I. Review of renormalization in QED and QCD

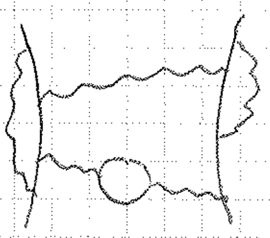
Loop diagrams in QFTs are plagued by UV divergences. The procedure of renormalization is a systematic way of removing these divergences by means of a finite number of redefinitions of parameters. We will study this formalism with the example of QED:

$$\mathcal{L}_{QED} = \bar{\Psi}_0 (i\not{\partial} - m_0)\Psi_0 - \frac{1}{4} F_{\mu\nu,0} F_0^{\mu\nu} - e_0 \bar{\Psi}_0 \gamma^\mu \Psi_0 A_{\mu,0}$$

"bare parameters"

"bare" fields

Superficial degree of divergence:



$$\sim \int \frac{d^4k_1 \dots d^4k_L}{(\not{k}_i - m)\dots \; k_j^2 \dots}$$

$D =$ power of loop momenta in numerator minus those in denominator

$$= 4L - P_e - 2P_\gamma$$

L: loops, P_e: electron propagators, P_γ: photon propagators

One naively expects:
(but things can be better in specific cases)

$D > 0 \;\rightarrow\;$ power divergence $\sim \Lambda_{UV}^D$

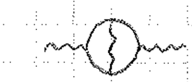
$D = 0 \;\rightarrow\;$ logarithmic divergence $\sim \ln \Lambda_{UV}$

$D < 0 \;\rightarrow\;$ no UV divergence

Now use:

$$L = P_e + P_\gamma - V + 1$$

V: vertices

e.g. 

$L = 4 + 1 - 4 + 1 = 2$

And:

$$V = \underset{\substack{\uparrow \\ \text{external photons}}}{2P_\gamma + N_\gamma} = \frac{1}{2}\left(2P_e + N_e\right)$$

N_e: external electrons

since each vertex involves one photon line and two electron lines and each propagator connects two vertices

Combining these relations, we get:

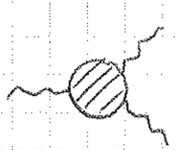
$$\begin{aligned} D &= 4\left(P_e + P_\gamma - V + 1\right) - P_e - 2P_\gamma \\ &= 4 - 4V + 3P_e + 2P_\gamma \\ &= 4 - 4V + 3\left(V - \tfrac{1}{2}N_e\right) + \left(V - N_\gamma\right) \\ &= 4 - \tfrac{3}{2}N_e - N_\gamma \end{aligned}$$

We see that only diagrams with a small number of external legs can have $D \geq 0$. We can restrict ourselves to 1PI and amputated graphs, since external legs do not enter the potentially divergent integrals. 1-particle reducible graphs are simply products of 1PI graphs. This leaves us with:

 $D = 4$ badly divergent, but no contribution to S-matrix elements (unobservable shift of vacuum energy)

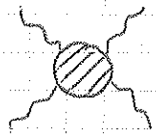
 $D = 3$ vanishes (Furry's theorem) → C invariance

 $D = 2$ vacuum polarization, only logarithmically divergent thanks to gauge invariance

 $D = 1$ vanishes (Furry's theorem)

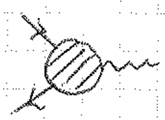
$(g^{\mu\nu}q^2 - q^\mu q^\nu)\,\Pi(q^2)$

subtracts -2 from D_{eff}



$D = 0$, but UV-finite thanks to gauge invariance ($D_{\text{eff}} = -4$)

$D = 1$ self energy, only logarithmically divergent thanks to chiral symmetry (no mass shift for $m = 0$)



$D = 0$ logarithmically divergent

Hence only the vacuum polarization, self energy and vertex function need to be renormalized. We define:

$$\psi_0 = Z_2^{1/2}\, \psi$$

renormalized fields

$$A_0^\mu = Z_3^{1/2}\, A^\mu$$

renormalized mass parameter

$$Z_2\, m_0 = Z_m\, m$$

$$Z_2 Z_3^{1/2} e_0 = Z_1\, e\, \mu^{\frac{4-d}{2}} \quad \text{(defined in Thomson limit } q^\mu \to 0\text{)}$$

renormalized, dimensionless coupling constant (scale μ appears in dimensional regularization, see below)

