Heavy Quark Effective Theory: a predictive EFT on the lattice

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Lattice for EFTs, EFTs for the lattice, and EFT on the lattice

1.1 Overview

For concreteness we consider 3+1 dimensions. Lattice field theories then approximate the 4-d (Euclidean) space by the discrete set of points of a lattice. A hypercubic lattice,

$$\Lambda = a\mathbb{Z}^4 = \{x_\mu = an_\mu |\, n_\mu \in \mathbb{Z}\,,\ \mu = 0,1,2,3\}\,, \tag{1.1}$$

has enhanced symmetry which is important for renormalisation. For numerical computations one considers a finite lattice,

$$\Lambda = \{x_\mu = an_\mu |\, n_\mu = 0,1,\ldots L_\mu/a - 1\} \tag{1.2}$$

with mostly

$$L_0 = T\,,\ L_1 = L_2 = L_3 = L\,. \tag{1.3}$$

The action which appears in the path integral weight has the form

$$S = \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{cont}}(x) \to a^4 \sum_{x\in\Lambda} \mathcal{L}_{\mathrm{lat}}(x) \tag{1.4}$$

$$\equiv a^4 \sum_{n_0,n_1,n_2,n_3} \mathcal{L}_{\mathrm{lat}}(an_0, an_1, an_2, an_3)\,. \tag{1.5}$$

We discuss Lagrangians $\mathcal{L}_{\mathrm{lat}}$ later.

A feeling for relevant scales in numerical lattice QCD is provided by Table 1.1.

Table 1.1 Typical scales in lattice QCD. Quark masses are often varied. Restrictions are due to limits in computer power ($L/a = 32 - 128$) and the need to avoid finite size effects ($m_\pi L \gtrsim 4$).

energy scale		typical range	remark
mass gap	m_π	130 MeV — 500 MeV	infrared scale $L \gg m_\pi^{-1}$
cutoff	$1/a$	1.5 GeV — 6 GeV	in large volume $L = 3 - 6$fm
b-quark mass	m_{b}	$\approx 5\,\mathrm{GeV}$	

Euclidean two-point functions

$$G_2^\Phi(x,y) = \langle \Phi^\star(x)\Phi(y)\rangle \tag{1.6}$$

allow for the extraction of energies E_n and matrix elements $\mathcal{M}_n = \langle n|\hat{\Phi}(\mathbf{x})|0\rangle$ through

$$\tfrac{1}{L^3} a^6 \sum_{\mathbf{xy}} G_2^\Phi(x,y) = \sum_{n\geq 1} |\mathcal{M}_n|^2 \, \mathrm{e}^{-|x_0-y_0|E_n} \tag{1.7}$$

$$[\text{ e.g. } \Phi = \Phi_\pi = i\bar{u}\gamma_5 d\,,\ \Phi^* = i\bar{d}\gamma_5 u\,]\,. \tag{1.8}$$

At large $x_0 - y_0$, the lowest terms dominate and one can determine the low lying energies and matrix elements. In particular when the two-point function is projected to space momentum zero, $\mathbf{p} = 0$, as done here by $a^3\sum_{\mathbf{x}} \mathrm{e}^{i0\cdot\mathbf{x}}$, and when there is a bound state in the channel excited by Φ (as opposed to just resonances), then

$$E_1 = m(a) \tag{1.9}$$

is a particle mass and $\mathcal{M}_1$ is a vacuum-to-one-particle matrix element. The physical mass is given by the continuum limit

$$m_{\rm cont} = \lim_{a\to 0} m(a)\,. \tag{1.10}$$

The big point is that the Euclidean, latticised, path integral can be evaluated by Monte Carlo “simulations”(Luscher, 2010,Schaefer, 2009) and thus expansions in coupling or

Table 1.2 Examples for the interplay of EFT and lattice QCD.

rôle	EFT	range of EFT	range of lattice QCD
1. Lattice for EFT			
determine LEC’s	Chiral PT	low energy QCD	“all energies” w. $E \ll a^{-1}$
2. EFT for Lattice			
discretisation effects	Symanzik EFT	$E \ll a^{-1}$	
finite volume effects	Chiral PT	$L^{-1} \ll m_\pi, \Lambda_{\rm QCD}$	
quark mass effects	Chiral PT	$m_{\rm u}, m_{\rm d} \ll \Lambda_{\rm QCD}$	
	Heavy Meson Chiral PT	$m_{\rm b} \gg \Lambda_{\rm QCD}$ $m_{\rm u}, m_{\rm d} \ll \Lambda_{\rm QCD}$	
combined effects	HMrsChPT	$m_{\rm u}, m_{\rm d} \ll \Lambda_{\rm QCD}$, $m_{\rm b} \gg \Lambda_{\rm QCD}$, $E \ll a^{-1}$	
3. EFT on the Lattice			
NP EFT	$\mathrm{QCD}^{(3)}$	$E \ll m_{\rm c}, m_{\rm b}, m_{\rm t}$	
NP EFT	HQET	E, $\Lambda_{\rm QCD} \ll m_{\rm b}$	$E \ll a^{-1}$, $m_{\rm b} \gtrsim a^{-1}$
NP EFT	NRQCD	E, $\Lambda_{\rm QCD} \ll a^{-1} \ll m_{\rm c}, m_{\rm b}$	
NP EFT	Nuclear EFT	I am not an expert	

quark mass are *not* needed. Thus a lattice field theory gives us access to the non-perturbative spectrum and selected matrix elements. It provides the non-perturbative definition of the QFT and through MC simulations also a means for its numerical solution.

The question arises, why then consider EFT's in the context of lattice field theories. There are roughly speaking three reasons.

1. Lattice QCD can provide observables which add to experimental ones, in order to determine the parameters of EFTs, e.g. $m_{\rm quark} \neq$ Nature, $\alpha_{\rm em} \neq$ Nature.

2. EFTs can help extrapolations of numerical results of lattice QCD. Prominent examples are the extrapolations $L \to \infty$ and to physical quark masses (Chiral PT) and to the continuum limit (Symanzik EFT). In some cases there are parameter-free asymptotic formulae, more generally the EFT dictates the functional form of extrapolations which will involve a (hopefully small) number of free parameters.

3. The third category are EFTs which are *not* solvable analytically, because they are strongly interacting, such as HQET which describes heavy quarks interacting non-perturbatively with the other QCD fields. Here the theory itself is discretised and simulated on a lattice.

Table 1.2 contains examples, the relevant scales, and their relations which are necessary for the EFT to apply.

1.2 Why EFT on the lattice?

Apart from the theoretical interest in describing EFTs non-perturbatively, the advantage of using HQET and not just QCD with the heavy b-quark as a relativistic Dirac field is that the cutoff, i.e. the inverse lattice spacing does not need to be much larger than the quark mass,

$$m_{\rm b} > \Lambda_{\rm cut} = a^{-1} \;\; \text{is ok in HQET.} \tag{1.11}$$

Thus b-quarks become treatable without extrapolations or other tricks. A feeling for the relevant scales is provided by Table 1.1.

In these two lectures we focus on HQET and emphasize general features of non-perturbative EFTs and the question how the parameters of the EFT can be determined without loosing predictions. We also give a flavor of Symanzik EFT, our basis for understanding how lattice field theories approach the continuum.

2

Heavy Quark Effective Theory at zero velocity

This theory was introduced by T. Mannel (Mannel, 2017). Our notation is

$$\psi_{\mathrm{h}} = \phi_0|_{\mathrm{Mannel}} \,,$$

$$\psi_{\bar{\mathrm{h}}} = \chi_0|_{\mathrm{Mannel}} \,.$$

We repeat the definitions of the special case of zero velocity in Euclidean time and in our notation. We still work in the formal (no regularisation) continuum theory.

2.1 Lagrangian and propagator

The Lagrangian is

$$\mathcal{L} = \mathscr{L}_{\mathrm{h}}^{\mathrm{stat}} + \mathscr{L}_{\mathrm{h}}^{(1)} \tag{2.1}$$

$$+ \mathscr{L}_{\bar{\mathrm{h}}}^{\mathrm{stat}} + \mathscr{L}_{\bar{\mathrm{h}}}^{(1)} + \mathrm{O}(\tfrac{1}{m_{\mathrm{h}}^2}) \tag{2.2}$$

$$\mathscr{L}_{\mathrm{h}}^{\mathrm{stat}} = \overline{\psi}_{\mathrm{h}}(m_{\mathrm{h}} + D_0)\psi_{\mathrm{h}} \,, \qquad [\, \mathscr{L}_{\bar{\mathrm{h}}}^{\mathrm{stat}} = \overline{\psi}_{\bar{\mathrm{h}}}(m_{\mathrm{h}} - D_0)\psi_{\bar{\mathrm{h}}} \,] \,, \tag{2.3}$$

$$\mathscr{L}_{\mathrm{h}}^{(1)} = -\tfrac{1}{2m_{\mathrm{h}}}(\mathcal{O}_{\mathrm{kin}} + \mathcal{O}_{\mathrm{spin}}) \,. \tag{2.4}$$

Due to the constraints

$$\psi_{\mathrm{h}} = P_+\psi_{\mathrm{h}} \,, \quad \psi_{\bar{\mathrm{h}}} = P_-\psi_{\bar{\mathrm{h}}} \,. \tag{2.5}$$

with

$$P_\pm = \tfrac{1}{2}(1 \pm \gamma_0) \,, \quad P_+P_- = 0, \tag{2.6}$$

the fields formally have four components but only two degrees of freedom. The mass-dimension five fields composing $\mathscr{L}_{\mathrm{h}}^{(1)}$ are

$$\mathcal{O}_{\mathrm{kin}}(x) = \psi_{\bar{\mathrm{h}}}(x)\, \mathbf{D}^2\, \psi_{\mathrm{h}}(x) \,, \tag{2.7}$$

$$\mathcal{O}_{\mathrm{spin}}(x) = \psi_{\bar{\mathrm{h}}}(x)\, \boldsymbol{\sigma} \cdot \mathbf{B}(x)\, \psi_{\mathrm{h}}(x) \,, \tag{2.8}$$

with

$$\sigma_k = \tfrac{1}{2}\epsilon_{ijk}\sigma_{ij} \,, \quad B_k = i\tfrac{1}{2}\epsilon_{ijk}[D_i, D_j] \,. \tag{2.9}$$

Analogous expressions for $\mathscr{L}_{\bar{\mathrm{h}}}^{(1)}$ are skipped here.

