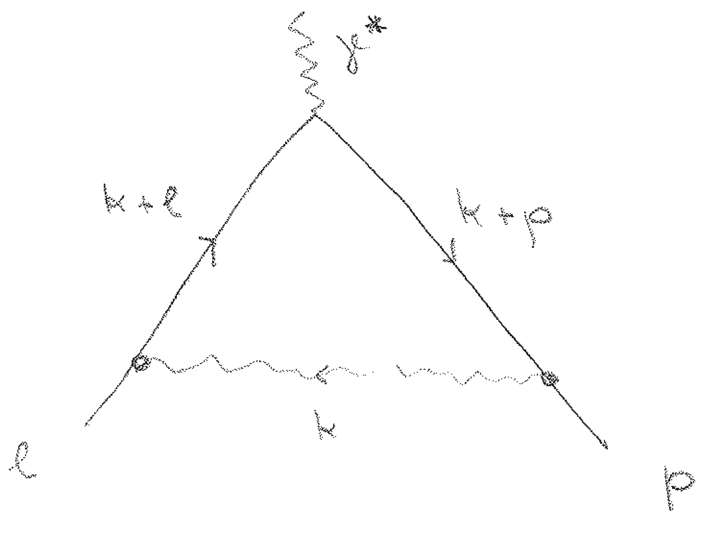
III. Method of regions for the Sudakov form factor

We now want to study the simplest problem in which both soft and collinear particles play a role. To do so, we will analyze the Sudakov form factor. By itself this is not a physical quantity, but it arises as a crucial element in many collider processes.

Consider



$$L^2 = -\ell^2 - i\varepsilon$$
$$P^2 = -p^2 - i\varepsilon$$
$$Q^2 = -(\ell-p)^2 - i\varepsilon$$

We will analyze this form factor for

$$L^2 \sim P^2 \ll Q^2$$

This is exactly the kinematics for which SCET is designed: very energetic particles, but small invariant masses.

To keep things simple, we will analyze again just the scalar integral

$$I = \int d^d k \, \frac{1}{k^2 \, (k+l)^2 \, (k+p)^2}$$

but the same momentum regions are relevant for tensor integrals.

To perform the expansion, it is useful to introduce light-like reference vectors along p^μ and l^μ, in analogy to the vector v^μ we introduced in our discussion of soft photons.

To be explicit, let's choose

$$n^\mu = (1, 0, 0, 1) \approx p^\mu / p^0$$

$$\bar{n}^\mu = (1, 0, 0, -1) \approx \ell^\mu / \ell^0$$

So that $n^2 = \bar{n}^2 = 0$; $n \cdot \bar{n} = 2$

Any vector can be decomposed as

$$p^\mu = n \cdot p \, \frac{\bar{n}^\mu}{2} + \bar{n} \cdot p \, \frac{n^\mu}{2} + p_\perp^\mu$$

$$= p_+^\mu + p_-^\mu + p_\perp^\mu$$

("light-cone coordinates")

Note $p^2 = n \cdot p \, \bar{n} \cdot p + p_\perp^2$.

Define expansion parameter $\lambda^2 \sim p^2/Q^2 \sim \ell^2/Q^2 \ll 1$

Due to $p^2 \sim \lambda^2 Q^2$ and $p^\mu = p_-^\mu + O(\lambda)$,
$= Q/2 \, n^\mu + O(\lambda)$

the components scale as

$$(n \cdot p \, , \, \bar{n} \cdot p \, , \, p_\perp^\mu)$$

$$p^\mu \sim (\lambda^2 \, , \, 1 \, , \, \lambda) \, Q$$

$$\ell^\mu \sim (1 \, , \, \lambda^2 \, , \, \lambda) \, Q$$

We now perform the region expansion of our integral. The following scalings of the loop momentum contribute

$$
\begin{array}{lllll}
 & & n\cdot k & \bar{n}\cdot k & k_\perp \\
\text{Hard (h)} & k^\mu \sim (& 1 \;, & 1 \;, & 1 \;)\, Q \\
\text{Collinear to } p \text{ (c)} & k^\mu \sim (& \lambda^2 \;, & 1 \;, & \lambda \;)\, Q \\
\text{Collinear to } \ell \; (\bar{c}) & k^\mu \sim (& 1 \;, & \lambda^2 \;, & \lambda \;)\, Q \\
\text{Soft (s)} & k^\mu \sim (& \lambda^2 \;, & \lambda^2 \;, & \lambda^2 \;)\, Q
\end{array}
$$

All other scalings $(\lambda^a, \lambda^b, \lambda^c)$ lead to scaleless integrals. (Pick one and check!)

Note that $k_s^2 \sim \lambda^4 Q^2 \sim \frac{P^2 L^2}{Q^2}$. Since $k_s^2 \ll k_c^2$, this mode is also called ultra-soft.

In some cases, the soft mode scales as $(\lambda, \lambda, \lambda)$. This version of SCET is called $SCET_{II}$, as opposed to our case, called $SCET_I$.

Let us now expand the integrand in the different regions to leading power.

$$I_h = \int d^d k \, \frac{1}{k^2 (k+p_-)^2 (k+l_+)^2} \sim \frac{1}{l_+ p_-} (l_+ p_-)^{-\varepsilon}$$

Note $p_-^2 = l_+^2 = 0$: $I_h \equiv$ on-shell form factor

$$I_c = \int d^d k \, \frac{1}{k^2 (k+p)^2 \, 2 k_- l_+} \sim \frac{1}{l_+ \cdot p_-} (p^2)^{-\varepsilon}$$

$(k+l)^2 = 2 k_- l_+$

$$I_s = \int d^d k \, \frac{1}{k^2 (2 p_- \cdot k_+ + p^2)(2 l_+ k_- + l^2)}$$

see slides for result of the computation!

Indeed, one recovers the full integral by adding the contributions of the different regions.