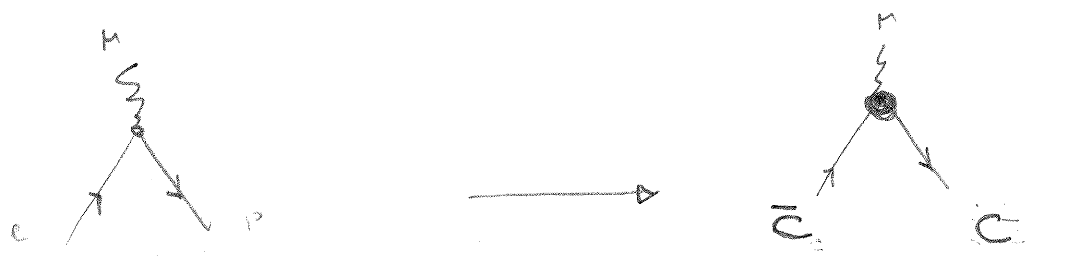
V. External operators

We have constructed $\mathcal{L}_c$, $\mathcal{L}_s$, $\mathcal{L}_{s+c}$. What is missing are terms which connect $c + \bar{c}$ fields. In the Sudakov problem, it is only the e.m. current J^μ which does this



$$J^\mu = \bar{\psi} \gamma^\mu \psi \xrightarrow[\text{tree level}]{} \bar{\xi}_c \gamma^\mu \xi_{\bar{c}}$$

(I now put labels "c", "$\bar{c}$" on ξ to distinguish the two types of collinear fields.)

Note:

$$\bar{\xi}_c \gamma^\mu \xi_{\bar{c}} = \bar{\xi}_c \left[n^\mu \frac{\bar{\not{n}}}{2} + \bar{n}^\mu \frac{\not{n}}{2} + \gamma_\perp^\mu \right] \xi_{\bar{c}} = \bar{\xi}_c \gamma_\perp^\mu \xi_{\bar{c}}$$

The rest of todays lecture will be spent on writing the most general leading power operator in SCET.

The problem is the following. Usually operators with derivatives are power suppressed, but in SCET

$$\bar{n}\cdot\partial\,\phi_c \sim \lambda^0 \underset{\swarrow q,g}{Q}\,\phi_c$$

we need to include operators with an arbitrary number of derivatives!

Consider:

$$\phi_c(x+t\bar{n}) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\bar{n}\cdot\partial)^n\,\phi_c(x)$$

$$\int dt\, C(t)\,\phi_c(x+t\bar{n})$$

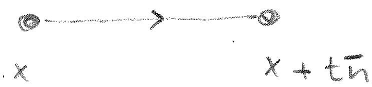
$$= \sum_{n=0}^{\infty} \frac{a_n}{n!}(\bar{n}\cdot\partial)^n\,\phi_c(x)$$

with $a_n = \int dt\, C(t)\, t^n$

Rather then including arbitrary derivatives, we can <u>smear</u> the field ϕ_c over the light-cone.

However, when putting operators at different points, we have to be careful to maintain gauge invariance. Consider, e.g.

$$\bar{\xi}_c(x+t\bar{n})\,[x+t\bar{n}, x]\,\frac{\not{n}}{2}\,\xi_c(x)$$



↑ "link field": Wilson line (see e.g. Peskin Schröder)

$$[x+t\bar{n}, x] = \mathbb{P}\exp\left[ig\int_0^t dt'\,\bar{n}\cdot A_c(x+t\bar{n})\right]$$

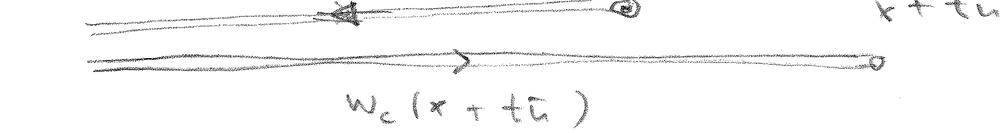
↑ orders color matrices along path: later ≡ left

$$[x+t\bar{n}, x] \to V_c(x+t\bar{n})\,[x+t\bar{n}, x]\,V_c^\dagger(x)$$

[Note SCET book has appendix on Wilson lines.]

Note: the matrix element of the above operator between protons defines the quark PDF.

In SCET, it is useful to take a "detour" and define $W_c(x) = [x, x-\infty\bar{n}]$.



So that $[x+t\bar{n}, x] = W_c(x+t\bar{n}) W_c^\dagger(x)$

then we define a new quark field

$$\chi_c(x) = W_c^\dagger(x)\,\xi_c(x)$$

$$\chi_c(x) \xrightarrow{V_c} V_c(\infty)\,\chi_c(x)$$

The field χ is invariant under collinear gauge transformations vanishing at infinity*. Similarly, one defines

$$\mathcal{A}_c^\mu(x) = W_c^\dagger(D_\mu^c W_c)$$

(Note $\bar{n}\cdot\mathcal{A}_c = 0$)

The fields χ_c, $\mathcal{A}_c^\mu$ are then used to construct SCET operators.

