max}, since operator matrix elements of the operator matrix elements in dim. reg. scale like:

$$\langle f | O_i^{(n)} | i \rangle \sim v^{n+\delta}, \; \Lambda_{QCD}^{n+\delta}$$ (or powers $E^{n+\delta}$ with $E \ll M$ the energy of our experiments)

where δ is set by the external states. Truncating the sum at n_{max} makes an error of order $\left(\frac{v}{M_{NP}}\right)^{n_{max}+1}, \left(\frac{\Lambda_{QCD}}{m_Q}\right)^{n_{max}+1}$ relative to the leading term.

- In essence, we split up the contribution from virtual particles into short-distance and long-distance modes:

$$\int_0^\infty \frac{d\omega}{\omega} = \underbrace{\int_M^\infty \frac{d\omega}{\omega}}_{\substack{\text{sensitive to}\\ \text{UV physics}\\ C_i^{(n)}}} + \underbrace{\int_0^M \frac{d\omega}{\omega}}_{\substack{\text{sensitive only}\\ \text{to IR physics}\\ \langle O_i^{(n)} \rangle}}$$

Now imagine that we are performing a measurement at a characteristic energy scale E, such that (in the first example) $v \ll E \ll M_{NP}$. We can then <u>integrate out high-energy fluctuations of the light fields</u> (with frequencies $\omega \gg E$) from the generating functional, because they will not be needed as source terms.

This yields a <u>different</u> effective Lagrangian, but one in which the operators $O_i^{(n)}$ are the same as before; what changes is the splitting up of modes, which now reads

$$\int_0^\infty \frac{d\omega}{\omega} = \int_E^\infty \frac{d\omega}{\omega} + \int_0^E \frac{d\omega}{\omega}$$

As a consequence, the values of the Wilson coefficients need to be differnt, i.e.:

$$\mathcal{L}_{EFT} = \sum_n \sum_i \frac{C_i^{(n)}(M)}{M^n} \underbrace{O_i^{(n)}(M)}_{\substack{\text{matrix elements}\\ \text{contain } \omega \leq M}} = \sum_n \sum_i \frac{C_i^{(n)}(E)}{M_n} \underbrace{O_i^{(n)}(E)}_{\substack{\text{matrix elements}\\ \text{contain } \omega \leq E}}$$

We are thus led to study the effective Lagrangian:

$$\mathcal{L}_{EFT} = \sum_{n=0}^{n_{max}} \sum_i \frac{C_i^{(n)}(\mu)}{M^n} O_i^{(n)}(\mu) \overset{!}{=} \mu\text{-independent}$$

Here $O_i^{(n)}(\mu)$ are renormalized composite operators, $C_i^{(n)}(\mu)$ are the associated running couplings (like $\alpha_s(\mu)$ or $m_q(\mu)$ in QCD), and μ is an arbitrary renormalization scale.

Comments:

- The terms with $n=0$ are just the renormalizable Lagrangians of the low-energy theories; hence, parameters such as $\alpha_s(\mu)$ or $m_q(\mu)$ might, in fact, contain some information about short-distance physics (e.g. GUTs).

- The higher-dimensional operators $O_i^{(n)}$ with $n \geq 1$ are interesting, because their coefficients tell us something about the high-energy scale M (e.g. weak interactions: $G_F \sim \frac{1}{m_W^2}$ $(D=6)$, neutrino masses $m_\nu \sim \frac{1}{M_{GUT}}$ $(D=5)$, ...).

- At any fixed n, the set $\{O_i^{(n)}\}$ of composite operators can be renormalized in the standard way, allowing however for the possibility of operator mixing:

$$\underset{\substack{\uparrow \\ \text{bare operators}}}{O_{i,0}^{(n)}} = \sum_j Z_{ij}^{(n)}(\mu) \underset{\substack{\uparrow \\ \text{renormalized operators}}}{O_j^{(n)}(\mu)}$$

[note: $Z_{ij}^{(n)}$ includes a factor $Z_{WF}^{1/2}$ for each component field]

In other words, in the renormalization of the operator $O_i^{(n)}$ one might need $O_{j\neq i}^{(n)}$ as counter-terms!

Important facts about the renorm. of composite ops.:

(see e.g. Pascual & Tarrach, p. 138–139)

We distinguish three types of operators:

- class-I operators are gauge invariant and do not vanish by virtue of the classical equations of motion (EoMs)
- class-II operators are gauge invariant but vanish by the classical EoMs
- class-III operators are not gauge invariant

It is convenient to use the background-field gauge, which is a method for renormalizing gauge theories while preserving explicit gauge invariance. Then the following observations apply:

i) The renormalization of class-I operators involves other class-I operators and class-II operators, but <u>not</u> class-III operators:

$$O_{I,0} = Z_I\, O_I + Z_{I\to II}\, O_{II} + \underbrace{Z_{I\to III}\, O_{III}}_{\text{absent!}} \quad \text{with } Z_{\underline{III}} = 0 \qquad (*)$$

ii) Class-II and class-III operators are renormalized among themselves:

$$\underset{(**)}{O_{II,0} = Z_{II}\, O_{II}}\,, \qquad O_{III,0} = Z_{III}\, O_{III}$$

Since $\langle O_{II} \rangle = 0$ by the EoMs, the contribution of O_{II} in (*) has no physical consequences. In background-field gauge, class-III operators never arise. Importantly, class-I operators do not appear in (**), and hence class-II operators can be ignored for all practical purposes!

