2. Chiral Perturbation Theory

- **Chiral Symmetry Breakings & External Sources**
- **Lowest-Order χPT**
- **Weinberg's Power Counting**
- **Loops**
- **χPT at $\mathcal{O}(p^4)$ and Beyond**

Explicit Symmetry Breaking

$$\mathcal{L}_{QCD} \equiv \mathcal{L}^0_{QCD} + \bar{\mathbf{q}}\,(\not{\mathbf{v}} + \not{\mathbf{a}}\,\gamma_5)\,\mathbf{q} - \bar{\mathbf{q}}\,(\mathbf{s} - i\,\gamma_5\,\mathbf{p})\,\mathbf{q}$$

$$= \mathcal{L}^0_{QCD} + \bar{\mathbf{q}}_L\,\not{\mathbf{l}}\,\mathbf{q}_L + \bar{\mathbf{q}}_R\,\not{\mathbf{r}}\,\mathbf{q}_R - \bar{\mathbf{q}}_R\,(\mathbf{s} + i\,\mathbf{p})\,\mathbf{q}_L - \bar{\mathbf{q}}_L\,(\mathbf{s} - i\,\mathbf{p})\,\mathbf{q}_R$$

Explicit Symmetry Breaking

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$$\mathbf{l}_\mu \equiv \mathbf{v}_\mu - \mathbf{a}_\mu = e\,\mathcal{Q}\,A_\mu + \cdots$$

$$\mathbf{r}_\mu \equiv \mathbf{v}_\mu + \mathbf{a}_\mu = e\,\mathcal{Q}\,A_\mu + \cdots \qquad \mathcal{Q} \equiv \tfrac{1}{3}\,\mathrm{diag}(2,-1,-1)$$

$$\mathbf{s} = \mathcal{M} + \cdots \qquad ; \qquad \mathcal{M} \equiv \mathrm{diag}(m_u, m_d, m_s)$$

Explicit Symmetry Breaking

$$\mathcal{L}_{QCD} \equiv \mathcal{L}^0_{QCD} + \bar{\mathbf{q}}\,(\not{\mathbf{v}} + \not{\mathbf{a}}\,\gamma_5)\,\mathbf{q} - \bar{\mathbf{q}}\,(\mathbf{s} - i\,\gamma_5\,\mathbf{p})\,\mathbf{q}$$

$$= \mathcal{L}^0_{QCD} + \bar{\mathbf{q}}_L\,\not{\mathbf{l}}\,\mathbf{q}_L + \bar{\mathbf{q}}_R\,\not{\mathbf{r}}\,\mathbf{q}_R - \bar{\mathbf{q}}_R\,(\mathbf{s} + i\,\mathbf{p})\,\mathbf{q}_L - \bar{\mathbf{q}}_L\,(\mathbf{s} - i\,\mathbf{p})\,\mathbf{q}_R$$

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$$\mathbf{s} = \mathcal{M} + \cdots \quad ; \quad \mathcal{M} \equiv \mathrm{diag}(m_u, m_d, m_s)$$

Local $SU(3)_L \otimes SU(3)_R$ Symmetry:

$$\mathbf{q}_L \to g_L\,\mathbf{q}_L \qquad \mathbf{q}_R \to g_R\,\mathbf{q}_R$$

$$\mathbf{l}_\mu \to g_L\,\mathbf{l}_\mu\,g_L^\dagger + i\,g_L\,\partial_\mu g_L^\dagger$$

$$\mathbf{r}_\mu \to g_R\,\mathbf{r}_\mu\,g_R^\dagger + i\,g_R\,\partial_\mu g_R^\dagger$$

$$(\mathbf{s} + i\,\mathbf{p}) \to g_R\,(\mathbf{s} + i\,\mathbf{p})\,g_L^\dagger$$

