IV Soft-Collinear Effective Theory: Lagrangian

We have identified the relevant momentum regions for the Sudakov problem.

We now introduce EFT fields $\psi_c, \psi_{\bar{c}}, \psi_s$ with the appropriate momentum scaling for each region and construct an EFT, which reproduces the expansion of the diagrams.

At tree level, we can simply substitute

$$\psi \to \psi_c + \psi_{\bar{c}} + \psi_s$$
$$A \to A_c + A_{\bar{c}} + A_s \qquad (*)$$

expand in suppressed terms and read off the effective Lagrangian. At the loop level, matching corrections arise, but we will see that these only affect terms involving both

$c + \bar{c}$ fields.

To set scaling of fields consider propogators. Note $p \cdot x \sim 1$

$\langle 0 | T \{ A_\mu(x) A_\nu(0) \} | 0 \rangle$

$$= \int d^4p \, e^{-ipx} \frac{i}{p^2} \left\{ -g_{\mu\nu} + \xi \frac{p^\mu p^\nu}{p^2} \right\}$$

$\rightarrow$ A_μ's scale like p_μ's

$\rightsquigarrow$ $A_s^\mu \sim \lambda^2$ $\quad \left[\int d^4p_s \sim \lambda^8 ; \frac{1}{p^2} \sim \lambda^{-4} \right]$

$(n \cdot A_c , \bar{n} \cdot A_c , A^\mu_{\perp c}) \sim (\lambda^2 , 1 , \lambda)$

$\left[\int d^4p_c \sim \lambda^4 ; \frac{1}{p^2} \sim \frac{1}{\lambda^2} \right]$

$\langle 0 | T \{ \psi(x) \bar{\psi}(0) \} | 0 \rangle$

$$= \int d^4p \, e^{-ipx} \frac{i \not{p}}{p^2}$$

$$\rightsquigarrow \psi_s \sim \lambda^3$$

Collinear case is more complicated:

$$\not p = n\cdot p \, \frac{\not{\bar n}}{2} + \bar n \cdot p \, \frac{\not n}{2} + \not p_\perp$$

write $\psi_c = \frac{\not n \not{\bar n}}{4}\psi + \frac{\not{\bar n}\not n}{4}\psi = (P_+ \psi) + (P_- \psi)$

$$= \xi + \eta$$

$$P_+ + P_- = \frac{\not n \not{\bar n}}{4} + \frac{\not{\bar n}\not n}{4} = \frac{\not n \not{\bar n}}{4} - \frac{\not n \not{\bar n}}{4} + \frac{2 n\cdot \bar n}{4}$$

$$= \underline{1}$$

$P_+^2 = 1 \; ; \; P_-^2 = 1$

$$\langle 0 | \xi(x) \bar\xi(0) | 0 \rangle = \int d^4p \, e^{-ipx} \, \frac{\frac{\not n \not{\bar n}}{4} \not p \frac{\not{\bar n}\not n}{4}}{p^2}$$

$$= \int d^4p \, e^{-ipx} \, \frac{\bar n \cdot p \, \frac{\not n}{2}}{p^2}$$

$$\sim \lambda^4 \cdot \frac{1}{\lambda^2} \sim \lambda^2$$

$\rightarrow \xi(x) \sim \lambda$. Analogously $\eta(x) \sim \lambda^2$

Now that we know the scaling, we can plug decomposition into $S = \int d^4x\, \mathcal{L}_{QCD}$.

$$S = \underbrace{S_s}_{\text{purely soft}} + \underbrace{S_c}_{\text{purely coll.}} + \underbrace{S_{c+s}}_{\text{sc - ints}} + \dots$$

$$S_s = \int \overset{\lambda^{-8}}{d^4x}\, \overset{\lambda^3}{\bar\psi_s}\, \overset{\lambda^2}{i\not{D}_s}\, \overset{\lambda^3}{\psi_s} - \frac{1}{4} F^s_{\mu\nu} F^{\mu\nu s}$$

$$iD_s = i\partial_s + A_s$$

equivalent to standard QCD, nothing to expand.