In terms of these quantities the Lagrangian reads:

$$\mathcal{L}_{QED} = Z_2\, \bar{\psi}\, i\not{\partial}\, \psi - Z_m\, m\, \bar{\psi}\psi - Z_3\, \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - Z_1\, \mu^{\frac{4-d}{2}}\, e\, \bar{\psi}\not{A}\psi$$

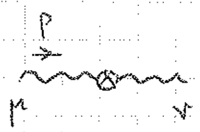
$$= \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - e\,\bar{\psi}\not{A}\psi\,\mu^{\frac{4-d}{2}} \quad \leftarrow \text{renormalized quantities}$$

$$+ \bar{\psi}(\delta_2\, i\not{\partial} - \delta_m)\psi - \frac{\delta_3}{4} F_{\mu\nu}F^{\mu\nu} - \mu^{\frac{4-d}{2}}\,\delta_1\, e\,\bar{\psi}\not{A}\psi \quad \leftarrow \text{counterterms}$$

Here:

$$\delta_2 = Z_2 - 1$$
$$\delta_3 = Z_3 - 1$$
$$\delta_1 = Z_1 - 1$$
$$\delta_m = (Z_m - 1)\, m$$

In addition to the usual QED Feynman rules we must now add Feynman rules for the CTs:



$$= -i\,\delta_3\,(p^2 g^{\mu\nu} - p^\mu p^\nu)$$

$$= i\,(\delta_2 \not{p} - \delta_m)$$

$$= -i\,\delta_1\,\mu^{\frac{4-d}{2}}\, e\, \gamma^\mu$$

The sum of all Feynman graphs including those involving CTs is free of UV divergences!

$$\Downarrow$$

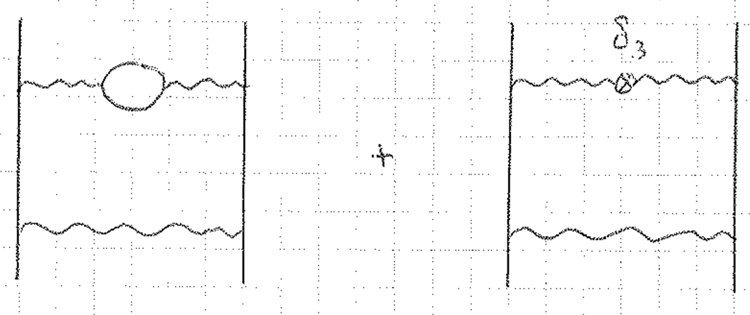
"renormalized perturbation theory"

This works to all orders in perturbation theory:

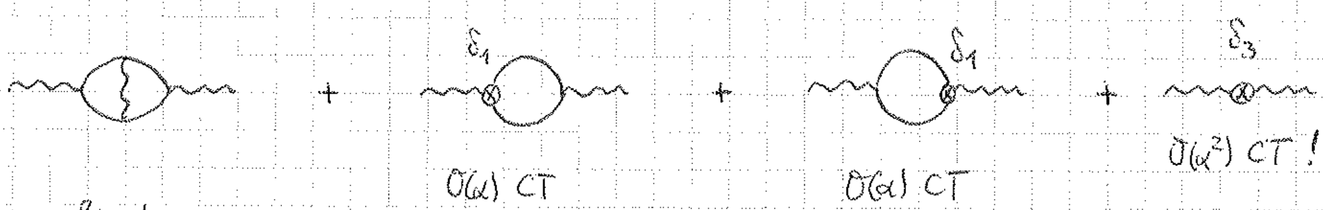
BPHZ theorem

(Bogliubov & Parasiuk; Hepp; Zimmermann)

Examples:

a) 

= finite

b) 

= finite

Calculation of Z_i renormalization factors:

To deal with UV divergences we must first introduce a <u>regularization scheme</u>. The only known scheme that preserves:

- Lorentz invariance
- gauge invariance
- analytic structure of scattering amplitude
- invariance under redefinitions of integration variables

is <u>dimensional regularization</u>:

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \int \frac{d^dk}{(2\pi)^d} \quad \text{with } d = 4 - 2\epsilon \qquad \left(\begin{array}{l}\text{also } d\text{-dimensional} \\ \text{Dirac algebra, see later}\end{array}\right)$$

$\rightarrow$ lowers D by $-2\epsilon L$ and renders logarithmically divergent diagrams finite

→ loop integrals are expressed in terms of functions of d or ε, which can be analytically continued to non-integer values

→ treat ε as an infinitesimally small parameter and expand diagrams about $\varepsilon = 0$

Mass dimensions for $d \neq 4$:

$$[\psi] = \frac{d-1}{2}, \quad [A^\mu] = \frac{d-2}{2}, \quad [m_0] = 1, \quad [e_0] = \frac{4-d}{2} = \varepsilon$$

canonical — no longer dimensionless!

follows from $[S] = 0$, $[\mathcal{L}] = d$

In order to define a dimensionless renormalized coupling one must introduce an auxiliary mass scale μ, such that:

$$e_0 = \mu^\epsilon e_{bare} = \mu^\epsilon Z_1 Z_2^{-1} Z_3^{-1/2} e(\mu) = \mu\text{-independent !}$$

The condition that no observable quantity should depend on μ gives rise to renormalization-group equations (RGEs). We will discuss this in detail later.

Recap of standard one-loop techniques:

1) combine denominators using Feynman parameters:

$$\frac{1}{A \cdot B} = \int_0^1 dx \, \frac{1}{(xA + \bar{x} B)^2} \quad \text{etc.}$$

→ can be generalized to $\dfrac{1}{A^a B^b C^c \ldots}$

2) introduce a shifted loop momentum

$$\ell = k + \sum_i c_i(x, y, \ldots)\, p_i$$

$\uparrow$ external momenta

to obtain the standard form:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^n} \left[N_0 + N_1^\mu \ell_\mu + N_2^{\mu\nu} \ell_\mu \ell_\nu + \ldots \right]$$

where Δ and N_i depend on external momenta, masses and Feynman parameters (note that Δ contains the Feynman $i0$)

3) use Lorentz invariance to replace:

$\ell_\mu \longrightarrow 0$ and similar for all odd powers

$\ell_\mu \ell_\nu \longrightarrow \frac{g_{\mu\nu}}{d} \ell^2$ and similar for higher even powers

4) perform the loop integral using the master formula:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{(\ell^2)^\alpha}{(\ell^2-\Delta)^\beta} = \frac{i(-1)^{\alpha-\beta}}{(4\pi)^{d/2}} (+\Delta)^{\alpha-\beta+d/2} \frac{\Gamma(\alpha+\frac{d}{2})\,\Gamma(\beta-\alpha-\frac{d}{2})}{\Gamma(\beta)\,\Gamma(\frac{d}{2})}$$

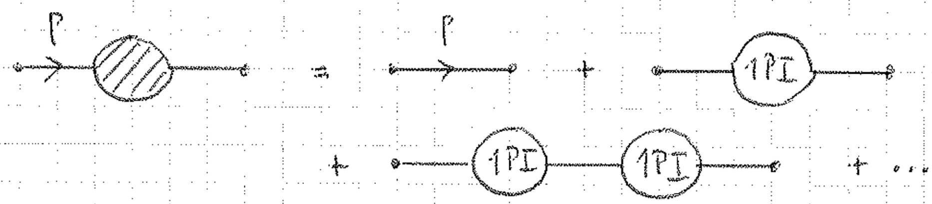
5) perform the integrals over Feynman parameters (yields at most dilogarithms at one-loop order) after expansion around $\varepsilon = 0$

$\Downarrow$

UV divergences show up as $\frac{1}{\varepsilon}$ pole terms

Electron self energy:

Complete propagator:

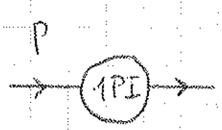


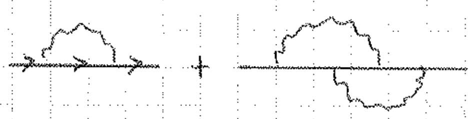
$$= \frac{i}{\not p - m_0 + i0} + \frac{i}{\not p - m_0 + i0}(-i\Sigma)\frac{i}{\not p - m_0 + i0} + \frac{i}{\not p - m_0 + i0}(-i\Sigma)\frac{i}{\not p - m_0 + i0}(-i\Sigma)\frac{i}{\not p - m_0 + i0}$$

$$+ \dots$$

$$= \frac{i}{\not p - m_0 - \Sigma(\not p, m_0) + i0}$$ geometric series

here:

$-i\Sigma =$ 

$=$  $+ \dots$

The resummed propagator has a pole at the physical value m with a residue given by Z_2:

$$\overset{\not p \to m}{\propto} \frac{i Z_2}{\not p - m + i0} + \text{nonsingular terms}$$

It follows that:

$$m = m_0 + \Sigma(\not p = m, m_0) \quad \rightarrow \text{ implicit equation for } m \text{ (pole mass)}$$

$$Z_2^{-1} = 1 - \left.\frac{\partial \Sigma(\not p, m_0)}{\partial \not p}\right|_{\not p = m} \quad \text{WFR constant (on shell)}$$

At one-loop order one finds:

renormalized parameters

IR divergent (not our concern here)

$$Z_2 = 1 - \frac{\alpha}{4\pi}\left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{m^2} - 2\ln\frac{m^2}{\mu_{IR}^2} + c\right)$$

$$m_0 = m\left[1 - \frac{3\alpha}{4\pi}\left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{m^2} + \frac{4}{3}\right)\right] \rightarrow Z_m = Z_2\frac{m_0}{m}$$

(*) discuss after p.11

The precise definition of Z_2 depends on the <u>subtraction scheme</u>. Often the most convenient scheme is the <u>modified minimal subtraction scheme</u>, $\overline{MS}$, where Z_2 contains only the poles in:

$$\frac{1}{\hat{\varepsilon}} \equiv \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \quad \left(\begin{array}{l}\text{alternatively, replace:}\\ \mu^{2\varepsilon} \rightarrow \mu^{2\varepsilon}(4\pi)^{-\varepsilon} e^{\varepsilon\gamma_E}\end{array}\right)$$

In this scheme:

holds in Feynman gauge ($\lambda = 1$)

$$Z_2 = 1 - \frac{\alpha}{4\pi\hat{\varepsilon}} + \mathcal{O}(\alpha^2) \quad \Rightarrow \quad \delta_2 = -\frac{\alpha}{4\pi\hat{\varepsilon}} + \ldots$$

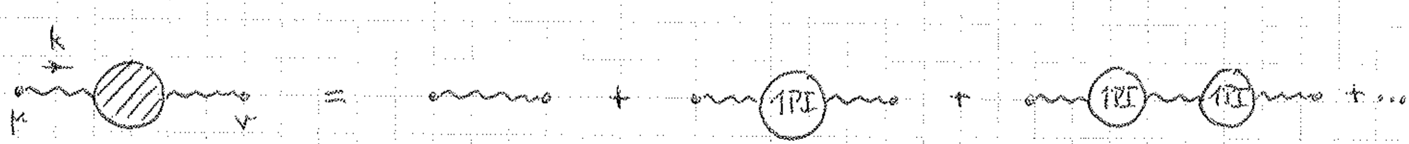
$$Z_m = Z_2\frac{m_0}{m}\bigg|_{\text{pole terms}} = 1 - \frac{\alpha}{\pi\hat{\varepsilon}} + \mathcal{O}(\alpha^2) \quad \Rightarrow \quad \delta_m = \left(-\frac{\alpha}{\pi\hat{\varepsilon}} + \ldots\right)m$$

These "minimal" expressions for the Z_i factors and CTs are sufficient to remove all terms that are singular for $\varepsilon \to 0$.