Important note:

It is often stated that the use of the classical EoMs to eliminate operators from the basis $\{O_i^{(n)}\}$ is not justified beyond tree level. This statement is incorrect. Class-II operators can always be removed using field redefinitions. (In some cases this generates class-I operators of higher dimension.)

Special care must be taken when the field redefinitions change the measure of the functional integral (→ anomalies).

In any event, the lesson is that at fixed $n \geq 1$ class-II operators can simply be removed from the operator basis!

Anomalous dimensions of composite operators:

From the fact that the bare operators are μ independent, it follows that: (sum over repeated indices is implicit)

$$0 = \frac{dZ_{ij}^{(n)}(\mu)}{d\ln\mu} O_j^{(n)}(\mu) + Z_{ij}^{(n)}(\mu) \frac{dO_j^{(n)}(\mu)}{d\ln\mu}$$

$$\Leftrightarrow \quad \frac{dO_k^{(n)}(\mu)}{d\ln\mu} = - \underbrace{Z_{ki}^{(n)-1}(\mu) \frac{dZ_{ij}^{(n)}(\mu)}{d\ln\mu}}_{\gamma_{kj}^{(n)}} O_j^{(n)}(\mu) \equiv - \gamma_{kj}^{(n)}(\alpha_s(\mu)) \, O_j^{(n)}(\mu)$$

Matrix notation:

$$\frac{d\vec{O}^{(n)}}{d\ln\mu} = - \hat{\gamma}^{(n)} \vec{O}^{(n)}$$

anomalous-dimension matrix

As previously, the anomalous-dimension matrix $\hat{\gamma}^{(n)}$ can be obtained from the coefficient of the $1/\varepsilon$ pole term in $\hat{Z}^{(n)}(\alpha_s)$, via the exact relation:

$$\hat{\gamma}^{(n)}(\alpha_s) = -2\alpha_s \frac{d\left(\hat{Z}^{(n)}(\alpha_s)\right)^{(1)}}{d\alpha_s} \quad \swarrow 1/\varepsilon \text{ pole}$$

RGE for Wilson coefficients:

The fact that the effective Lagrangian is μ independent implies, for fixed $n \geq 0$:

$$\sum_i \left[\frac{dC_i^{(n)}(\mu)}{d\ln\mu}\, O_i^{(n)}(\mu) + C_i^{(n)}(\mu)\, \frac{dO_i^{(n)}(\mu)}{d\ln\mu}\right] = 0$$

$$\Leftrightarrow \quad \frac{d\vec{C}^{(n)}(\mu)}{d\ln\mu}\, \vec{O}^{(n)}(\mu) - \vec{C}^{(n)}(\mu)\, \hat{\gamma}^{(n)}\, \vec{O}^{(n)}(\mu) = 0$$

From the linear independence of the basis operators $\{O_i^{(n)}\}$ it follows that:

$$\frac{d\vec{C}^{(n)}(\mu)}{d\ln\mu} = \hat{\gamma}^{(n)T}\, \vec{C}^{(n)}(\mu)$$

Expressing $\vec{C}^{(n)}$ and $\hat{\gamma}^{(n)}$ as functions of the running coupling $\alpha_s(\mu)$, one finds the formal solution:

$$\vec{C}^{(n)}(\mu) = T_\alpha \exp\left[\int_{\alpha_s(Q)}^{\alpha_s(\mu)} d\alpha_s\, \frac{\hat{\gamma}^{(n)T}(\alpha_s)}{\beta(\alpha_s)}\right] \vec{C}^{(n)}(Q)$$

T_α: ordering in α_s such that $\alpha_s(\mu)$ stands left

Q: reference scale; usually choose $Q \approx M$ equal to the high scale

At leading order (but not beyond) the ordering symbol becomes irrelevant and one obtains:

$$\vec{C}^{(n)}(\mu) \overset{\text{LO}}{=} \exp\left[-\frac{\hat{\gamma}_0^{(n)T}}{2\beta_0} \ln \frac{\alpha_s(\mu)}{\alpha_s(Q)}\right] \vec{C}^{(n)}(Q)$$

matrix exponential

Resummation of large logarithms:

Starting from:

$$\mathcal{L}_{EFT} \simeq \sum_{n=0}^{n_{max}} \sum_i \frac{C_i^{(n)}(\mu)}{M^n} O_i^{(n)}(\mu) \quad ,$$

we choose $\mu \sim E_{expt.}$ such that the operator matrix elements $\langle O_i^{(n)}(\mu) \rangle$ do not contain any large logarithms. These large logs are then resummed and contained in the values of the Wilson coefficients. For the simplest case of only 1 operator, we have at leading order:

$$\underset{\substack{\uparrow \\ \text{expt'l energy}}}{C(\mu = E)} = \left(\frac{\alpha_s(E)}{\alpha_s(M)}\right)^{-\gamma_0/2\beta_0} C(M)$$

↑ high-energy matching condition from integrating out heavy dofs.