$$S_c = \int \overset{\lambda^{-4}}{d^4x}\, (\bar\xi + \bar\eta)\, \underbrace{i\not{D}_c}_{in\cdot D_c \frac{\not{\bar n}}{2} + i\bar n\cdot D_c \frac{\not{n}}{2} + i\not{D}_{\perp c}} (\xi + \eta)$$

$$= \int \overset{\frac{1}{\lambda^4}}{d^4x}\, \overset{\lambda}{\bar\xi}\, \overset{\lambda^2}{in\cdot D_c} \frac{\not{\bar n}}{2} \overset{\lambda}{\xi} + \bar\xi\, i\not{D}_{\perp c}\, \eta + \bar\eta\, i\not{D}_{\perp c}\, \xi$$

$$+ \underset{\lambda^2}{\bar\eta}\, \underset{\lambda^0}{i\bar n\cdot D} \frac{\not{n}}{2} \underset{\lambda^2}{\eta} \sim \lambda^0$$

It is inconvenient to have two field ξ, η with different power counting mix.

Can be avoided by integrating out η, since the action is quadratic (for external collinear states ξ is sufficient).

To do so, one shift:

$$\eta \to \eta - \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D} i\not{D}_\perp \xi$$

Note: EOM for η is

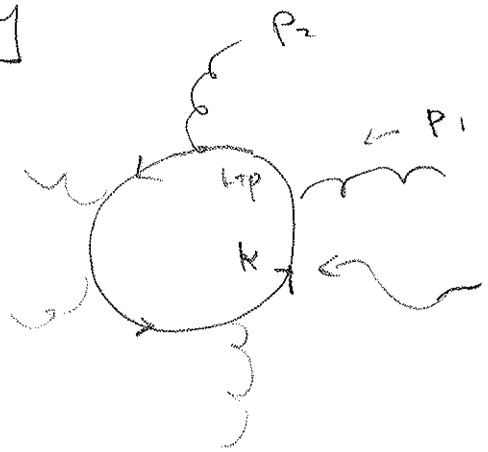
$$\frac{\delta \mathcal{L}}{\delta \bar{\eta}} = 0 \Rightarrow \frac{\not{\bar{n}}}{2} i\bar{n}\cdot D\, \eta = -i\not{D}_\perp \xi$$

$$\eta = -\frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D} i\not{D}_\perp \xi$$

$$\Rightarrow \mathcal{L}_c = \bar{\xi} \frac{\not{n}}{2} in\cdot D \xi + \bar{\eta} \frac{\not{\bar{n}}}{2} i\bar{n}\cdot D\, \eta$$
$$- \bar{\xi}\, i\not{D}_\perp \frac{1}{i\bar{n}\cdot D} \frac{\not{\bar{n}}}{2} i\not{D}_\perp \xi$$

Then integrate out η. Leaves $\det\left(\frac{\not{n}}{2}\, i\bar{n}\cdot D\right)$.

Diagrammatically



$$\frac{1}{\bar{n}\cdot k + i\varepsilon}$$

All propagator poles for $\mathrm{Im}(\bar{n}\cdot k) < 0$.

Close $\bar{n}\cdot k$ contour in upper plane: integral vanishes.

$\det\left(\frac{\not{n}}{2}\, i\bar{n}\cdot D\right)$ is trivial.

So the end result is

$$\mathcal{L}_c = \bar{\xi}\,\frac{\not{\bar{n}}}{2}\left[in\cdot D + i\not{D}_\perp \frac{1}{i\bar{n}\cdot D}\, i\not{D}_\perp\right]\xi$$

$$- \frac{1}{4}\left(F_{\mu\nu}^{c,a}\right)^2$$

Note: $\bar{n}\cdot D$ is large for collinear field,

inverse derivative is unproblematic (can put

arbitrary $i\varepsilon$).