Consider the lowest order $\mathcal{L} = \mathscr{L}_{\mathrm{h}}^{\mathrm{stat}}$. The propagator, G^{stat}, of the field ψ_{h} is defined by

$$(D_0 + m)G^{\mathrm{stat}}(x, y; A) = (\partial_0 + A_0(x) + m)G^{\mathrm{stat}}(x, y) = \delta(x - y)\, P_+ \,. \qquad (2.10)$$

Note that this is the propagator in the presence of a gauge-field $A_\mu(x)$ that one integrates over in the path-integral. It is easy to write down explicitly:

$$G^{\mathrm{stat}}(x, y; A) = \theta(x_0 - y_0)\, \delta(\mathbf{x} - \mathbf{y})\, P_+ \, \exp(-m\,(x_0 - y_0))\, \mathcal{P}(x \leftarrow y)\,, \qquad (2.11)$$

$$\mathcal{P}(x \leftarrow y) = \mathbf{P}_{\mathrm{ord}} \exp\left\{ - \int_{y_0}^{x_0} \mathrm{d}z_0 A_0(z_0, \mathbf{x}) \right\} . \qquad (2.12)$$

Here $\mathbf{P}_{\mathrm{ord}}$ denotes path ordering. Any 2-point function, with a heavy quark, e.g.

$$C_{\mathrm{AA}}^{\mathrm{stat}}(x_0) = a^3 \sum_{\mathbf{x}} \left\langle A_0^{\mathrm{stat}}(x) (A_0^{\mathrm{stat}})^\dagger(0) \right\rangle_{\mathrm{stat}} , \quad A_0^{\mathrm{stat}} = \overline{\psi}\gamma_0\gamma_5\psi_{\mathrm{h}} \qquad (2.13)$$

$$= -a^3 \sum_{\mathbf{x}} \frac{1}{Z} \int \mathrm{D}[A] \qquad (2.14)$$

$$\mathrm{tr} \left\{ G^{\mathrm{stat}}(x, 0; A_\mu)\gamma_5\gamma_0 \; \underbrace{G^{\mathrm{light}}(0, x; A_\mu)}_{\text{sol. of Dirac eq.}} \; \gamma_0\gamma_5 \right\} \mathrm{e}^{-S_{\mathrm{eff}}[A]}$$

satisfies

$$C^{\mathrm{stat}}(x, y; m) = C^{\mathrm{stat}}(x, y; 0)\, \exp(-m\,(x_0 - y_0))\,. \qquad (2.15)$$

Therefore the term m in the Lagrangian may be removed (strictly speaking we need $m \to \epsilon > 0$ and then consider the limit $\epsilon \to 0$) and then all energies shifted by it:

$$E_n = E_n|_{m=0} + m\,. \qquad (2.16)$$

This is exact; we did not use perturbation theory.

2.2 Symmetries

Symmetries

1. Flavor

If there are F heavy quarks, we just add a corresponding flavor index and use a notation

$$\psi_{\mathrm{h}} \to \psi_{\mathrm{h}} = (\psi_{\mathrm{h}1}, \ldots, \psi_{\mathrm{h}F})^T\,, \quad \overline{\psi}_{\mathrm{h}} \to \overline{\psi}_{\mathrm{h}} = (\overline{\psi}_{\mathrm{h}1}, \ldots, \overline{\psi}_{\mathrm{h}F}) \qquad (2.17)$$

$$\mathcal{L}_{\mathrm{h}}^{\mathrm{stat}} = \overline{\psi}_{\mathrm{h}}(D_0 + \epsilon)\psi_{\mathrm{h}}\,. \qquad (2.18)$$

Then we obviously have the symmetry

$$\psi_{\mathrm{h}}(x) \to V\,\psi_{\mathrm{h}}(x)\,, \quad \overline{\psi}_{\mathrm{h}}(x) \to \overline{\psi}_{\mathrm{h}}(x)V^\dagger\,, \qquad V \in \mathrm{SU}(F) \qquad (2.19)$$

and the same for the anti-quarks. Note that this symmetry emerges in the large mass limit irrespective of how the limit is taken. For example we may take ($F=2$ with the first heavy flavor identified with charm and the second with beauty)

$$m_\mathrm{b} - m_\mathrm{c} = c \times \Lambda_\mathrm{QCD}\,, \quad \text{or} \quad m_\mathrm{b}/m_\mathrm{c} = c'\,, \quad m_\mathrm{b} \to \infty \tag{2.20}$$

with either c or c' fixed when taking $m_\mathrm{b} \to \infty$.

2. Spin

We further note that for each field there are also the two spin components but the Lagrangian contains no spin-dependent interaction. The associated SU(2) rotations are generated by the spin matrices eq. (2.9) (remember that $\psi_\mathrm{h}\,,\ \overline{\psi}_\mathrm{h}$ are kept as 4-component fields with 2 components vanishing)

$$\sigma_k = \frac{1}{2}\epsilon_{ijk}\sigma_{ij} \equiv \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}\,, \tag{2.21}$$

where the symbol σ_k is used at the same time for the Pauli matrices and the 4×4 matrix. We here are in the Dirac representation where

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,\ P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,,\ P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{2.22}$$

The spin rotation is then

$$\psi_\mathrm{h}(x) \to \mathrm{e}^{i\alpha_k\sigma_k}\,\psi_\mathrm{h}(x)\,, \qquad \overline{\psi}_\mathrm{h}(x) \to \overline{\psi}_\mathrm{h}(x)\mathrm{e}^{-i\alpha_k\sigma_k}\,, \tag{2.23}$$

with arbitrary real parameters α_k. It acts on each flavor component of the field. Obviously, the symmetry is even bigger. We can take $V \in \mathrm{SU}(2F)$ in eq. (2.19). This plays a rôle in heavy meson ChPT (Wise, 1992, Grinstein *et al.*, 1992, Burdman and Donoghue, 1992).

3. Local Flavor-number

The static Lagrangian contains no space derivative. The transformation

$$\psi_\mathrm{h}(x) \to \mathrm{e}^{i\eta(\mathbf{x})}\,\psi_\mathrm{h}(x)\,, \qquad \overline{\psi}_\mathrm{h}(x) \to \overline{\psi}_\mathrm{h}(x)\mathrm{e}^{-i\eta(\mathbf{x})}\,, \tag{2.24}$$

is therefore a symmetry for any local phase $\eta(\mathbf{x})$. For every point $\mathbf{x}$ there is a corresponding Noether charge

$$Q_\mathrm{h}(x) = \overline{\psi}_\mathrm{h}(x)\psi_\mathrm{h}(x)\,[= \overline{\psi}_\mathrm{h}(x)\gamma_0\psi_\mathrm{h}(x)\,] \tag{2.25}$$

which we call local flavor number. It is conserved,

$$\partial_0 Q_\mathrm{h}(x) = 0 \;\; \forall x\,. \tag{2.26}$$

We can take these symmetries to be the defining properties of the (lowest order) EFT.

3 Non-perturbative formulation of EFT

We now describe the general concept and formulation of an effective field theory. The special features of HQET will be mentioned in the following subsection. We consider processes in a fundamental theory (QCD or the standard model of particle physics – the important feature is the renormalizability of the theory) at low energy. In particular we first focus on processes (scattering, decay) of particles with masses of this low energy or below it (in HQET also the large mass particles *are* involved as discussed by T. Mannel). In this situation, vacuum fluctuations involving much heavier particles are suppressed and a true creation of the heavier particles is energetically forbidden. One therefore expects to be able to describe the physics of these low energy processes by an effective field theory containing only the fields of the light particles (Weinberg, 1979). The leading order Lagrangian of the theory is formed first from the free field theory Lagrangians and all the renormalizable interactions. Restricting to just renormalizable interactions at the lowest order is not always possible. We will come back to that later. For now we consider universal EFTs where the lowest order theory is renormalizable.

3.1 Universal EFTs

Assuming the usual power counting, all local composite fields with mass dimension smaller or equal to four are allowed. Let us denote the Lagrangian by $\mathcal{L}_{\rm LO}$ and the Euclidean action is $S^{\rm LO} = \int \mathrm{d}^4x \mathcal{L}_{\rm LO}(x)$. Correlation functions are then defined by the standard path integral

$$\Phi^{\rm LO} = \langle O \rangle_{\rm LO} = \tfrac{1}{Z_{\rm LO}} \int_{\rm fields} \mathrm{e}^{-S^{\rm LO}} O\,, \quad S^{\rm LO} = \int \mathrm{d}^4x\, \mathcal{L}_{\rm LO}(x)\,, \tag{3.1}$$

$$\mathcal{L}_{\rm LO}(x) = \sum_i \omega_i^{\rm LO}\, \mathscr{O}_i^{\rm LO}(x)\,, \quad [\mathscr{O}_i^{\rm LO}] \leq 4 \quad [\omega_i^{\rm LO}] \geq 0. \tag{3.2}$$

with $\langle 1 \rangle_{\rm LO} = 1$ and O some multi-local product of fields, e.g.

$$O = \varphi(x)\varphi(y)\,. \tag{3.3}$$

In this way we start at LO with a *renormalizable* theory. For a lattice formulation this means that the *continuum limit* of the theory exists when a finite number of bare parameters is varied as a function of a with renormalized parameters kept fixed.