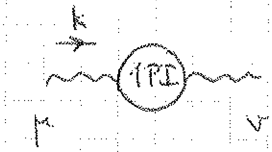
<u>Note:</u> With on-shell WFR, the renormalized propagator has a pole with residue 1 at the physical mass. In the $\overline{MS}$ scheme, on the other hand, one needs to include finite self-energy contributions after renormalization. Also, the renormalized mass is then no longer the position of the pole.

Vacuum polarization:

Complete propagator:



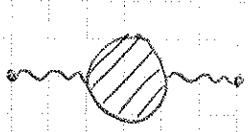
Define:

$$i\,\pi^{\mu\nu}(k) =$$ 

Gauge invariance implies $k_\mu \pi^{\mu\nu}(k) = 0 = k_\nu \pi^{\mu\nu}(k)$ (Ward identity), and hence we can write:

$$\pi^{\mu\nu}(k) = (g^{\mu\nu}k^2 - k^\mu k^\nu)\,\pi(k^2)$$

Performing the geometric sum in an arbitrary covariant gauge, one obtains:

 $$= \frac{-i}{k^2[1-\pi(k^2)]+i0}\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2+i0}\right] - \frac{i}{\underset{\uparrow\ \text{gauge parameter}}{\lambda}}\frac{k^\mu k^\nu}{(k^2+i0)^2}$$

The full propagator has a pole at $k^2=0$ (massless photon) with residue:

$$Z_3 = \frac{1}{1-\pi(0)}$$

From the relevant one-loop diagram:

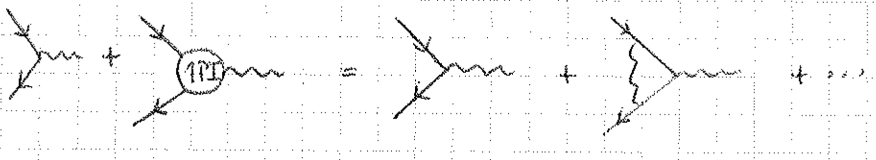
one obtains:

$$Z_3 = 1 - \frac{\alpha}{3\pi}\left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{m^2}\right)$$

In the $\overline{MS}$ scheme we have: $\quad Z_3 = 1 - \frac{\alpha}{3\pi\hat{\varepsilon}} \quad \Rightarrow \quad \delta_3 = -\frac{\alpha}{3\pi\hat{\varepsilon}}$

Charge renormalization:

The Ward-Takahashi identity of QED implies that $Z_1 = Z_2$ to all orders in perturbation theory, where Z_1 is computed from the vertex function:



evaluated at $q^\mu \to 0$

It then follows that:

$$e_0 = \mu^\varepsilon Z_1 Z_2^{-1} Z_3^{-1/2} e = \mu^\varepsilon Z_3^{-1/2} e$$

$$\overset{\overline{MS}}{=} \mu^\varepsilon \left(1 + \frac{\alpha}{6\pi\hat{\varepsilon}} + O(\alpha^2)\right) e$$

And:

$$Z_1 = Z_2 = 1 - \frac{\alpha}{4\pi\hat{\varepsilon}} + O(\alpha^2) \quad \Rightarrow \quad \delta_1 = \delta_2 = -\frac{\alpha}{4\pi\hat{\varepsilon}} + \dots$$

This completes the calculation of the one-loop renormalization factors and CTs of QED.

Renormalization Theory for Effective Field Theories

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Overview:

I. Review of renormalization in QED and QCD
II. Wilsonian RG and running couplings
} Lectures 1 & 2

III. Renormalization of composite operators
IV. Applications
} Lectures 3 & 4

Literature:

- many textbooks on QFT, e.g. Itzykson & Zuber, Peskin & Schroeder, Weinberg, ...
- an old jewel: "QCD: Renormalization for the Practitioner" by Pascual & Tarrach
- many lecture notes on EFTs
- MN: Phys. Rept. 245 (1994) 259 – 397