References

- EFT: `hep-ph/9606222`
- Power Counting: `arXiv:1601.07551`
- Matching in HQET and field redefinitions: `hep-ph/9701294`
- Invariants: `arXiv:0907.4763, 1010.3161, 1503.07537, 1512.03433, 1706.08520`
- SMEFT holomorphy: `1409.0868`

Problems: v5

1. Show that for a *connected* graph, $V - I + L = 1$. What is the formula if the graph has n connected components?

2. Work out the properties of fermion bilinears $\overline{\psi}(\mathbf{x},t)\Gamma P_L \chi(\mathbf{x},t)$ under C, P, T, where $\Gamma = 1, \gamma^\mu, \sigma^{\mu\nu}$. The results for $P_L \to P_R$ can be obtained by using $L \leftrightarrow R$.

3. Fierz identities are relations of the form

$$(\overline{A}\Gamma B)(\overline{C}\Gamma D) = \sum_i (\overline{C}\Gamma_i B)(\overline{A}\Gamma_i D)$$

   where A, B, C, D are fermion fields. They are much simpler if written in terms of chiral fields, where $\Gamma_i = 1, \gamma^\mu, \sigma^{\mu\nu}$. Work out the Fierz relations for

$$(\overline{A}P_L B)(\overline{C}P_L D),\ (\overline{A}P_L B)(\overline{C}P_R D),\ (\overline{A}\gamma^\mu P_L B)(\overline{C}\gamma_\mu P_L D), (\overline{A}\gamma^\mu P_L B)(\overline{C}\gamma_\mu P_R D),$$
$$(\overline{A}\sigma^{\mu\nu} P_L B)(\overline{C}\sigma_{\mu\nu} P_L D),\ (\overline{A}\sigma^{\mu\nu} P_L B)(\overline{C}\sigma_{\mu\nu} P_R D)$$

   The P_R P_R identities are given by using $L \leftrightarrow R$ on the L L identities. Do not forget the Fermi minus sign.

4. In $d = 4$ spacetime dimensions, work out the field content of Lorentz-invariant operators with dimension n for $n = 1, \ldots, 6$. At this point, do not try and work out which operators are independent, just the possible structure of allowed operators. Use the notation ϕ for a scalar, ψ for a fermion, $X_{\mu\nu}$ for a field strength, and D for a derivative. For example, an operator of type $\phi^2 D$ such as $\phi D_\mu \phi$ is not allowed because it is not Lorentz-invariant. An operator of type $\phi^2 D^2$ could be either $D_\mu \phi D^\mu \phi$ or $\phi D^2 \phi$, so a $\phi^2 D^2$ operator is allowed, and we will worry later about how many independent operators $\phi^2 D^2$ we can construct.

5. For $d = 2, 3, 4, 5, 6$ dimensions, work out the field content of operators with dimension $n \leq d$, i.e. the "renormalizable" operators.

6. Show that if $\alpha_s(\mu)$ is fixed at some high scale, say $\mu = 1\,\text{TeV}$, then $m_p \propto m_t^{2/27}$, where m_p is the proton mass and m_t is the top quark mass.

7. (a) Compute in dimensional regularization in $d = 4 - 2\epsilon$ dimensions

$$
\begin{aligned}
I_F &= -i\mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)(k^2 - M^2)} + \text{c.t.} \\
I_{\text{EFT}} &= -i\mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)} \left[-\frac{1}{M^2} - \frac{k^2}{M^4} + \ldots \right] + \text{c.t.}.
\end{aligned}
$$

Both integrals only have UV divergences, so the $1/\epsilon$ pieces are canceled by the counterterms. Determine the counterterm contributions $I_{F,ct}$, $I_{\text{EFT},ct}$.

(b) Compute $I_M \equiv (I_F + I_{F,ct}) - (I_{\text{EFT}} + I_{\text{EFT},ct})$ and show that it is analytic in m.

(c) Compute $I_F^{(\text{exp})}$, i.e. I_F with the IR m scale expanded out

$$
I_F^{(\text{exp})} = -i\mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - M^2)} \left[\frac{1}{k^2} + \frac{m^2}{k^4} + \ldots \right]
$$

Note that the first term in the expansion has a $1/\epsilon$ UV divergence, and the remaining terms have $1/\epsilon$ IR divergences.

(d) Compute $I_F^{(\text{exp})} + I_{F,ct}$. Show that the UV divergence cancels, and the remaining $1/\epsilon$ IR divergence is the same as the UV divergent counterterm $I_{\text{EFT},ct}$ in the EFT.