Higher order terms in the expansion of physical amplitudes (or correlation functions) in $\frac{1}{m_{\rm h}}$ are given by including fields with higher mass dimension, which is compensated by the appropriate factor of the large mass in the denominator,

$$\mathcal{L}_{\rm NLO} = \sum_j \omega_j \mathscr{O}_j\,, \quad \omega_j = \tfrac{1}{m_{\rm h}}\hat{\omega}_j\,, \quad [\mathscr{O}_j] = 5\,, \quad [\hat{\omega}_j] = 0\,, \tag{3.4}$$

where the parameters $\hat{\omega}_i$ are dimensionless. M. Neubert called these coefficients Wilson coefficients. The fields contained in the (multi-local) O are expanded in the same way as the action,

$$O_{\rm eff} = O_{\rm LO} + O_{\rm NLO} + \dots\,.$$

We now have to deal with interactions in eq. (3.4) which are not renormalizable (by power counting). However, we are only interested in the expansion $\Phi = \Phi_{\rm eff}^{\rm LO} + \Phi_{\rm eff}^{\rm NLO} + \dots$ of observables Φ in $m_{\rm h}^{-1}$. It is therefore sufficient to define the theory with the weight in the path integral expanded,

$$\mathrm{e}^{-S} \to \mathrm{e}^{-S^{\rm LO}}\{1 - S^{\rm NLO} + \dots\}\,, \tag{3.5}$$

in

$$\Phi = \langle O \rangle = \frac{\int_{\rm fields} \mathrm{e}^{-S} O}{\int_{\rm fields} \mathrm{e}^{-S}}\,, \tag{3.6}$$

At NLO accuracy the expansion is then given by

$$\begin{aligned} \Phi_{\rm eff}^{\rm LO} &= \langle O^{\rm LO} \rangle_{\rm LO} \\ \Phi_{\rm eff}^{\rm NLO} &= \langle O^{\rm NLO} \rangle_{\rm LO} - \Big(\langle O^{\rm LO} S^{\rm NLO} \rangle_{\rm LO} - \langle O^{\rm LO} \rangle_{\rm LO} \underbrace{\langle S^{\rm NLO} \rangle_{\rm LO}}_{\text{from } 1/Z} \Big) \end{aligned} \tag{3.7}$$

and

$$S^{\rm NLO} = \int \mathrm{d}^4 x\, \mathcal{L}_{\rm NLO}(x)\,.$$

The term $\Phi_{\rm eff}^{\rm NLO}$ is renormalizable with a finite number of counter terms which are equivalent to renormalizing the parameters ω_i (including the LO ones). Also parameters in the fields O are part of the list of ω_i.

Let us give a short reasoning why this is so. Consider O of eq. (3.3). Then

$$\varphi^{\rm LO}(x) \to \varphi_{\rm R}^{\rm LO}(x) \tag{3.8}$$

$$\varphi^{\rm NLO}(x) \to \varphi_{\rm R}^{\rm NLO}(x) = \sum_{\{\,i\,:\,[\varphi_i] \le [\varphi^{\rm NLO}]\,\}} Z_i^{\varphi}\, \varphi_i(x)\,, \qquad Z_i^{\varphi} = Z_i^{\varphi}(\Lambda_{\rm cut}, g_0) \tag{3.9}$$

renormalizes the first term. Assuming a single $\varphi^{\rm NLO}$ without mixing we can write for the second term

$$\langle O^{\rm LO} S^{\rm NLO} \rangle_{\rm LO} = \int \mathrm{d}^4 z\, \langle O^{\rm LO} \sum_j \omega_j \mathscr{O}_j(z) \rangle_{\rm LO} \tag{3.10}$$

$$\to \int \mathrm{d}^4 z\, \langle O^{\rm LO} \sum_{i,j} \omega_i' Z_{ij} \mathscr{O}_j(z) \rangle_{\rm LO} \tag{3.11}$$

$$+ \int \mathrm{d}^4 z\, \langle \sum_i w_i O_i \rangle_{\rm LO}\,,\,, \qquad Z_{ij} = Z_{ij}(\Lambda_{\rm cut}, g_0)\,,\; w_i = w_i(\Lambda_{\rm cut}, g_0)$$

where the last terms arise from contact terms $x \to z\,,\; y \to z$. The OPE (in the strict sense) dictates that the contact terms have this form.

The structure of this expression is such that one has to make all coefficients free functions of the cutoff and coupling g_0 in order to obtain a fully renormalized expression. Starting with a linear combination of all fields allowed by the symmetries in the action and in φ, this is all that is to do.

In other words,

$$\omega_i = \omega_i(\Lambda_{\rm cut}, g_0) \quad \to \quad \text{NLO EFT is finite} \tag{3.12}$$

$$i \in \text{terms in action} \cup \text{terms in fields } \varphi \tag{3.13}$$

The finite parts of the coefficients can be chosen to match to the fundamental theory up to $(\frac{1}{m_{\rm h}})^2$.

$$\omega_i = \omega_i(\Lambda_{\rm cut}, m_{\rm h}, g_0) \quad \to \quad \text{NLO EFT is QCD up to } \mathrm{O}((\tfrac{1}{m_{\rm h}})^2) \tag{3.14}$$

Renormalizability is particularly important for a non-perturbative evaluation of the path integral in a lattice formulation. The continuum limit of an effective theory only exists when we treat the higher dimensional interactions as insertions in correlation functions in the form of eq. (3.8). The continuum limit is then also expected to be universal, i.e. independent of the specific discretisation.

There are two important consequences of this discussion.

The first is that, given a renormalizable lowest order Lagrangian, it is irrelevant that higher order terms have mass dimension greater than d $(= 4)$. The only consequence is that their coefficients have to be determined, by either matching to experiment (phenomenological approach) or by matching to the fundamental theory. Whether coefficients are finite (have a limit as the cutoff is removed) or not is irrelevant.

The second is that the result of the predictions of the EFT are entirely universal in the following sense. They do not depend on the regularization, i.e. on the way the theory was discretized if we use a lattice.

3.2 EFTs with an intrinsic cutoff

There are cases where physics dictates that the lowest order Lagrangian has to contain non-renormalizable terms. One example, related to HQET, is the physics of quarkonia. Here NRQCD needs to be used as explained in (Mannel, 2017). The only difference between NRQCD and HQET is the following.

$$\text{HQET:} \;\; \mathcal{L}_{\rm LO} = \mathscr{L}_{\rm h}^{\rm stat}\,, \quad \mathcal{L}_{\rm NLO} = -\tfrac{1}{2m_{\rm h}}(\mathcal{O}_{\rm kin} + \mathcal{O}_{\rm spin}) \tag{3.15}$$

$$\text{NRQCD:} \;\; \mathcal{L}_{\rm LO} = \mathscr{L}_{\rm h}^{\rm stat} - \tfrac{1}{2m_{\rm h}}\mathcal{O}_{\rm kin}\,, \quad \mathcal{L}_{\rm NLO} = -\tfrac{1}{2m_{\rm h}}\mathcal{O}_{\rm spin}\,. \tag{3.16}$$

NRQCD is non-renormalizable. Removing the cutoff requires to add more and more (eventually infinitely many) interaction (or counter-) terms. Since this can't be done, one has to live with and discuss the dependence on the cutoff. In a lattice theory the cutoff is the inverse of the lattice spacing, $\Lambda_{\rm cut} = a^{-1}$, With a fixed number of terms in the Lagrangian one then has cutoff-effects a^k of both positive $k > 0$ (discretisation errors) and negative $k < 0$ (divergences).

A window,

$$E \ll a^{-1} \ll m_{\rm h}\,, \tag{3.17}$$

has to be present for both types of terms to be negligible.

Clearly this is a difficult situation. NRQCD on the lattice has been used frequently in the past, but less so nowadays.

4
Renormalization of HQET

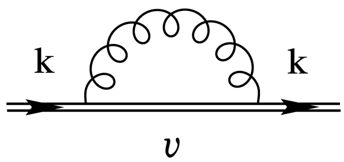


Fig. 4.1 One-loop self energy graph of a static quark. Graph from T. Mannel's lectures.

4.1 At leading order in $1/m_{\rm b}$

4.1.1 Action

Before coming to a general discussion, it is instructive to look at the simplest case of renormalisation and matching in perturbation theory. We start with the static effective theory. Consider the self-energy of the static quark in perturbation theory, namely the diagram Fig. 4.1. This diagram behaves like

$$\Sigma \sim g_0^2 \int \mathrm{d}^4 l \, \frac{1}{l^2(l_0 + k_0 + i\epsilon)} \sim \Lambda_{\rm cut} \sim \tfrac{1}{a} \,. \tag{4.1}$$

This divergence has to be compensated by a mass counterterm

$$\delta m \sim \frac{s_1 g_0^2 + s_2 g_0^4 + \dots}{a} \,, \tag{4.2}$$

which we just expect on the basis of dimensions anyway.

The Lagrangian therefore contains the counterterm and reads

$$\mathscr{L}_{\rm h}^{\rm stat} = \psi_{\bar{\rm h}}(m_{\rm bare} + D_0)\psi_{\rm h} \,, \quad m_{\rm bare} = \delta m + m_{\rm finite} \,, \quad \delta m \sim \Lambda_{\rm cut} \sim \tfrac{1}{a} \tag{4.3}$$

with a single parameter, $m_{\rm bare}$. $\psi_{\rm h}$ etc are the bare fields in the regularized path integral. We note that the split into a finite mass (different for different heavy flavours f if they are present) and the divergent piece δm is arbitrary.

More importantly, note that asymptotic freedom tells us that

$$a \sim \mathrm{e}^{-1/(2b_0 g_0^2)} \Lambda_{\mathrm{lat}}^{-1} \tag{4.4}$$

where Λ_{lat} is the Λ-parameter in the lattice minimal subtraction scheme, see e.g. (Weisz, 2011). Therefore

$$\delta m \overset{a\to 0}{\sim} [s_1 g_0^2 + s_2 g_0^4 + \ldots]\, \mathrm{e}^{1/(2b_0 g_0^2)}\, \Lambda_{\mathrm{lat}}\,. \tag{4.5}$$

and a finite number of terms in a series of g_0^2 is not sufficient to determine δm such that energies are finite, i.e. they have a continuum limit. The counterterm δm, or better immediately the full combination m_{bare} needs to be determined non-perturbatively.