To see how this resums large logs, we can substitute the relation:

$$\frac{\alpha_s(E)}{\alpha_s(M)} = \frac{1}{1 + \beta_0 \frac{\alpha_s(M)}{4\pi} \ln \frac{E^2}{M^2}}$$

for the running couplings. This yields:

$$C(E) = \left(1 - \beta_0 \frac{\alpha_s(M)}{4\pi} \ln \frac{M^2}{E^2}\right)^{\gamma_0/2\beta_0} C(M)$$

large logs (resummed)

$$= \left\{ 1 - \frac{\gamma_0}{2} \frac{\alpha_s(M)}{4\pi} \ln \frac{M^2}{E^2} + \frac{\gamma_0(\gamma_0 - 2\beta_0)}{8} \left(\frac{\alpha_s(M)}{4\pi} \ln \frac{M^2}{E^2} \right)^2 + \dots \right\} C(M)$$

One last remark concerning the running coupling $\alpha_s(\mu)$:

In order for the expressions for the Wilson coefficients (see previous page) to make sense, we need to make sure that our formula for $\alpha_s(\mu)$ can be reliably evaluated at any value of μ. At leading order we found:

$$\alpha_s(\mu) = \frac{\alpha_s(Q)}{1 + \beta_0 \frac{\alpha_s(Q)}{4\pi} \ln \frac{\mu^2}{Q^2}}$$

what if this becomes a large log?

We need to show that higher-order corrections in the β-function do not spoil this formula by introducing additional large logs.

To see this, recall that:

$$\beta(\alpha_s) = \frac{d\alpha_s(\mu)}{d\ln\mu} = -2\alpha_s \left(\beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \beta_2 \left(\frac{\alpha_s}{4\pi}\right)^3 + \dots \right)$$

Let's study the effect of these higher-order terms. Separating variables, we have:

$$-\frac{d\alpha_s}{\alpha_s^2} \frac{1}{\left(1 + \frac{\beta_1}{\beta_0}\frac{\alpha_s}{4\pi} + \frac{\beta_2}{\beta_0}\left(\frac{\alpha_s}{4\pi}\right)^2 + \dots\right)} = \frac{\beta_0}{4\pi} d\ln\mu^2$$

no logs in this series!

single source of log

As long as we are in the perturbative regime, we can expand the perturbative series on the left-hand side and obtain, at NLO:

$$-\frac{d\alpha_s}{\alpha_s^2}\left(1-\frac{\beta_1}{\beta_0}\frac{\alpha_s}{4\pi}+\dots\right) = \frac{\beta_0}{4\pi}\ln\mu^2$$

Integrating this equation gives:

$$\frac{1}{\alpha_s(\mu)} - \underset{\substack{\uparrow \\ \text{reference scale}}}{\frac{1}{\alpha_s(Q)}} + \frac{\beta_1}{4\pi\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(Q)} + \mathcal{O}\left(\frac{\alpha_s(\mu)-\alpha_s(Q)}{16\pi^2}\right) = \frac{\beta_0}{4\pi}\ln\frac{\mu^2}{Q^2}$$

Multiplying with $\alpha_s(Q)$ gives:

$$\frac{\alpha_s(Q)}{\alpha_s(\mu)} - \frac{\beta_1}{\beta_0}\frac{\alpha_s(Q)}{4\pi}\ln\frac{\alpha_s(Q)}{\alpha_s(\mu)} + \mathcal{O}\left[\frac{\alpha_s(Q)}{4\pi}\frac{(\alpha_s(\mu)-\alpha_s(Q))}{4\pi}\right] = 1+\beta_0\frac{\alpha_s(Q)}{4\pi}\ln\frac{\mu^2}{Q^2}$$

only large log!

We can now insert here the LO solution for $\alpha_s(Q)/\alpha_s(\mu)$ to obtain:

$$\frac{\alpha_s(Q)}{\alpha_s(\mu)} = 1+\beta_0\frac{\alpha_s(Q)}{4\pi}\ln\frac{\mu^2}{Q^2} + \frac{\beta_1}{\beta_0}\frac{\alpha_s(Q)}{4\pi}\ln\left(1+\beta_0\frac{\alpha_s(Q)}{4\pi}\ln\frac{\mu^2}{Q^2}\right)+\dots$$

Even in the "large-log region" where $\frac{\alpha_s(Q)}{4\pi}\ln\frac{\mu^2}{Q^2} = \mathcal{O}(1)$ or bigger, the correction proportional to β_1 (2-loop β-function) is suppressed by at least $\frac{\alpha_s(Q)}{4\pi} \ll 1$ relative to the leading term! Higher-order terms are suppressed by $\left(\frac{\alpha_s(Q)}{4\pi}\right)^n$. The LO formula for $\alpha_s(\mu)$ is thus a decent approximation for all values of μ.

THANK YOU ☺