So, finally (!), we can construct the most general leading-power SCET current. It is

Wilson cofficient

$$J^\mu(0) = \int ds \int dt\, C(s,t)\, \bar{\chi}_{\bar{c}}(t\bar{n})\, \gamma_\perp^\mu\, \chi_c(sn)$$

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* and transforms in the standard way for $V_s(x)$

The coefficient $C(s,t)$ encodes the dependence on the large momentum components, i.e. $Q^2 = n\cdot\ell\, \bar{n}\cdot p$.

To see this, use $\phi(x) = e^{i\mathbb{P}x}\, \phi(0)\, e^{-i\mathbb{P}x}$.

Taking the matrix element

$$\langle p_c \,|\, J^\mu(0) \,|\, \ell \rangle$$

$$= \int ds \int dt \; e^{-isn\cdot\ell}\, e^{it\bar{n}\cdot p}\, C(s,t)\, \bar{u}(p)\, \gamma_\perp^\mu\, u(\ell)$$

$$= \tilde{C}(\underbrace{n\cdot\ell\; \bar{n}\cdot p}_{Q^2})\, \bar{u}(p)\gamma_\perp^\mu\, u(\ell)$$

We thus end up with a Q^2-dep. Wilson coefficient.

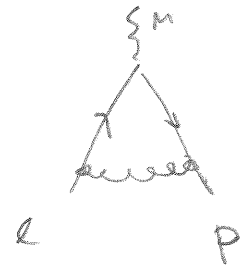
As in any EFT, the Wilson coefficient depends logarithmically on the large scale.

Note that SCET is invariant under $n \to \alpha n$, $\bar{n} \to \frac{1}{\alpha}\bar{n}$.

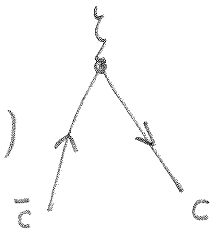
Due to this "reparametrization invariance", the Fourier coefficient $\tilde{C}$ only depends on $Q^2 = n\ell\,\bar{n}p$, not on the individual momentum components.

The final step is to determine $\tilde{C}(Q^2)$ from a matching computation. At one loop, we have:

QCD



$\equiv \frac{\alpha_s}{4\pi}\tilde{C}_V^{(1)}(Q^2)$



From expanding Wilson line $W_{\bar{c}}^{\dagger}$

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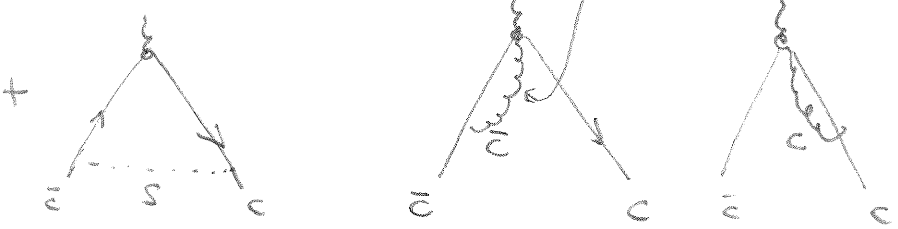

Set $p^2 = \ell^2 = 0$: All loop diagrams in EFT vanish because they become scaleless. The coefficient $\tilde{C}(Q^2)$ is obtained from the on-shell form factor in QCD.

Computation yields

$$\tilde{C}_V^{\text{bare}}(Q^2,\varepsilon) = 1 + \frac{\overset{\overset{Z_\alpha \alpha_s(\mu)\mu^{2\varepsilon} e^{\varepsilon\gamma_E}(4\pi)^{\varepsilon}}{\shortparallel}}{\alpha_s^0}}{4\pi} C_F\left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{\pi^2}{6} + O(\varepsilon)\right)\left(\frac{e^{\gamma}Q^2}{4\pi}\right)^{-\varepsilon}$$

Renormalize:

$$\tilde{C}_V(Q^2,\mu) = \lim_{\varepsilon\to 0} Z^{-1}(\varepsilon, Q^2,\mu)\,\tilde{C}_V(\varepsilon, Q^2)$$

In $\overline{MS}$ scheme, we get

$$C_V(Q^2,\mu^2) = 1 + \frac{C_F\,\alpha_s(\mu)}{4\pi}\left(-\ln^2\frac{Q^2}{\mu^2} + 3\ln\frac{Q^2}{\mu^2} + \frac{\pi^2}{6} - 8\right)$$
$$+ \mathcal{O}(\alpha_s^2)$$

Fulfills RG equation

$$\frac{d}{d\ln\mu}C_V(Q^2,\mu^2) = \left[C_F\,\gamma_{cusp}(\alpha_s)\ln\frac{Q^2}{\mu^2} + \gamma_V(\alpha_s)\right]C_V(Q^2,\mu^2)$$

At one loop $\gamma_{cusp} = 4\frac{\alpha_s}{4\pi}$,

$$\gamma_V = -6C_F\frac{\alpha_s}{4\pi},$$

but all ingredients are known to 3 loops, since the on-shell FF is known to this accuracy.