However, the explicit form of the heavy quark propagator, eq. (5.13), shows that m_{bare} drops out of all observables (at LO) except for the relation between the QCD quark mass and one energy level in the static theory, say the mass of the B-meson. All energy differences and all properly normalized[1] matrix elements are independent of m_{bare}.

$$E_n(m_{\mathrm{bare}}) - E_m(m_{\mathrm{bare}}) = E_n(0) - E_m(0)\,, \tag{4.6}$$

$$\mathcal{M}_n^{\mathrm{bare}}(m_{\mathrm{bare}}) = \mathcal{M}_n^{\mathrm{bare}}(0)\,. \tag{4.7}$$

4.1.2 Composite fields

Interesting, non-trivial, renormalisation (and matching) happens for composite fields. We choose here the time-component of the axial current,

$$\text{QCD:} \qquad A_0^{\mathrm{R}}(x) = Z_{\mathrm{A}} \overline{\psi}_{\mathrm{u}}(x)\gamma_5\gamma_0\psi_{\mathrm{b}}(x)\,. \tag{4.8}$$

To distinguish it from the HQET field we label the heavy quark in QCD by b. The normalization factor is

$$Z_{\mathrm{A}} = 1 + Z_{\mathrm{A}}^{(1)} g_0^2 + \ldots \tag{4.9}$$

with a pure number (no renormalization scale dependence) $Z_{\mathrm{A}}^{(1)}$, which can be chosen (in any regularization) such that the chiral Ward identities hold. The matrix element,

$$\langle 0|\, \mathbb{A}_0^{\mathrm{R}}(0)|B(\mathbf{p}=0)\rangle = m_{\mathrm{B}}^{1/2} F_{\mathrm{B}}, \tag{4.10}$$

of the associated Hilbert space operator $\mathbb{A}_0^{\mathrm{R}}$ defines the decay constant F_{B}, the only hadronic parameter determining the decay rate $B \to \ell\nu$.

Now go to HQET at LO, the static theory. For the moment just write down the structure of the EFT expression at 1-loop order,

$$\begin{aligned} \mathcal{M}_{\mathrm{QCD}}(m_{\mathrm{b}}) = C^{\mathrm{Wils}}(m_{\mathrm{b}}) \left(\frac{2b_0\bar{g}^2(\mu)}{2b_0\bar{g}^2(m_{\mathrm{b}})}\right)^{-\gamma_0/(2b_0)} & \mathcal{M}_{\mathrm{stat}}(\mu) \\ \times \left(1 + \mathrm{O}(g^2) + \mathrm{O}(|\mathbf{p}|/m_{\mathrm{b}})\right)\,, \qquad & \mathcal{M} = \langle\alpha|\, \mathbb{A}_0^{\mathrm{R}}|\beta\rangle \end{aligned} \tag{4.11}$$

[1] A proper mass-independent non-relativistic normalization has to be chosen. The standard one is $\langle B(\mathbf{p}')|B(\mathbf{p})\rangle = 2(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')$.

for any matrix element $\mathcal{M}$ of the current. We rewrite this as (to be precise take $m_{\rm b} = \overline{m}_{\overline{\rm MS}}(m_{\rm b})$)

$$\mathcal{M}_{\rm QCD}(m_{\rm b}) = \underbrace{C^{\rm RGI}(m_{\rm b})}_{C^{\rm Wils}(m_{\rm b})/\varphi(\bar{g}(m_{\rm b}))} \quad \underbrace{\mathcal{M}_{\rm stat}^{\rm RGI}}_{\varphi(\bar{g}(\mu))\,\mathcal{M}_{\rm stat}(\mu)} \times (1 + {\rm O}(|\mathbf{p}|/m_{\rm b}))\,, \qquad \mathcal{M} = \langle\alpha|\,\mathbb{A}_0^{\rm R}|\beta\rangle \tag{4.12}$$

with

$$\varphi_{\rm stat}(\bar{g}) \;=\; \left[\,2b_0\bar{g}^2\,\right]^{-\gamma_0/2b_0} \underbrace{\exp\left\{-\int_0^{\bar{g}} {\rm d}x\left[\frac{\gamma(x)}{\beta(x)} - \frac{\gamma_0}{b_0 x}\right]\right\}}_{1+{\rm O}(\bar{g}^2)}, \tag{4.13}$$

Note that $\mathcal{M}_{\rm stat}^{\rm RGI}$ does *not* depend on μ or $m_{\rm b}$. Nor does it depend on a renormalisation scheme [show this as an exercise; hint: take μ large]. It is a pure number, a NP property of the EFT. There are no corrections to eq. (4.13); corrections appear when β, γ are approximated by perturbation theory at a certain order. [2]

There are methods to compute $\mathcal{M}_{\rm stat}^{\rm RGI}$ with negligible perturbative truncation error. They involve "step scaling strategies".

A complete (but here irrelevant) transition to RGIs is given by expressing $C^{\rm Wils}/\varphi$ as a function $C_{\rm PS}(M/\Lambda) = C^{\rm Wils}(m_{\rm b})/\varphi(\bar{g}(m_{\rm b}))$, with M the renormalization group invariant mass and Λ the Λ-parameter of QCD.

Mass scaling. The $\bar{g}(m_{\rm b})$-dependence is equivalently to the mass dependence and one may define another RG function $\gamma_{\rm match}$,[3]

$$\gamma_{\rm match}(\bar{g}) \equiv \frac{m_{\rm b}}{\mathcal{M}_{\rm QCD}}\frac{\partial\mathcal{M}_{\rm QCD}}{\partial m_{\rm b}} \overset{\bar{g}\to 0}{\sim} -\gamma_0\bar{g}^2 - \gamma_1^{\rm match}\bar{g}^4 + \ldots\,. \tag{4.14}$$

An interesting application is the asymptotics of the decay constant of a heavy-light pseudo-scalar (e.g. B):[4]

$$F_{\rm PS} \overset{M\to\infty}{\sim} \frac{[\ln(M/\Lambda)]^{\gamma_0/2b_0}}{\sqrt{m_{\rm PS}}}\mathcal{M}_{\rm stat}^{\rm RGI} \times [1 + {\rm O}([\ln(M/\Lambda)]^{-1})]\,, \tag{4.15}$$

which one obtains easily by inserting the leading order RG functions. Higher orders yield the indicated logarithmic corrections.

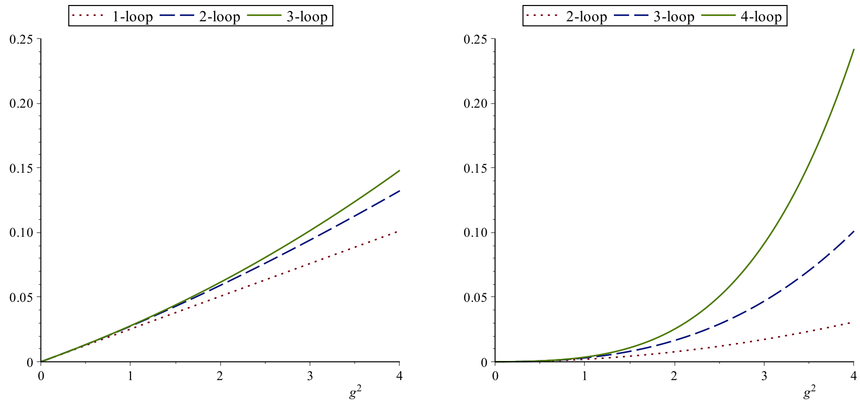


Fig. 4.2 The function $\gamma_{\rm match}(g)$ as a function of $g^2 = \bar{g}^2(m_{\rm b})$ for $N_{\rm f} = 3$ flavors in the $\overline{\rm MS}$ scheme. On the left we show $\gamma_{\rm match} = \gamma^{A_0}_{\rm match}$ for the time component of the axial current, on the right we show the difference $\gamma^{V_k}_{\rm match} - \gamma^{A_0}_{\rm match}$. Note that at 1-loop order the latter vanishes.

4.1.3 On the accuracy of perturbation theory

When one evaluates functions such as $C_{\rm PS}$ in a given order of perturbation theory, various quantities enter such as the β-function, the quark mass anomalous dimension. Apart from $\gamma_{\rm match}$, these all have a well behaved perturbative expansion in the $\overline{\rm MS}$ scheme as seen in the following table reproduced from appendix A.2.2 of (Sommer, 2011).

Keeping this in mind, we just discuss $\gamma_{\rm match}$. In the left graph in Fig. 4.2 we plot different orders of $\gamma_{\rm match}$. For the larger values $\bar{g}^2$ in the plot one may get worried about neglecting higher order terms. Note that $\bar{g}^2$ is around 2.5 for the b-quark and it is out of the range of the graph for the charm quark.

However, a more serious reason for concern derives from the right hand side graph. There the difference of the anomalous dimensions for V_k and A_0 is shown. For such differences perturbation theory is known to one loop higher (Bekavac *et al.*, 2010) and the perturbative coefficients do grow further. Asymptotic convergence seems to be useful only for rather small couplings or masses far above the b-quark mass. At the b-quark mass every known perturbative order contributes about an equal amount. Since we do not understand the reason for this behavior, it raises concern about

[2]When β is inserted in $n+1$-loop approximation and the other functions in n-loop approximation, the r.h.s. is correct up to $\mathrm{O}(\alpha(\mu)^n)$ uncertainties.