(e) Compute $I_{\text{EFT}}^{(\text{exp})}$, i.e. I_{EFT} with the IR m scale expanded out. Show that it is a scaleless integral which vanishes. Using the known UV divergence from (a), write it in the form

$$
I_{\text{EFT}}^{(\text{exp})} = \frac{1}{16\pi^2} \left[\frac{C}{\epsilon_{\text{UV}}} - \frac{C}{\epsilon_{\text{IR}}} \right]
$$

and that the IR divergence agrees with that in $I_F^{(\text{exp})} + I_{F,ct}$.

(f) Compute $\left(I_F^{(\text{exp})} + I_{F,ct} \right) - \left(I_{\text{EFT}}^{(\text{exp})} + I_{\text{EFT},ct} \right)$ and show that all the $1/\epsilon$ divergences (both UV and IR) cancel, and the result is equal to I_M found in (b).

(g) Make sure you understand why you can compute I_M simply by taking $I_F^{(\text{exp})}$ and dropping all $1/\epsilon$ terms (both UV and IR).

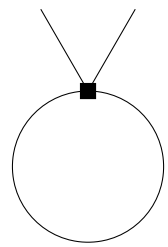
8. Show that for $SU(N)$,

$$
[T^A]^\alpha{}_\beta [T^A]^\lambda{}_\sigma = \frac{1}{2} \delta^\alpha_\sigma \delta^\lambda_\beta - \frac{1}{2N} \delta^\alpha_\beta \delta^\lambda_\sigma
$$

and the color Fierz identities

$$
\begin{aligned}
\delta^\alpha{}_\beta \delta^\lambda{}_\sigma &= \frac{1}{N} \delta^\alpha_\sigma \delta^\lambda_\beta + 2 [T^A]^\alpha{}_\sigma [T^A]^\lambda{}_\beta \\
[T^A]^\alpha{}_\beta [T^A]^\lambda{}_\sigma &= \frac{N^2 - 1}{2N^2} \delta^\alpha_\sigma \delta^\lambda_\beta - \frac{1}{N} [T^A]^\alpha{}_\sigma [T^A]^\lambda{}_\beta
\end{aligned}
$$

9. Compute the one-loop scalar graph with a scalar of mass m and interaction $-\lambda\phi^4/4!$ in the $\overline{\text{MS}}$ scheme.



10. $*$ Compute the decay rate $\Gamma(b \to c e^- \overline{\nu}_e)$ with the interaction Lagrangian

$$L = -\frac{4G_F}{\sqrt{2}} V_{cb} (\bar{c}\gamma^\mu P_L b)(\overline{\nu}_e \gamma_\mu P_L e)$$

with $m_e \to 0$, $m_\nu \to 0$, but retaining the dependence on $\rho = m_c^2/m_b^2$. It is convenient to write the three-body phase space in terms of the variables $x_1 = 2E_e/m_b$ and $x_2 = 2E_\nu/m_b$.

11. Compute the anomalous dimension of $\overline{q}q$ in QCD. Start with massless QCD, and treat $L = -m\overline{q}q$ as an operator insertion.

12. $*$ Compute the anomalous dimension mixing matrix of

$$O_1 = (\overline{b}^\alpha \gamma^\mu P_L c_\alpha)(\overline{u}^\alpha \gamma^\mu P_L d_\alpha) \qquad O_2 = (\overline{b}^\alpha \gamma^\mu P_L c_\beta)(\overline{u}^\beta \gamma^\mu P_L d_\alpha)$$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Two other often used bases are

$$Q_1 = (\bar{b}\gamma^\mu P_L c)(\overline{u}\gamma^\mu P_L d) \qquad Q_2 = (\bar{b}\gamma^\mu P_L T^A c)(\overline{u}\gamma^\mu P_L T^A d)$$

and

$$O_\pm = O_1 \pm O_2$$

So let

$$\mathcal{L} = c_1 O_1 + c_2 O_2 = d_1 Q_1 + d_2 Q_2 = c_+ O_+ + c_- O_-$$

and work out the transformation between the anomalous dimensions for $d_{1,2}$ and $c_{+,-}$ in terms of those for $c_{1,2}$,

13. The equation of motion for $\lambda\phi^4$ theory,

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \qquad\qquad E = (-\partial^2 - m^2)\phi - \frac{\lambda}{3!}\phi^3$$

The EOM Ward identity for $\theta = F(\phi)E$ is

$$\langle 0|T\left\{\phi(x_1)\ldots\phi(x_n)\theta(z)\right\}|0\rangle = i\sum_{r=1}^{n}\delta(z-x_r)\langle 0|T\left\{\phi(x_1)\ldots\cancel{\phi(x_r)}\ldots\phi(x_n)F(z)\right\}|0\rangle$$