The distinguishing feature of this RG (and others in SCET) is the presence of a logarithm $\ln\left(\frac{Q^2}{\mu^2}\right)$ in the anomalous dimension Γ. It is very important, that there is only a single log to all orders in perturbation theory, otherwise the expansion of Γ would break down for $\mu \ll Q$.

The linearity follows from factorization, see later.

To achieve factorization we now perform the same decoupling transformation, we applied in soft-photon EFT in the first lecture. Redefine

$$\xi_c = S_n(x_-)\xi_c^{(0)}$$

$$A_c^\mu = S_n(x_-)A_c^{(0)\mu}\, S_n^\dagger(x_-)$$

where

$$S_n(x) = \mathbb{P}\exp\left\{ig\int_{-\infty}^{0} ds\, n\cdot A_s(x+sn)\right\}$$

Then

$$\mathcal{L}_{c+s} = \bar{\xi}\frac{\not{\bar{n}}}{2}\underbrace{in\cdot D}_{in\cdot\partial + gA_c + gA_s(x_-)}\xi = \bar{\xi}^{(0)}\frac{\not{\bar{n}}}{2}\, in D_c^{(0)}\,\xi^{(0)}$$

There are no longer any s-c interactions in the Lagrangian, but (as in the soft-photon case discussed in I), they reappear in the operator J^μ:

$$\bar{\chi}_c(t\bar{n})\,\gamma_\perp^\mu\,\chi(sn)$$

$$= \bar{\chi}_c^{(0)}(t\bar{n})\,S_n^+(0)\,\gamma_\perp^\mu\,S_{\bar{n}}(0)\,\chi_{\bar{c}}^{(0)}(sn)$$

to decouple the $\xi_{\bar{c}}$ fields, we use a soft Wilson line along $\bar{n}$.

The Sudakov form factor thus factorizes into the form

$$\tilde{C}_V(Q^2,\mu^2)\;J(P^2,\mu^2)\;\bar{J}(L^2,\mu^2)\;S(\Lambda_s^2,\mu^2)$$

$\frac{P^2L^2}{Q^2}$

collinear matrix element

soft matrix element

It is easy to solve the RG equation. Let us first neglect the running coupling. In this case, the solution would simply be

$$\tilde{C}_V(Q^2,\mu^2) = \exp\left[-\frac{1}{2}\gamma_{cusp}(\alpha_s)\ln^2\left(\frac{Q^2}{\mu^2}\right) - \gamma_V(\alpha_s)\ln\left(\frac{Q^2}{\mu^2}\right)\right] \cdot \tilde{C}_V(Q^2,\mu^2=Q^2)$$

This makes it obvious, that the leading logarithms in C_V have the form $\alpha_s^n \ln^{2n}\left(\frac{Q^2}{\mu^2}\right)$. These are the so-called Sudakov logarithms.

Including the running coupling, one gets

$$\tilde{C}_V(Q^2,\mu^2) = \exp\left\{\int_{\mu_h}^{\mu}\frac{d\mu'}{\mu'}\left[C_F\,\gamma_{cusp}(\alpha_s)\ln\left(\frac{Q^2}{\mu'^2}\right)+\gamma_V\right]\right\} \cdot \tilde{C}_V(Q^2,\mu_h^2)$$

$$= U(\mu_h,\mu)\,\tilde{C}_V(Q^2,\mu_h)$$

Using $\dfrac{d\alpha_s(\mu)}{d\ln\mu} = \beta(\alpha_s(\mu))$, one can rewrite $\int_{\mu_h}^{\mu} \frac{d\mu}{\mu} = \int_{\alpha(\mu_h)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)}$. Expanding the exponent in α_s, one then gets the result for $\mathcal{U}(\mu_h, \mu)$ in RG-improved PT (see Matthias Neubert's lecture).

One finds

$$\mathcal{U}(\mu_h, \mu) = \exp\left[2C_F\, S(\mu_h, \mu) - A_{\gamma_V}(\mu_h, \mu) \right] \cdot \left(\frac{Q^2}{\mu_h^2} \right)^{-C_F A_{\gamma_{cusp}}(\mu_h, \mu)}$$

where

$$S(\nu, \mu) = - \int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \, \frac{\gamma_{cusp}(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\nu)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}$$

$$A_{\gamma}(\nu, \mu) = - \int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \, \frac{\gamma(\alpha)}{\beta(\alpha)}$$

To get explicit results, one then expands $\gamma(\alpha_s)$, $\beta(\alpha_s)$ in α_s and integrates.