[3]Note that $\gamma_{\rm match}$ is just the anomalous dimension of the axial current $A_0^{\rm stat}$ in a special renormalization scheme. In this scheme, at scale $\mu = \overline{m}(m_{\rm b})$, its matrix elements are equal to the QCD ones up to order $\frac{1}{m_{\rm h}}$,

$$\mathcal{M}_{\rm QCD} = \mathcal{M}_{\rm match}(m_{\rm b}) + \mathrm{O}(\tfrac{1}{m_{\rm h}})\,.$$

We therefore refer to it as the matching scheme.

[4]Note the slow, logarithmic, decrease of the corrections in eq. (4.15). We will see below, in the discussion of Fig. 4.2, that the perturbative evaluation of $C_{\rm PS}(M_{\rm b}/\Lambda)$ is somewhat problematic.

coefficient	$i=1$	$i=2$	$i=3$	$i=4$
$(4\pi)^i b_{i-1}$	0.71620	0.40529	0.32445	0.47367
$(4\pi)^i d_{i-1}$	0.63662	0.76835	0.80114	0.90881
$(4\pi)^i\,\gamma_{{\rm stat},i-1}$	-0.31831	-0.26613	-0.25917	
$(4\pi)^i\,\gamma^{\gamma_0\gamma_5}_{{\rm match},i-1}$	-0.31831	-0.57010	-0.94645	
$(4\pi)^i\,\gamma^{\gamma_k}_{{\rm match},i-1}$	-0.31831	-0.87406	-3.12585	
$(4\pi)^i\,[\gamma^{\gamma_0\gamma_5}_{{\rm match},i-1} - \gamma^{\gamma_k}_{{\rm match},i-1}]$	0	0.30396	2.17939	14.803

Table 4.1 We list coefficients of β-function (b_i), anomalous dimension of the mass (d_i), anomalous dimension of the static-light current, (γ_i) and the mass scaling function $\gamma_i^{\rm match}$ for various currents. The first three refer to the $\overline{\rm MS}$ scheme and also $\gamma_i^{\rm match}$ are expansion coefficients of the $\overline{\rm MS}$-coupling, but the currents are defined such that they match QCD, i.e. are physical.
The normalisation (4π-factors) of the numbers is such that these are coefficients of α^i.

using perturbation theory for the matching functions. How does one estimate their uncertainty?

We emphasize, that the bad behavior is easily traced back to the function $C_{\rm match}$ and was noted in (Bekavac *et al.*, 2010). We tried earlier (Sommer, 2011) to rearrange the perturbative series in order to find a more stable perturbative prediction, but we did not succeed. Possibly the concept of 'renormalon subtraction' helps, but to our knowledge this has not been shown for the case at hand.

Summary. Let us summarize the most important facts about renormalization and matching at leading order in $1/m_{\rm b}$.

- The Lagrangian has a single parameter, the **quark mass**, which needs to be renormalized. It is **linearly divergent** and can therefore not be computed in PT. However, it only enters the relation $m_{\rm b} \leftrightarrow m_{\rm B}$. All energy differences are independent of $m_{\rm bare}$.
- Asymptotic freedom allows, in principle, to fully renormalize and match the electroweak currents in PT. One may split the renormalization and matching transparently into the definition of RGI operators in the EFT and a matching function. In practice, looking at three non-trivial orders, higher orders become smaller than the lower ones only when the quark mass is significantly above $m = 5\,\text{GeV}$. For HQET applied to the b-quark it seems necessary to do this step non-perturbatively.

4.2 At higher orders in $1/m_{\rm b}$

For quantitative phenomenological results one has to compute also $\frac{1}{m_{\rm h}}$ corrections in HQET. Is it consistent to match perturbatively as we discussed in the previous sections? We saw that the uncertainty due to a truncation of the perturbative matching expressions at n-loop order corresponds to a relative *error*

$$\frac{\delta_{n-\mathrm{loops}}(\mathcal{M}_{\mathrm{QCD}})}{\mathcal{M}_{\mathrm{QCD}}} = \frac{\delta_{n-\mathrm{loops}}(C_{\mathrm{PS}})}{C_{\mathrm{PS}}} \propto [\bar{g}^2(m_{\mathrm{b}})]^n \sim \left[\frac{1}{2b_0 \ln(m_{\mathrm{b}}/\Lambda)}\right]^n .$$

As m_{b} is made large, this perturbative error decreases only logarithmically. It becomes dominant over the power correction which one wants to include by pushing the HQET expansion to NLO,

$$\frac{\delta_{n-\mathrm{loops}}(\mathcal{M}_{\mathrm{QCD}})}{\mathcal{M}_{\mathrm{QCD}}} = \frac{\delta_{n-\mathrm{loops}}(C_{\mathrm{PS}})}{C_{\mathrm{PS}}} \overset{m_{\mathrm{b}}\gg\Lambda}{\gg} \frac{\Lambda}{m_{\mathrm{b}}} . \tag{4.16}$$

With a perturbative matching function, one does not perform a consistent NLO expansion such that errors decrease as $(1/m_{\mathrm{b}})^2$.

A practically even more serious issue is that at NLO one has to deal with the mixing of operators with lower dimensional ones. For example $\mathcal{O}_{\mathrm{kin}} = \overline{\psi}_{\mathrm{h}}\mathbf{D}^2\psi_{\mathrm{h}}$ mixes with $\overline{\psi}_{\mathrm{h}}D_0\psi_{\mathrm{h}}$ and $\overline{\psi}_{\mathrm{h}}\psi_{\mathrm{h}}$. In this situation mixing coefficients are power divergent $\sim a^{-n}$. In the example we have $n = 1, 2$. Subtracting power divergences in perturbation theory and then computing the matrix elements non-perturbatively always leaves a divergent remainder. Matrix elements of perturbatively subtracted operators do not have a non-perturbative continuum limit.

We are lead to conclude that it is necessary to perform matching and renormalization non-perturbatively. The only alternative is to supplement the theory by assumptions. Namely one may assume that at the lattice spacings available in practice, power divergences of the form $g_0^{2l}/(m_{\mathrm{b}}a)^n$ are small since $m_{\mathrm{b}}a > 1$. This then has to be combined with the assumption that the b-quark is not large enough to be in the asymptotic region of eq. (4.16).

5
The lattice formulation

We work on a hyper-cubic Euclidean lattice as specified before.

5.1 QCD on the lattice

Fermion fields $\psi(x), \overline{\psi}(x)$ are associated to the points of the lattice. However, to formulate gauge-covariant derivatives, one has to parallel-transport the fermion fields from one site to the other,

$$P(x \leftarrow x + a\hat{\mu}) = U(x,\mu) \quad = \quad \longleftarrow \quad \in \mathrm{SU}(3)\,, \tag{5.1}$$

$$P(x + a\hat{\mu} \leftarrow x) = U^\dagger(x,\mu) \quad = \quad \longrightarrow \quad \in \mathrm{SU}(3)\,. \tag{5.2}$$

The parallel transporters transform under gauge transformations, $\psi(x) \to \Omega(x)\psi(x)$, as

$$U(x,\mu) \;\to\; \Omega(x)U(x,\mu)\Omega(x+a\hat{\mu})^\dagger \tag{5.3}$$

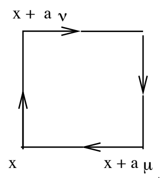
such that the (finite difference, backward) derivative

$$(\nabla^*_\mu \psi)(x) = \frac{1}{a}[\psi(x) - U^\dagger(x - a\hat{\mu},\mu)\psi(x - a\hat{\mu})] \tag{5.4}$$

is gauge covariant. The lattice Dirac operator is formulated in terms of these covariant derivatives.

The gauge action has to be local, gauge invariant, lattice (i.e. 90 degree) rotational invariant, formed in terms of the gauge fields U. The trace of the parallel-transporter around a plaquette (elementary square),

$$O_{\mu\nu}(x) = \ \mathrm{tr}\ U(x,\mu)U(x+a\hat{\mu},\nu)U^{-1}(x+a\hat{\nu},\mu)U^{-1}(x,\nu) =$$



$, \tag{5.5}$

is the most local object and summing over all x, μ, ν leads to the most natural action, the Wilson gauge action. Assuming the lattice gauge fields U to be constructed from a smooth continuum gauge field, the action reproduces the continuum up to effects of order a^2,

$$a^4 \sum_{\mu,\nu,x} (1 - \tfrac{1}{N_c} O_{\mu\nu}(x)) \sim \int \mathrm{d}^4 x \, \mathrm{tr}\, F_{\mu\nu}(x) F_{\mu\nu}(x) + \mathrm{O}(a^2)\,. \tag{5.6}$$

Since in the path integral gauge fields are not smooth, this is called the naive continuum limit of the action; it means we have the right classical theory.

Unfortunately I have no time to explain more of lattice QCD.

5.2 HQET on the lattice

5.2.1 Static Lagrangian

We simply use

$$D_0\, \psi_\mathrm{h}(x) = \nabla_0^* \psi_\mathrm{h}(x) \tag{5.7}$$

and, for later convenience, insert a specific normalization factor, defining the static lattice Lagrangian

$$\mathcal{L}_\mathrm{h} = \frac{1}{1 + a\delta m} \overline{\psi}_\mathrm{h}(x) [\nabla_0^* + \delta m] \psi_\mathrm{h}(x)\,. \tag{5.8}$$

The following points are worth noting, at least for the students working in lattice FT.

- There are no doubler modes (it is not easy to write down a Dirac operator on the lattice with chiral symmetry, describing a single fermion. However, here there is no chiral symmetry).
- There is a positive hermitian transfer matrix and thus a hermitian Hamiltonian.
- The lattice action preserves all the continuum heavy quark symmetries discussed in the continuum HQET section. Those formulae hold without a change.