Lowest-Order Effective Lagrangian:

$$\mathcal{L} = \frac{f^2}{4} \langle D_\mu \mathbf{U} \, D^\mu \mathbf{U}^\dagger + \chi \, \mathbf{U}^\dagger + \mathbf{U} \, \chi^\dagger \rangle$$

$$D_\mu \mathbf{U} = \partial_\mu \mathbf{U} - i \, \mathbf{r}_\mu \, \mathbf{U} + i \, \mathbf{U} \, \mathbf{l}_\mu$$

$$\chi \equiv 2 \, B_0 \, (\mathbf{s} + i \, \mathbf{p})$$

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Currents:

$$\mathbf{J}_L^\mu = \frac{\partial}{\partial\, \mathbf{l}_\mu} \mathcal{L}_2 = \frac{i}{2} f^2\, D^\mu \mathbf{U}^\dagger\, \mathbf{U} = \frac{f}{\sqrt{2}}\, D^\mu \mathbf{\Phi} + \cdots$$

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Currents:

$$\mathbf{J}^\mu_{L} = \frac{\partial}{\partial\, \mathbf{l}_\mu} \mathcal{L}_2 = \frac{i}{2} f^2\, D^\mu \mathbf{U}^\dagger\, \mathbf{U} = \frac{f}{\sqrt{2}}\, D^\mu \mathbf{\Phi} + \cdots$$

$$\mathbf{J}^\mu_{R} = \frac{\partial}{\partial\, \mathbf{r}_\mu} \mathcal{L}_2 = \frac{i}{2} f^2\, D^\mu \mathbf{U}\, \mathbf{U}^\dagger = -\frac{f}{\sqrt{2}}\, D^\mu \mathbf{\Phi} + \cdots$$

Lowest-Order Effective Lagrangian:

$$\mathcal{L} = \frac{f^2}{4} \langle D_\mu \mathbf{U}\, D^\mu \mathbf{U}^\dagger + \chi\, \mathbf{U}^\dagger + \mathbf{U}\, \chi^\dagger \rangle$$

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$$\langle 0 |\, (J^\mu_A)_{12}\, | \pi^+(p) \rangle = i \sqrt{2}\, f\; p^\mu \quad \Longrightarrow \quad f = f_\pi \approx 92.4\ \mathrm{MeV}$$

$(\pi^+ \to \mu^+ \nu_\mu)$

Lowest-Order Effective Lagrangian:

$$\mathcal{L} = \frac{f^2}{4} \langle D_\mu \mathbf{U} \, D^\mu \mathbf{U}^\dagger + \chi \, \mathbf{U}^\dagger + \mathbf{U} \, \chi^\dagger \rangle$$

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$$\langle 0 | \, (J^\mu_A)_{12} \, | \pi^+(p) \rangle = i \sqrt{2} \, f \; p^\mu \quad \longrightarrow \quad f = f_\pi \approx 92.4 \text{ MeV}$$

$(\pi^+ \to \mu^+ \nu_\mu)$

$$\bar{\mathbf{q}}^j_L \, \mathbf{q}^i_R = -\frac{\partial \, \mathcal{L}_2}{\partial (\mathbf{s} - i \, \mathbf{p})^{ji}} = -\frac{f^2}{2} B_0 \, \mathbf{U}^{ij} \quad \longrightarrow$$

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Quark Masses: $\frac{f^2}{4}\langle \chi\, \mathbf{U}^\dagger + \mathbf{U}\, \chi^\dagger \rangle \quad \rightarrow \quad \mathcal{L}_m = -B_0\, \langle \mathcal{M}\, \Phi^2 \rangle$

Quark Masses:

$$\frac{f^2}{4}\,\langle \chi\,\mathbf{U}^\dagger + \mathbf{U}\,\chi^\dagger\rangle \quad\to\quad \mathcal{L}_m = -B_0\,\langle \mathcal{M}\,\Phi^2\rangle$$

$$\mathcal{L}_m \;=\; -B_0\left\{(m_u+m_d)\left[\pi^+\pi^- + \frac{1}{2}\,\pi^0\pi^0\right] + (m_u+m_s)\,K^+K^- \right.$$

$$\left. +\;(m_d+m_s)\,K^0\bar{K}^0 \;+\; \frac{1}{6}\,(m_u+m_d+4\,m_s)\,\eta^2 \;+\; \frac{1}{\sqrt{3}}\,(m_u-m_d)\,\pi^0\eta\right\}$$

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$$+ (m_d + m_s)\, K^0 \bar{K}^0 + \frac{1}{6}\,(m_u + m_d + 4\, m_s)\,\eta^2 + \frac{1}{\sqrt{3}}\,(m_u - m_d)\,\pi^0\eta \Big\}$$

