Renormalization factors in QCD:

The QCD Lagrangian is structurally similar to QED:

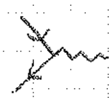
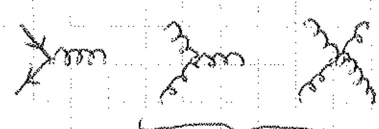
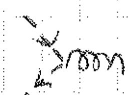
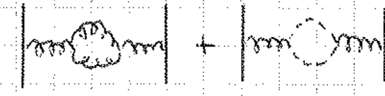
$$\mathcal{L}_{QCD} = \sum_q \bar{q}(i\not{D} - m_q)\,q - \frac{1}{4} F_{\mu\nu,a} F_a^{\mu\nu} + \theta\text{-term } (?)$$

$$+ \text{ gauge fixing } + \text{ Faddeev-Popov ghosts}$$

where:

$$i D^\mu = i\partial^\mu + \underset{\substack{\uparrow \\ \text{strong coupling}}}{g_s} A_a^\mu \, t_a \; \nwarrow \text{SU(3) generators}$$

Similarities and differences:

	QED	QCD
gauge group:	U(1) abelian	SU(N_c=3) non-abelian
vertices:		 gluon self-couplgs. 
Faddeev-Popov ghosts: (anticommuting scalar fields → wrong spin-statistics relation → <u>not</u> as external states)	—	needed to cancel unphysical gluon polarizatns  = physical
non-trivial group factors:	—	$C_F = \frac{N_c^2-1}{2N_c}$, $C_A = N_c$, $T_F = \frac{1}{2}$
superficial degree of divergence:	$D = 4 - \frac{3}{2} N_e - N_\gamma$	$D = 4 - \frac{3}{2} N_q - N_g - \frac{3}{2} \underset{\substack{\uparrow \\ \text{ghosts}}}{N_c}$
phenomenology:	weak coupling at all relevant energies (Landau pole > M_{PL})	<u>strong coupling</u> at low energies → confinement; weak coupling at high energies (<u>asymptotic freedom</u>)

Important note:

While the on-shell renormalization scheme is useful for many (but not all → see below) calculations in QED and the EW theory, it is not a viable scheme for QCD:

- quarks and gluons are never on-shell due to confinement
- coupling g_s cannot be renormalized at $q^2 = 0$, since QCD is strongly coupled at low energy and quarks and gluons are not the relevant degrees of freedom there
- masses of light quarks ($q = u, d, s$) satisfy $m_q \ll \Lambda_{QCD}$ and hence can be set to zero
  $\Rightarrow$ will not consider heavy quarks here, which are usually dealt with using EFTs (several lecture courses in this school!)

↳ will use $\overline{\text{MS}}$ scheme in QCD!

Superficial degree of divergence (exercise): (skip this!)

i) $D = 4L - P_q - 2P_g - P_c + V_{3g}$ ↖ contains one power of momentum

ii) $L = P_q + P_g + P_c - (V_{qg} + V_{3g} + V_{4g} + V_{cg}) + 1$

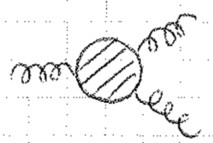
iii) $V_{qg} = \frac{1}{2}(2P_q + N_q)$; $V_{cg} = \frac{1}{2}(2P_c + N_c)$;

$V_{qg} + 3V_{3g} + 4V_{4g} + V_{cg} = 2P_g + N_c$

Combining this yields

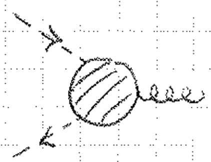
$$D = 4 - N_g - \frac{3}{2}N_q - \frac{3}{2}\underset{\substack{\uparrow \\ \text{ghosts}}}{N_c} \qquad (\text{note: } N_c = 0 \text{ for physical amplitudes})$$

In addition to QED, the following amplitudes have logarithmic UV divergences and require renormalization:



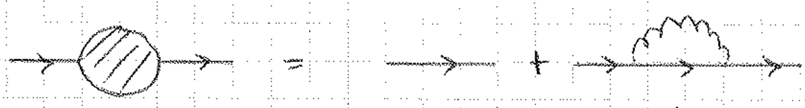
$D=1$, but log. div. due to gauge invariance (Furry's theorem does not apply to QCD)

$D=0$, log. divergent



$D=0$, log. divergent

Quark self-energy:

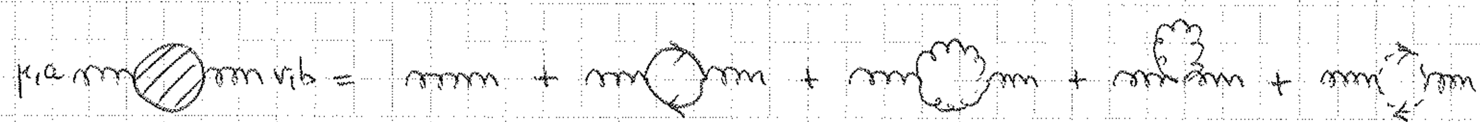


same as QED ($\alpha \to \alpha_s$) times color factor $\mathrm{tr}(t_a t_a) = C_F$

hence:

$$Z_2 = 1 - \frac{C_F \alpha_s}{4\pi \hat{\varepsilon}} \qquad (\overline{MS})$$

Vacuum polarization:



same as QED (with $\alpha \to \alpha_s$) times color factor $\mathrm{tr}(t_a t_b) = T_F \delta^{ab}$

genuinely non-abelian!

$$= \delta^{ab} \left(g^{\mu\nu} q^2 - q^\mu q^\nu \right) \Pi(q^2)$$

find:

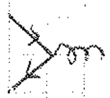
$$\Pi(q^2) = \frac{\alpha_s}{4\pi} \left\{ \left[\left(\frac{13}{6} - \frac{\xi}{2} \right) C_A - \frac{4}{3} T_F n_q \right] \left(\frac{1}{\hat{\varepsilon}} - \ln \frac{-q^2 - i0}{\mu^2} \right) + \ldots \right\}$$

gauge parameter ($\xi = 1/\lambda$)

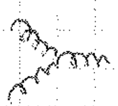
hence:
$$Z_3 = \frac{1}{1-\Pi(q^2)}\Big|_{\text{pole term}}$$
$$= 1 + \frac{\alpha_s}{4\pi\hat{\epsilon}}\left[\left(\frac{13}{6}-\frac{\xi}{2}\right)C_A - \frac{4}{3}T_F n_q\right]$$

Charge renormalization:

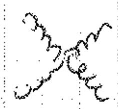
Gauge invariance requires that the same charge appears in all four vertices:



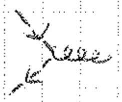
$$g_{s,0} = \mu^\epsilon \underbrace{Z_1 Z_2^{-1}}_{\text{no longer} = 1 \text{ in QCD!}} Z_3^{-1/2}\, g_s$$



$$g_{s,0} = \mu^\epsilon Z_1^{3g} Z_3^{-3/2}\, g_s$$



$$g_{s,0} = \mu^\epsilon Z_1^{4g} Z_3^{-2}\, g_s$$



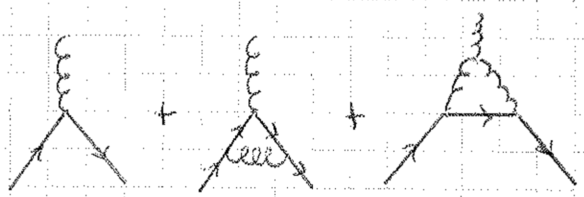
$$g_{s,0} = \mu^\epsilon Z_1^{c} Z_{2c}^{-1} Z_3^{-1/2}\, g_s$$ ($\rightarrow$ simplest way to calculate charge-ren. constant)

It follows that:

$$Z_1^{3g} = Z_1 Z_3 Z_2^{-1}\;;\quad Z_1^{4g} = Z_1 Z_2^{-1} Z_3^{3/2}\;;\quad Z_1^{c} = Z_1 Z_{2c} Z_2^{-1}$$

$\rightarrow$ exact relations between renormalization constants

We work with the first relation and compute the renormalization constant Z_1 of the quark-gluon vertex:



needs: $f^{abc}\, t^b t^c = \frac{i}{2} C_A t^a$

One finds:

$$Z_1 = 1 - \frac{\alpha_s}{4\pi\hat{\varepsilon}}\left[C_F + \frac{3+\xi}{4} C_A \right] \neq Z_2$$

similar to QED $\qquad$ non-abelian (gauge-dependent)

We now obtain:

$$g_{s,0} = \mu^{\varepsilon}\, Z_1 Z_2^{-1} Z_3^{-1/2}\, g_s$$

different signs!

with:

$$Z_1 Z_2^{-1} Z_3^{-1/2} = 1 - \frac{\alpha_s}{4\pi\hat{\varepsilon}}\left(\frac{11}{6} C_A - \frac{2}{3} T_F n_q \right)$$

similar

$$\left[\text{cf. QED:} \qquad 1 + \frac{\alpha}{6\pi\hat{\varepsilon}}\, n_\ell \right]$$