5.2.2 Propagator

From the Lagrangian eq. (5.8) we have the defining equation for the propagator

$$\frac{1}{1 + a\,\delta m} (\nabla_0^* + \delta m) G_\mathrm{h}(x,y) = \delta(x-y) P_+ \equiv a^{-4} \prod_\mu \delta_{\frac{x_\mu}{a} \frac{y_\mu}{a}} P_+\,. \tag{5.9}$$

Obviously $G_\mathrm{h}(x,y)$ is proportional to $\delta(\mathbf{x}-\mathbf{y})$. Writing $G_\mathrm{h}(x,y) = g(n_0,k_0;\mathbf{x})\delta(\mathbf{x}-\mathbf{y})P_+$ with $x_0 = an_0\,,\ y_0 = ak_0$, the above equation yields a simple recursion for $g(n_0+1,k_0;\mathbf{x})$ in terms of $g(n_0,k_0;\mathbf{x})$ which is solved by

$$g(n_0,k_0;\mathbf{x}) = \theta(n_0-k_0)(1+a\delta m)^{-(n_0-k_0)} \mathcal{P}(x \leftarrow y)\,, \tag{5.10}$$

$$\mathcal{P}(x,x) = 1\,, \quad \mathcal{P}(x \leftarrow y + a\hat{0}) = \mathcal{P}(x \leftarrow y)\, U(y,0)\,, \tag{5.11}$$

where

$$\theta(n_0-k_0) = \begin{cases} 0 & n_0 < k_0 \\ 1 & n_0 \geq k_0\,. \end{cases} \tag{5.12}$$

The static propagator then reads

$$G_{\rm h}(x,y) = \theta(x_0-y_0)\, \delta(\mathbf{x}-\mathbf{y})\; \exp\big(-\widehat{\delta m}\,(x_0-y_0)\big)\, \mathcal{P}(x\leftarrow y)\, P_+\,, \tag{5.13}$$

$$\widehat{\delta m} = \tfrac{1}{a}\ln(1+a\delta m)\,. \tag{5.14}$$

$\mathcal{P}(x,y)$ is just the lattice parallel transporter. Note that the derivation fixes $\theta(0)=1$ for the lattice θ-function. As in the continuum, the mass counter term δm just yields an energy shift; now, on the lattice, the shift is

$$E^{\rm QCD}_{\rm h/\bar{h}} = E^{\rm stat}_{\rm h/\bar{h}}\Big|_{\delta m=0} + m_{\rm bare}\,, \quad m_{\rm bare} = \widehat{\delta m} + m\,. \tag{5.15}$$

It is valid for all energies of states with a single heavy quark or anti-quark. As in the continuum the split between δm and the finite m is convention dependent.

5.2.3 Symmetries

All HQET symmetries are preserved on the lattice. The symmetry transformations can literally be carried over from the continuum, e.g. the local flavor number eq. (2.24). One just replaces the continuum fields by the lattice ones.

Note that these HQET symmetries are defined in terms of transformations of the heavy quark fields while the light quark fields and gauge fields do not change (unlike e.g. standard parity). Integrating out just the quark fields in the path integral while leaving the integral over the gauge fields, they thus yield identities for the integrand or one may say for "correlation functions in any fixed gauge background field".

5.2.4 Symanzik EFT

According to the — by now well tested — Symanzik conjecture, the cutoff (= discretisation) effects of a lattice theory can be described in terms of an effective *continuum* theory. (Symanzik, 1983*a*, Symanzik, 1983*b*, Lüscher *et al.*, 1996). Once the terms in Symanzik's effective Lagrangian are known, the cutoff effects can be canceled by adding terms of the same form to the lattice action, resulting in an improved action.

For a static quark, Symanzik's effective action is (Kurth and Sommer, 2001)

$$S_{\rm eff} = S_0 + aS_1 + \ldots\,, \quad S_i = \int {\rm d}^4x\, \mathcal{L}_{\rm i}(x) \tag{5.16}$$

where $\mathcal{L}_0(x) = \mathcal{L}^{\rm stat}_{\rm h}(x)$ is the continuum static Lagrangian of eq. (**??**) and

$$\mathcal{L}_1(x) = \sum_{i=3}^{5} c_i\, \mathcal{O}_{\rm i}(x)\,, \tag{5.17}$$

is given in terms of local fields with mass dimension $[\mathcal{O}_{\rm i}(x)] = 5$. Their coefficients c_i are functions of the bare gauge coupling. Assuming for simplicity mass-degenerate light quarks with a mass $m_{\rm l}$, the set of possible dimension five fields, which share the symmetries of the lattice theory, is

$$\mathcal{O}_3 = \overline{\psi}_{\rm h} D_0 D_0 \psi_{\rm h}\,, \quad \mathcal{O}_4 = m_{\rm l}\, \overline{\psi}_{\rm h} D_0 \psi_{\rm h}\,, \quad \mathcal{O}_5 = m_{\rm l}^2\, \overline{\psi}_{\rm h} \psi_{\rm h}\,. \tag{5.18}$$

Note that $P_+ \sigma_{0j} P_+ = 0$ means there is no term $\overline{\psi}_{\rm h} \sigma_{0j} F_{0j} \psi_{\rm h}$, and $\overline{\psi}_{\rm h} D_j D_j \psi_{\rm h}$ can't occur because it violates the local phase invariance eq. (2.24). Finally $\overline{\psi}_{\rm h} \sigma_{jk} F_{jk} \psi_{\rm h}$ is not invariant under the spin rotations eq. (2.23).

Furthermore, we are only interested in on-shell correlation functions and energies. For this class of observables $\mathcal{O}_3$, $\mathcal{O}_4$ do not contribute (Lüscher and Weisz, 1985, Lüscher *et al.*, 1996) because they vanish by the equation of motion,

$$D_0 \psi_{\rm h} = 0\,. \tag{5.19}$$

The only remaining term, $\mathcal{O}_5$, induces a redefinition of the mass counter-term δm which therefore depends explicitly on the light quark mass.

We note that for almost all applications, δm is explicitly canceled in the relation between physical observables and one thus has automatic on-shell $\mathrm{O}(a)$ improvement for the static action. No parameter has to be tuned to guarantee this property. Still, the improvement of matrix elements and correlation functions requires to also consider composite fields in the effective theory. For the axial current one finds one $\mathrm{O}(a)$ operator. For a detailed description we refer to (Sommer, 2011).

5.2.5 $\frac{1}{m_{\rm h}}$ terms.

Valid discretisations for these terms are easily written down:

$$\mathcal{O}_{\rm kin}(x) = \psi_{\bar{\rm h}}(x)\, \nabla_k^{*} \nabla_k\, \psi_{\rm h}(x)\,, \tag{5.20}$$

$$\mathcal{O}_{\rm spin}(x) = \psi_{\bar{\rm h}}(x)\, \boldsymbol{\sigma} \cdot \mathbf{B}(x)\, \psi_{\rm h}(x)\,. \tag{5.21}$$

6

Non-perturbative HQET

6.1 $1/m_\mathrm{b}$-expansion of correlation functions, masses, and matrix elements

Let us consider just one composite field (only a single $\frac{1}{m_\mathrm{h}}$-term is needed when we consider the field at $\mathbf{p}=0$)

$$Z_\mathrm{A}^\mathrm{HQET}\,(\,A_0^\mathrm{stat}+\omega_\mathrm{A}A_0^{(1)}\,)\,,\quad A_0^{(1)}=\overline{\psi}_\mathrm{l}\overleftarrow{D}_j\gamma_j\gamma_5\psi_\mathrm{h}\,. \tag{6.1}$$

For now we assume that the coefficients

$$\begin{aligned} \mathrm{O}(1)\;:\;& m_\mathrm{bare}\,,\;Z_\mathrm{A}^\mathrm{HQET}\,,\\ \mathrm{O}(1/m_\mathrm{b})\;:\;& \omega_\mathrm{kin}\,,\;\omega_\mathrm{spin}\,,\;\omega_\mathrm{A}\,, \end{aligned} \tag{6.2}$$

are known as a function of the bare coupling g_0 and the quark mass m. Their non-perturbative determination will be discussed later.

The rules of the $1/m_\mathrm{b}$-expansion are illustrated on the example

$$C_\mathrm{AA,R}^\mathrm{QCD}(x_0)=a^3\sum_\mathbf{x}\left\langle A_0^\mathrm{R}(x)(A_0^\mathrm{R})^\dagger(0)\right\rangle \tag{6.3}$$

One uses eq. (6.1) and expands the expectation value consistently in $1/m_\mathrm{b}$, counting powers of $1/m_\mathrm{b}$ as in eq. (6.2). At order $1/m_\mathrm{b}$, terms proportional to $\omega_\mathrm{kin}\times\omega_\mathrm{A}$ etc. are to be dropped. [They have to be dropped! Q: why?] As a last step, we have to take the energy shift between HQET and QCD into account. Therefore correlation functions with a time separation x_0 obtain an extra factor $\exp(-x_0\,m)$, where the scheme dependence of m is compensated by a corresponding one in δm.