Isospin limit: $\mathbf{m_u = m_d = \hat{m}}$

$$\frac{M_\pi^2}{2\,\hat{m}} = \frac{M_K^2}{\hat{m} + m_s} = \frac{3\, M_\eta^2}{2\,\hat{m} + 4\, m_s} = B_0$$

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- **Gell-Mann–Okubo:** $4\,M_K^2 = M_\pi^2 + 3\,M_\eta^2$

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- **Gell-Mann–Okubo:** $4\,M_K^2 = M_\pi^2 + 3\,M_\eta^2$
- **Gell-Mann–Oakes–Renner:** $f^2\,M_\pi^2 = -\hat{m}\,\langle 0|\bar{u}\,u + \bar{d}\,d|0\rangle$

Quark Mass Ratios:

Dashen Theorem

$$\left(M_{K^0}^2 - M_{K^\pm}^2\right)_{\rm em} = \left(M_{\pi^0}^2 - M_{\pi^\pm}^2\right)_{\rm em} + \mathcal{O}(e^2 p^2)$$

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Dashen Theorem

$$\left(M_{K^0}^2 - M_{K^\pm}^2\right)_{\rm em} = \left(M_{\pi^0}^2 - M_{\pi^\pm}^2\right)_{\rm em} + \mathcal{O}(e^2 p^2)$$

Proof: $e^2 \langle \mathcal{Q}_{\rm R} \mathrm{U} \mathcal{Q}_{\rm L} \mathrm{U}^\dagger \rangle = -\frac{2e^2}{f^2} \left(\pi^+\pi^- + K^+K^-\right) + \mathcal{O}(\phi^4) \quad ; \quad \mathcal{Q}_X \to g_X\, \mathcal{Q}_X\, g_X^\dagger \qquad \square$

Quark Mass Ratios:

Dashen Theorem

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$$\frac{m_d - m_u}{m_d + m_u} = \frac{\left(M^2_{K^0} - M^2_{K^\pm}\right) - \left(M^2_{\pi^0} - M^2_{\pi^\pm}\right)}{M^2_{\pi^0}} \approx 0.29$$

Quark Mass Ratios:

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$$\frac{m_d - m_u}{m_d + m_u} = \frac{\left(M^2_{K^0} - M^2_{K^\pm}\right) - \left(M^2_{\pi^0} - M^2_{\pi^\pm}\right)}{M^2_{\pi^0}} \approx 0.29$$

$$\frac{m_s - m_u}{m_u + m_d} = \frac{M^2_{K^0} - M^2_{\pi^0}}{M^2_{\pi^0}} \approx 12.6$$

Quark Mass Ratios:

Dashen Theorem

$$\left(M^2_{K^0} - M^2_{K^\pm}\right)_{\rm em} = \left(M^2_{\pi^0} - M^2_{\pi^\pm}\right)_{\rm em} + \mathcal{O}(e^2 p^2)$$

Proof: $e^2 \langle \mathcal{Q}_{\rm R} {\rm U} \mathcal{Q}_{\rm L} {\rm U}^\dagger \rangle = -\frac{2e^2}{f^2} \left(\pi^+\pi^- + K^+K^-\right) + \mathcal{O}(\phi^4)$; $\mathcal{Q}_X \to g_X\, \mathcal{Q}_X\, g_X^\dagger$ □

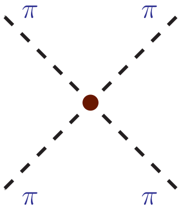
$$\frac{m_d - m_u}{m_d + m_u} = \frac{\left(M^2_{K^0} - M^2_{K^\pm}\right) - \left(M^2_{\pi^0} - M^2_{\pi^\pm}\right)}{M^2_{\pi^0}} \approx 0.29$$

$$\frac{m_s - m_u}{m_u + m_d} = \frac{M^2_{K^0} - M^2_{\pi^0}}{M^2_{\pi^0}} \approx 12.6$$