Next we consider soft-collinear interaction terms in S_{eff}. The general construction* is somewhat involved, but since we only need the leading-power terms, it is quite simple.

* q_s is suppressed, compared to ξ_c.
  → No soft quarks arise in leading power interactions
* $A_{\perp s}$ and $\bar{n} \cdot A_s$ are suppressed: only the component $n \cdot A_s$ can arise.

The interactions can be obtained by substituting

$$A_c^\mu \longrightarrow A_c^\mu + n \cdot A_s \frac{\bar{n}^\mu}{2}$$

in S_c.

The final step in the construction is to perform a derivative expansion, i.e. expand in small momentum components.

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*see hep-ph/0211358

Consider the term

$$\int d^4x \; \bar{\xi}(x) \; n \cdot A_s(x) \frac{\bar{n}\!\!\!/}{2} \xi(x)$$

The product of a soft and collinear field scales like a collinear field since

$$\begin{array}{lll} & & n\cdot p \quad \bar{n}\cdot p \quad p_\perp \\ p_c^\mu + p_s^\mu \sim p_c^\mu & \sim & (\lambda^2, \; 1, \; \lambda) \\ \to \quad x^\mu & \sim & (1, \; 1/\lambda^2, \; 1/\lambda) \\ p_s^\mu & \sim & (\lambda^2, \; \lambda^2, \; \lambda^2) \end{array}$$

So
$$p_s \cdot x = \underbrace{\tfrac{1}{2} n\cdot p_s \; \bar{n}\cdot x}_{O(1)} + \underbrace{\tfrac{1}{2} \bar{n}\cdot p_s \; n\cdot x}_{O(\lambda^2)} + \underbrace{p_\perp x_\perp}_{O(\lambda)}$$

$$= 2 p_{s+} \cdot x_- + 2 p_{s-} \cdot x_+ + p_\perp \cdot x_\perp$$

$$\int d^4x \; \bar{\xi}(x) \frac{\bar{n}\!\!\!/}{2} \xi(x) \; n \cdot A_s(x)$$

$$= \int d^4x \; \bar{\xi}(x) \frac{\bar{n}\!\!\!/}{2} \xi(x) \left[1 + \underbrace{x_\perp \cdot \partial_\perp}_{O(\lambda)} + \underbrace{x_+ \cdot \partial_{x_+}}_{O(\lambda^2)} + \dots \right] nA(x) \Big|_{x = x_-}$$

$$= \int d^4x \; \bar{\xi}(x) \frac{\bar{n}\!\!\!/}{2} \xi(x) \; n \cdot A(x_-) + O(\lambda)$$

So we arrive at the final form of the effective Lagrangian for s+c field and their leading power interactions:

$$\mathcal{L}_{SCET} = \bar{\psi}_s \, i\not{D}_s \, \psi_s + \bar{\xi} \left[\frac{\not{n}}{2} \, in \cdot D + i\not{D}_c \frac{1}{i\bar{n} \cdot D_c} i\not{D}_c \right] \xi$$

$$- \frac{1}{4} (F_{\mu\nu}^{s,a})^2 - \frac{1}{4} (F_{\mu\nu}^{c,a})^2$$

Where: $$iD_\mu^s = i\partial_\mu + g A_\mu^s$$

$$iD_\mu^c = i\partial_\mu + g A^c_\mu$$

$$in \cdot D = in \cdot \partial + g n A_c + g n \cdot A_s (x_-)$$

and $$D^\mu = n \cdot D \frac{\bar{n}^\mu}{2} + \bar{n} \cdot D_c \frac{n^\mu}{2} + D^\mu_{c\perp}$$

$$ig F_{\mu\nu}^c = [iD_\mu , iD_\nu]$$

$$ig F_{\mu\nu}^s = [iD_\mu^s , iD_\nu^s]$$