Dropping all terms $\mathrm{O}(1/m_\mathrm{b}^2)$ without further notice, one arrives at the expansion

$$\begin{aligned} C_\mathrm{AA}^\mathrm{QCD}(x_0) &= \mathrm{e}^{-mx_0}(Z_\mathrm{A}^\mathrm{HQET})^2\Big[C_\mathrm{AA}^\mathrm{stat}(x_0)+\omega_\mathrm{A}\,C_{\delta\mathrm{AA}}^\mathrm{stat}(x_0) && (6.4)\\ &\qquad +\omega_\mathrm{kin}\,C_\mathrm{AA}^\mathrm{kin}(x_0)+\omega_\mathrm{spin}\,C_\mathrm{AA}^\mathrm{spin}(x_0)\Big]\\ &\equiv \mathrm{e}^{-mx_0}(Z_\mathrm{A}^\mathrm{HQET})^2\,C_\mathrm{AA}^\mathrm{stat}(x_0)\Big[1+\omega_\mathrm{A}\,R_{\delta A}^\mathrm{stat}(x_0) && (6.5)\\ &\qquad +\omega_\mathrm{kin}\,R_\mathrm{AA}^\mathrm{kin}(x_0)+\omega_\mathrm{spin}\,R_\mathrm{AA}^\mathrm{spin}(x_0)\Big] \end{aligned}$$

with

$$
\begin{aligned}
C^{\rm stat}_{\delta{\rm AA}}(x_0) &= a^3\sum_{\mathbf{x}}\langle A_0^{\rm stat}(x)(A_0^{(1)}(0))^\dagger\rangle_{\rm stat} \;+\; a^3\sum_{\mathbf{x}}\langle A_0^{(1)}(x)(A_0^{\rm stat}(0))^\dagger\rangle_{\rm stat}\,,\\
C^{\rm kin}_{\rm AA}(x_0) &= a^3\sum_{\mathbf{x}}\langle A_0^{\rm stat}(x)(A_0^{\rm stat}(0))^\dagger a^4\sum_z \mathcal{O}_{\rm kin}(z)\rangle\\
C^{\rm spin}_{\rm AA}(x_0) &= a^3\sum_{\mathbf{x}}\langle A_0^{\rm stat}(x)(A_0^{\rm stat}(0))^\dagger a^4\sum_z \mathcal{O}_{\rm spin}(z)\rangle\,.
\end{aligned}
$$

It is now a straight forward exercise to obtain the expansion of the B-meson mass.

$$
\begin{aligned}
m_{\rm B} &= -\lim_{x_0\to\infty}\widetilde{\partial}_0 \ln C^{\rm QCD}_{\rm AA}(x_0) && (6.6)\\
&= m_{\rm bare} + E^{\rm stat} + \omega_{\rm kin}E^{\rm kin} + \omega_{\rm spin}E^{\rm spin} + \omega_{\rm A}\times 0\,, && (6.7)\\
E^{\rm stat} &= -\lim_{x_0\to\infty}\widetilde{\partial}_0\,\ln C^{\rm stat}_{\rm AA}(x_0)\Big|_{\delta m=0}\,, \qquad [E^{\rm stat}] = 1\,, && (6.8)\\
E^{\rm kin} &= -\lim_{x_0\to\infty}\widetilde{\partial}_0\,R^{\rm kin}_{\rm AA}(x_0)\,,\quad E^{\rm spin} = -\lim_{x_0\to\infty}\widetilde{\partial}_0\,R^{\rm spin}_{\rm AA}(x_0)\,,\quad [E^{\rm kin}] = [E^{\rm spin}] = 2\,.
\end{aligned}
$$

Again we have made the dependence on δm explicit through $m_{\rm bare} = m_{\rm b} + \widehat{\delta m}$ and then all quantities are defined in the theory with $\delta m = 0$. Note that the ratios $R^x_{\rm AA}$ (and therefore $E^{\rm kin}, E^{\rm spin}$) do not depend on δm; the quantities $E^{\rm kin}, E^{\rm spin}$ have mass dimension two and we have already anticipated eq. (6.15). These equations tell us also that

$$
C^{\rm kin}_{\rm AA}(x_0) \overset{\Lambda_{\rm QCD}x_0\gg 1}{\sim} \text{constant}\times x_0\,{\rm e}^{-E^{\rm stat}x_0} \qquad (6.9)
$$

just like $C^{\rm spin}_{\rm AA}$.

The expansion for the decay constant is

$$
\begin{aligned}
F_{\rm B}\sqrt{m_{\rm B}} &= \lim_{x_0\to\infty}\left\{2\exp(m_{\rm B}x_0)\,C^{\rm QCD}_{\rm AA}(x_0)\right\}^{1/2} && (6.10)\\
&= Z^{\rm HQET}_{\rm A}\,\Phi^{\rm stat}\left\{1+\rho\right\} && (6.11)\\
\Phi^{\rm stat} &= \lim_{x_0\to\infty}\left\{2\exp(E^{\rm stat}x_0)\,C^{\rm stat}_{\rm AA}(x_0)\right\}^{1/2}\\
\rho &= \tfrac{1}{2}\lim_{x_0\to\infty}\Big[\,\omega_{\rm kin}\,(x_0E^{\rm kin} + R^{\rm kin}_{\rm AA}(x_0))\\
&\qquad + \omega_{\rm spin}(x_0E^{\rm spin} + R^{\rm spin}_{\rm AA}(x_0))\\
&\qquad + \omega_{\rm A}R^{\rm stat}_{\delta A}(x_0)\,\Big]\,. && (6.12)
\end{aligned}
$$

Inserting $\mathbb{1} = \sum_n |B,n\rangle\langle B,n|$, with finite volume normalization $\langle B,n|B,n\rangle = 2L^3$, and n labeling the eigenstates of the Hamiltonian, one further observes that (do it as an exercise)

$$
\begin{aligned}
E^{\rm kin} &= -\frac{1}{2L^3}\langle B|a^3\sum_{\mathbf{z}}\mathcal{O}_{\rm kin}(0,\mathbf{z})|B\rangle_{\rm stat} = -\frac{1}{2}\langle B|\mathcal{O}_{\rm kin}(0)|B\rangle_{\rm stat} && (6.13)\\
E^{\rm spin} &= -\frac{1}{2}\langle B|\mathcal{O}_{\rm spin}(0)|B\rangle_{\rm stat}\,, && (6.14)\\
0 &= \lim_{x_0\to\infty}\widetilde{\partial}_0 R^{\rm stat}_{\delta A}(x_0)\,, && (6.15)
\end{aligned}
$$

with $|B\rangle \equiv |B,0\rangle$. As expected, only the parameters of the action are relevant in the expansion of hadron masses. Note that in lattice regularization there is a so-called transfer matrix, which allows to define the Hamiltonian and the Hilbert space rigorously and thus $\mathbb{1} = \sum_n |B,n\rangle\langle B,n|$ is rigorous.

A correct split of the terms in eq. (6.7) and eq. (6.12) into leading order and next to leading order pieces which are separately renormalized and which hence *separately have a continuum limit* requires more thought on the renormalization of the $1/m_\mathrm{b}$-expansion. We turn to this soon.

First, let us discuss how the HQET-parameters can be determined such that the effective theory yields the $1/m_\mathrm{b}$ expansion of the QCD observables.

6.2 Strategy for non-perturbative matching

How can one non-perturbatively match HQET to QCD. Consider the action as well as A_0 (just at $\mathbf{p} = 0$) and denote the free parameters of the effective theory by $\omega_i\,,\ i = 1 \ldots N_\mathrm{HQET}$.

In static approximation we have the parameter vector

$$\omega^\mathrm{stat} = (\, m_\mathrm{bare}^\mathrm{stat}\,,\ [\ln(Z_\mathrm{A})]^\mathrm{stat}\,)^t\,, \quad N_\mathrm{HQET} = 2 \tag{6.16}$$

and including the first order terms in $1/m$ together with the static ones, the HQET parameters are

$$\omega^\mathrm{HQET} = (\, m_\mathrm{bare}\,,\ \ln(Z_\mathrm{A}^\mathrm{HQET})\,,\, \omega_\mathrm{A}\,,\, \omega_\mathrm{kin}\,,\, \omega_\mathrm{spin}\,)^t \quad N_\mathrm{HQET} = 5\,. \tag{6.17}$$

The pure $1/m_\mathrm{b}$ parameters may be defined as $\omega^{(1/m)} = \omega^\mathrm{HQET} - \omega^\mathrm{stat}$, with all of them, e.g. also $m_\mathrm{bare}^{(1/m)}$, non-zero. In fact our discussion of renormalization of the $1/m_\mathrm{b}$ terms shows that $m_\mathrm{bare}^{(1/m)}$ diverges as $1/(a^2 m)$.

With suitable observables, (M_b = RGI mass of the heavy quark)

$$\Phi_i(L, M_\mathrm{b}, a)\,,\ i = 1 \ldots N_\mathrm{HQET}\,,$$

in a finite volume with $L = T = L_1 \approx 0.5\,\mathrm{fm}$, we require matching[1]

$$\Phi_i(L, M_\mathrm{b}, a) = \Phi_i^\mathrm{QCD}(L_1, M_\mathrm{b}, 0)\,,\ i = 1 \ldots N_\mathrm{HQET}\,. \tag{6.18}$$

Note that the continuum limit is taken in QCD, while in HQET we want to extract the bare parameters of the theory from the matching equation and thus have a finite value of a. It is convenient to pick observables with HQET expansions linear in ω_i, e.g. $\Phi = \log(C_\mathrm{AA})$,

$$\Phi(L, M_\mathrm{b}, a) = \eta(L,a) + \underbrace{\phi(L,a)}_{N_\mathrm{HQET}\times N_\mathrm{HQET}} \times\, \omega(M_\mathrm{b}, a)\,, \tag{6.19}$$

in terms of a $N_\mathrm{HQET} \times N_\mathrm{HQET}$ coefficient matrix ϕ. A natural choice for the first two observables is to choose them as finite volume observables which converge (exponentially) to interesting physical observables that one wants to predict,

[1] Recall that observables without a superscript refer to HQET.

$$\Phi_1 \equiv L m_{\rm B}(L) \overset{L\to\infty}{\sim} L m_{\rm B} \tag{6.20}$$

$$\Phi_2 \equiv \ln(\langle 0; L|A_0^{\rm R}|B, L\rangle) = \log(L^{3/2} F_{\rm B}(L)\sqrt{m_{\rm B}(L)/2})$$
$$\overset{L\to\infty}{\sim} \log(L^{3/2} F_{\rm B}\sqrt{m_{\rm B}/2})\,. \tag{6.21}$$

In static approximation these determine directly ω_1 and ω_2. We will not discuss the other Φ_i, here. The explicit form of η, ϕ is

$$\eta = \begin{pmatrix} \Gamma^{\rm stat} \\ \zeta_{\rm A} \\ \ldots \end{pmatrix}, \quad \phi = \begin{pmatrix} L & 0 \ldots \\ 0 & 1 \ldots \\ \ldots & \end{pmatrix} \tag{6.22}$$

with

$$\Gamma^{\rm stat} = -L\, m_{\rm B}^{\rm stat}(L)\big|_{m_{\rm bare}=0}\,, \quad \zeta_{\rm A} = \ln\left(L^{3/2} F_{\rm B}(L)\sqrt{m_{\rm B}(L)/2}\right)^{\rm stat}_{m_{\rm bare}=0}\,. \tag{6.23}$$

In static approximation, the structure of the matrix ϕ is perfect: one observable determines one parameter. This is possible since there is no (non-trivial) mixing at that order.