In momentum space, integrate both sides with

$$\int \mathrm{d}z e^{-iq\cdot z}\prod_i \int \mathrm{d}x_i e^{-ip_i\cdot x_i}$$

to give the momentum space version

$$\langle 0|T\left\{\widetilde{\phi}(p_1)\ldots\widetilde{\phi}(p_n)\widetilde{\theta}(q)\right\}|0\rangle = i\sum_{r=1}^{n}\delta(z-x_r)\langle 0|T\left\{\widetilde{\phi}(p_1)\ldots\cancel{\widetilde{\phi}(p_r)}\ldots\widetilde{\phi}(x_n)\widetilde{F}(q+p_r)\right\}|0\rangle$$

(a) Consider the equation of motion operator

$$\theta_1 = \phi\, E = \phi(-\partial^2 - m^2)\phi - \frac{\lambda}{3!}\phi^4$$

and verify the Ward identity by explicit calculation at order λ (i.e. tree level) for $\phi\phi$ scattering, i.e. a graph with four ϕ fields, $n = 4$.

(b) Take the on-shell limit $p_r^2 \to m^2$ at fixed $q \neq 0$ of

$$\prod_r (-i)(p_r^2 - m^2) \times \text{Ward Identity}$$

and verify that both sides of the Ward identity vanish. Note that both sides do not vanish if one first takes $q = 0$ and then takes the on-shell limit.

(c) $*$ Repeat the above calculation to order λ^2, i.e. one loop.

(d) $*$ Repeat (to one loop) for the equation of motion operator

$$\theta_2 = \phi^3\, E = \phi^3(-\partial^2 - m^2)\phi - \frac{\lambda}{3!}\phi^6$$

14. Take the heavy quark Lagrangian

$$\begin{aligned}\mathcal{L}_v &= \bar{Q}_v\left\{iv\cdot D + i\cancel{D}_\perp \frac{1}{2m + iv\cdot D} i\cancel{D}_\perp\right\}Q_v \\ &= \bar{Q}_v\left\{iv\cdot D - \frac{1}{2m}\cancel{D}_\perp\cancel{D}_\perp + \frac{1}{4m^2}\cancel{D}_\perp(iv\cdot D)\cancel{D}_\perp + \ldots\right\}Q_v\end{aligned}$$

and use a sequence of field redefinitions to eliminate the $1/m^2$ suppressed $v \cdot D$ term. $(iv \cdot D)Q_v = 0$ is the equation of motion for the heavy quark field, so this example shows how you eliminate equation of motion operators. Here v^μ is a velocity vector with $v \cdot v = 1$, and for a four-vector A,

$$D_\perp^\mu \equiv D^\mu - (v \cdot D)v^\mu$$

If you prefer, you can work in the rest frame of the heavy quark, where $v^\mu = (1, 0, 0, 0)$, $v \cdot D = D^0$ and $D_\perp^\mu = (0, \mathbf{D})$.

15. Compute the on-shell electron form factors $F_1(q^2)$ and $F_2(q^2)$ expanded to first order in q^2/m^2 using dimensional regularization to regulate the IR and UV divergences. This gives the one-loop matching to heavy-electron EFT. The non-Abelian version (in `hep-ph/9701294`) gives the one-loop matching to the HQET Lagrangian. Note that it is much simpler to *first* expand and then do the Feynman parameter integrals. $F_{1,2}(q^2)$ are given in many field theory textbooks, but usually not in pure dim reg.

16. The SCET matching for the vector current $\overline{\psi}\gamma^\mu\psi$ can be done by repeating the previous problem with external masses $m \to 0$ and $p^2 \to 0$, and doing the integral in pure dim reg with $Q^2 = -q^2 \neq 0$. Here Q^2 is the big scale, whereas in the previous problem q^2 was the small scale. The spacelike calculation $Q^2 > 0$ avoids having to deal with the $+i0^+$ terms in the Feynman propagator which lead to imaginary parts. The timelike result $Q^2 < 0$ can then be obtained by analytic continuation.

17. Show (by explicit calculation) for a general 2×2 matrix A that

$$0 = \frac{1}{6}\langle A \rangle^3 - \frac{1}{2}\langle A \rangle \langle A^2 \rangle + \frac{1}{3}\langle A^3 \rangle, \qquad 0 = \frac{1}{2}\langle A \rangle^2 - \frac{1}{2}\langle A^2 \rangle - \langle A \rangle A + A^2$$

and for general 2×2 matrices A, B, C that

$$0 = \langle A \rangle \langle B \rangle \langle C \rangle - \langle A \rangle \langle BC \rangle - \langle B \rangle \langle AC \rangle - \langle C \rangle \langle AB \rangle + \langle ABC \rangle + \langle ACB \rangle .$$

Identities analogous to this for 3×3 matrices are used to remove L_0 and replaced it by $L_{1,2,3}$ in χPT, as discussed by Pich in his lectures.