Next we consider soft-collinear interaction terms in S_{eff}. The general construction* is somewhat involved, but since we only need the leading-power terms, it is quite simple.

* ψ_s is suppressed, compared to ξ_c.
  → No soft quarks arise in leading power interactions
* $A_{\perp s}$ and $\bar{n}\cdot A_s$ are suppressed: only the component $n\cdot A_s$ can arise.

The interactions can be obtained by substituting

$$A_c^\mu \longrightarrow A_c^\mu + n\cdot A_s \frac{\bar{n}^\mu}{2}$$

in S_c.

The final step in the construction is to perform a derivative expansion, i.e. expand in small momentum components.

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* see hep-ph/0211358

Consider the term

$$\int d^4x\ \bar{\xi}(x)\ n\cdot A_s(x)\,\frac{\bar{n}\!\!\!/}{2}\,\xi(x)$$

The product of a soft and collinear field scales like a collinear field since

$$\begin{array}{llll} & n\cdot p & \bar{n}\cdot p & p_\perp \\ p_c^\mu + p_s^\mu \sim p_c^\mu \sim (& \lambda^2, & 1, & \lambda) \\ \to \quad x^\mu \sim (& 1, & 1/\lambda^2, & 1/\lambda) \\ p_s^\mu \sim (& \lambda^2, & \lambda^2, & \lambda^2) \end{array}$$

So $$p_s\cdot x = \underbrace{\tfrac{1}{2}\, n\cdot p_s\ \bar{n}\cdot x}_{O(1)} + \underbrace{\tfrac{1}{2}\,\bar{n}\cdot p_s\ n\cdot x}_{O(\lambda^2)} + \underbrace{p_\perp x_\perp}_{O(\lambda)}$$

$$= 2p_{s+}\cdot x_- + 2p_{s-}\cdot x_+ + p_\perp\cdot x_\perp$$

$$\int d^4x\ \bar{\xi}(x)\frac{\bar{n}\!\!\!/}{2}\xi(x)\ n\cdot A_s(x)$$

$$= \int d^4x\ \bar{\xi}(x)\frac{\bar{n}\!\!\!/}{2}\xi(x)\Big[1 + \underbrace{x_\perp\cdot\partial_\perp}_{O(\lambda)} + \underbrace{x_+\cdot\partial_{x_+}}_{O(\lambda^2)} + \dots\Big]\, nA(x)\Big|_{x=x_-}$$

$$= \int d^4x\ \bar{\xi}(x)\frac{\bar{n}\!\!\!/}{2}\xi(x)\ n\cdot A(x_-) + O(\lambda)$$

So we arrive at the final form of the effective Lagrangian for s+c field and their leading power interactions:

$$\mathcal{L}_{SCET} = \bar{\psi}_s i\not{D}_s \psi_s + \bar{\xi}\left[\frac{\not{n}}{2} in\cdot D + i\not{D}_{c\perp} \frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp}\right]\xi$$

$$- \frac{1}{4}\left(F^{s,a}_{\mu\nu}\right)^2 - \frac{1}{4}\left(F^{c,a}_{\mu\nu}\right)^2$$

Where:

$$iD^s_\mu = i\partial_\mu + g A^s_\mu$$

$$iD^c_\mu = i\partial_\mu + g A^c_\mu$$

$$in\cdot D = in\cdot\partial + g\, nA_c + g\, n\cdot A_s(x_-)$$

and $D^\mu = n\cdot D \frac{\bar{n}^\mu}{2} + \bar{n}\cdot D_c \frac{n^\mu}{2} + D^\mu_{c\perp}$

$$ig F^c_{\mu\nu} = [iD_\mu, iD_\nu]$$

$$ig F^s_{\mu\nu} = [iD^s_\mu, iD^s_\nu]$$

To finish our discussion of the Lagrangian, let us briefly discuss gauge transformations. Since we split the fields into different components, we can also consider two types of transformations

soft: $V_s(x) = \exp[\, i\, \alpha_s^a(x) t^a \,]$

collinear: $V_c(x) = \exp[\, i\, \alpha_c^a(x)\, t^a \,]$

where $\partial_\mu \alpha_s \sim \lambda^2 \alpha_s$, while $\partial_\mu \alpha_c \sim (\lambda^2, 1, \lambda)$

They act on the fields as follows

V_s: $\quad \psi_s(x) \longrightarrow V_s(x)\, \psi_s(x)$

$$D_s^\mu(x) \longrightarrow V_s(x)\, D^\mu_s\, V_s^\dagger(x)$$

$$\left[\text{or } A_s^\mu \to V_s A_s^\mu V_s^\dagger + \frac{i}{g} V_s (\partial_\mu V_s) \right]$$

$$\xi(x) \longrightarrow V_s(x_-)\, \xi(x)$$

multipole exp.!

$$A_c^\mu(x) \longrightarrow V_s^\dagger(x_-)\, A_c^\mu\, V_s^\dagger(x_-)$$

So that $D^\mu \to V_s^\dagger(x_-)\, D^\mu\, V_s(x_-)$

V_c : $\xi(x) \to V_c(x)\,\xi(x)$

$\psi_s(x) \to \psi_s(x)$

(otherwise power counting would fail!)

$D^\mu \to V_c^\dagger(x)\, D^\mu\, V(x)$

$D_s^\mu \to D_s^\mu$

The explicit form of the transformations of A_c^μ can be found in the SCET book.