Having specified the matching conditions, the HQET parameters $\omega_i(M_{\rm b}, a)$ can be obtained from eqs.(6.18,6.19), but only for rather small lattice spacings since a reasonable suppression of lattice artifacts requires $L/a \geq 10$ and thus $a \leq 0.05\,{\rm fm}$ (Think of L as $L \sim 1/|\mathbf{p}|$).

Larger lattice spacings as needed in large volume, can be reached by adding a step scaling strategy, illustrated in Fig. 6.1. Let us now go through the various steps of this strategy. The initial value of L is set to $L = L_1$.

(1) Take the continuum limit

$$S1:\ \Phi_i^{\rm QCD}(L_1, M_{\rm b}, 0) = \lim_{a/L_1\to 0} \Phi_i^{\rm QCD}(L_1, M_{\rm b}, a)\,. \tag{6.24}$$

This requires $L_1/a = 20 \ldots 40$, or $a = 0.025\,{\rm fm} \ldots 0.012\,{\rm fm}$.

(2a) Set the HQET observables equal to the QCD ones, eq. (6.18) and extract the parameters

$$S2:\ \tilde\omega(M_{\rm b}, a) \equiv \phi^{-1}(L_1, a)\,[\Phi(L_1, M_{\rm b}, 0) - \eta(L_1, a)] \tag{6.25}$$

The only restriction here is $L_1/a \gg 1$, so one can use $L_1/a = 10 \ldots 20$, which means $a = 0.05\,{\rm fm} \ldots 0.025\,{\rm fm}$.

(2b.) Choose, e.g., $L_2 = 2L_1$ and insert $\tilde\omega$ into $\Phi(L_2, M_{\rm b}, a)$:

$$S3:\ \Phi(L_2, M_{\rm b}, 0) = \lim_{a/L_2\to 0} \{\eta(L_2, a) + \phi(L_2, a)\,\tilde\omega(M_{\rm b}, a)\} \tag{6.26}$$

(3.) Repeat (2a.) for $L_1 \to L_2$:

$$S4:\ \omega(M_{\rm b}, a) \equiv \phi^{-1}(L_2, a)\,[\Phi(L_2, M_{\rm b}, 0) - \eta(L_2, a)]\,. \tag{6.27}$$

With the same resolutions $L_2/a = 10 \ldots 20$ one has now reached $a = 0.1\,{\rm fm} \ldots 0.05\,{\rm fm}$.

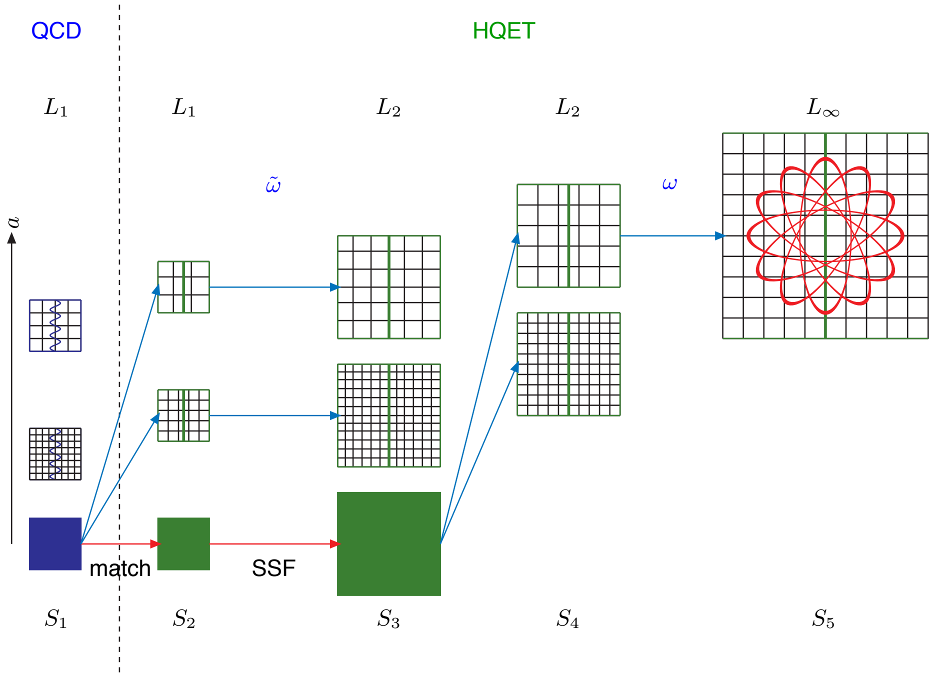


Fig. 6.1 Strategy for non-perturbative HQET (Blossier *et al.*, 2010). Note that in the realistic implementation(Blossier *et al.*, 2010) finer resolutions are used.

(4.) Finally insert ω into the expansion of large volume observables, e.g.

$$S5: \; m_{\mathrm{B}} = \lim_{a\to 0}[\omega_1(M_{\mathrm{b}}, a) + E^{\mathrm{stat}}(a)]\,. \tag{6.28}$$

In the chosen example the result is the relation between the RGI b-quark mass and the B-meson mass m_{B}. It is illustrative to put the different steps into one equation,

$$\begin{aligned} m_{\mathrm{B}} = & \\ & \lim_{a\to 0}[E^{\mathrm{stat}}(a) - \Gamma^{\mathrm{stat}}(L_2, a)] && a = 0.1\mathrm{fm} \ldots 0.05\mathrm{fm} && [S_4, S_5] \\ & + \lim_{a\to 0}[\Gamma^{\mathrm{stat}}(L_2, a) - \Gamma^{\mathrm{stat}}(L_1, a)] && a = 0.05\mathrm{fm} \ldots 0.025\mathrm{fm} && [S_2, S_3] \\ & + \frac{1}{L_1}\lim_{a\to 0}\Phi_1(L_1, M_{\mathrm{b}}, a) && a = 0.025\,\mathrm{fm} \ldots 0.012\,\mathrm{fm} && [S_1]\,. \end{aligned} \tag{6.29}$$

We have indicated the lattices drawn in Fig. 6.1 and the typical lattice spacings of these lattices. The explicit expression for the decay constant in static approximation is even more simple; write it down as an exercise!

Including $\frac{1}{m_{\mathrm{h}}}$ terms and also other components of the weak flavour currents works in the same way. A relevant point is that the matching conditions can be chosen to have some block structure; *not* everything depends on everything.

7
Messages

Here are the main results of our discussion and one point (dependence on matching conditions) a little bit elaborated

- ☺ An effective theory such as HQET can be implemented non-perturbatively. The result is universal in the following sense: it does not depend on the (lattice) regularisation.
- ☺ Results
  exist so far only for $N_\mathrm{f} = 0, 2$ (light quarks) QCD. The pattern found is that (with the matching as described)

  NLO = O(10%) LO

  No significant deviations of NLO to results using other methods are known. Unfortunately, the set of observables that have been evaluated is limited so far.
  LO + NLO: B leptonic decays and mass of b-quark. (Sommer, 2015)
  LO: $\mathrm{B_s}$ semi-leptonic decay at fixed, large, q^2 (Bahr *et al.*, 2016)

  Therefore all indications are that NLO HQET has a precision of around 1% for b-quarks (with momenta $\sim$ 500MeV).
- But there are intrinsic limitations in expansions such as the $1/m_\mathrm{b}$-expansion in QFTs. These have nothing to do with the lattice regularisation.
  Results at each order in the expansion are ambiguous by terms of the size of the next order. Ambiguous means that they depend on the matching condition imposed. As an example consider the often written formal HQET-expansion

$$m_\mathrm{B}^\mathrm{av} \equiv \frac{1}{4}[m_\mathrm{B} + 3m_{\mathrm{B}^*}] \tag{7.1}$$

$$= m_\mathrm{b} + \bar{\Lambda} + \frac{1}{2m_\mathrm{b}}\lambda_1 + \mathrm{O}(1/m_\mathrm{b}^2) \tag{7.2}$$

  with (ignoring renormalization)

$$\lambda_1 = \langle B|\mathcal{O}_\mathrm{kin}|B\rangle\,. \tag{7.3}$$

  The quantity $\bar{\Lambda}$ is referred to as "static binding energy" and λ_1 as the kinetic energy of the b-quark inside the B-meson. Depending on how we choose the matching conditions, we change λ_1 by a term of order Λ_QCD^2. In fact we could set

$$\begin{aligned}
\Phi_1 &= m_{\mathrm{B}}^{\mathrm{av}}(L) \text{ and take } L \text{ large} && (7.4)\\
\text{then} & && (7.5)\\
\text{LO:} &\quad m_{\mathrm{b}} + \bar{\Lambda} = m_{\mathrm{B}}^{\mathrm{av}} && (7.6)\\
\text{NLO:} &\quad m_{\mathrm{b}} + \bar{\Lambda} + \frac{1}{2m_{\mathrm{b}}}\lambda_1 = m_{\mathrm{B}}^{\mathrm{av}} && (7.7)\\
&\Rightarrow \lambda_1 = 0\,. && (7.8)
\end{aligned}$$

If we take L of order 0.5 fm and parametrize $m_{\mathrm{B}}^{\mathrm{av}}$ like eq. (7.2) we will have $\lambda_1 = c\Lambda_{\mathrm{QCD}}^2 \neq 0$. As a remark, this is similar to the case of the gluon condensate. A convenient non-perturbative definition is to set it to zero.

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