➡ $$\mathbf{m_u : m_d : m_s = 0.55 : 1 : 20.3}$$ Weinberg

$$\frac{f^2}{4}\,\langle \chi\, \mathbf{U}^\dagger + \mathbf{U}\, \chi^\dagger \rangle \;=\; -B_0\,\langle\, \mathcal{M}\, \Phi^2 \rangle \;+\; \frac{B_0}{6\, f^2}\,\langle\, \mathcal{M}\, \Phi^4 \rangle \;+\; \cdots$$

$$\frac{f^2}{4}\,\langle \chi\, \mathbf{U}^\dagger + \mathbf{U}\,\chi^\dagger \rangle \;=\; -B_0\,\langle\, \mathcal{M}\,\Phi^2\rangle \;+\; \frac{B_0}{6\,f^2}\,\langle\, \mathcal{M}\,\Phi^4\rangle \;+\; \cdots$$

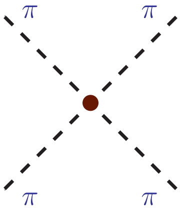


$$T\left(\pi^+\pi^0 \to \pi^+\pi^0\right) \;=\; \frac{t - M_\pi^2}{f_\pi^2}$$

$t \equiv (p'_+ - p_+)^2$

Weinberg

$$\frac{f^2}{4}\langle \chi\, \mathbf{U}^\dagger + \mathbf{U}\, \chi^\dagger \rangle = -B_0 \langle \mathcal{M}\, \Phi^2 \rangle + \frac{B_0}{6 f^2} \langle \mathcal{M}\, \Phi^4 \rangle + \cdots$$



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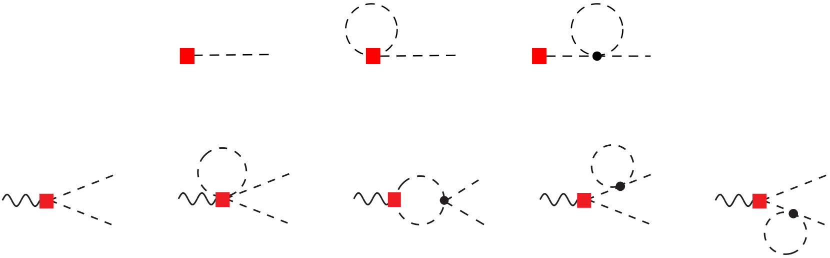
$t \equiv (p'_+ - p_+)^2$

Weinberg

$\mathcal{L}_2$ ⟷ Current Algebra 60's

Chiral Loops:



- Vertices ➔ p^2
- Propagators ➔ p^{-2}
- Loops ➔ p^4

➔ **1-loop diagrams contribute at $\mathcal{O}(\mathbf{p^4})$**

Scaling with Momenta:

A general connected diagram with N_d vertices of $\mathcal{O}(p^d)$, B_I propagators and L loops, scales as $\mathcal{O}(p^D)$ with

$$D = 4L - 2B_I + \sum_d N_d\, d$$

$$\#\ \text{vertices} \equiv V = \sum_d N_d \qquad , \qquad L = B_I - V + 1$$

$$\longrightarrow \qquad \boxed{D = 2L + 2 + \sum_d N_d\,(d-2)}$$

Chiral Power Counting

$\mathbf{U}$	$\mathcal{O}(p^0)$
$D_\mu \mathbf{U}\,,\ \mathbf{l}_\mu\,,\ \mathbf{r}_\mu$	$\mathcal{O}(p^1)$
$\chi\,,\ \mathbf{F}^{\mu\nu}_{L,R}$	$\mathcal{O}(p^2)$

$$\mathbf{F}_L^{\mu\nu} \equiv \partial^\mu \mathbf{l}^\nu - \partial^\nu \mathbf{l}^\mu - i\ [\mathbf{l}^\mu, \mathbf{l}^\nu]$$

$$\mathbf{F}_R^{\mu\nu} \equiv \partial^\mu \mathbf{r}^\nu - \partial^\nu \mathbf{r}^\mu - i\ [\mathbf{r}^\mu, \mathbf{r}^\nu]$$

Chiral Power Counting

$\mathbf{U}$	$\mathcal{O}(p^0)$
$D_\mu \mathbf{U}\,,\ \mathbf{l}_\mu\,,\ \mathbf{r}_\mu$	$\mathcal{O}(p^1)$
$\chi\,,\ \mathbf{F}^{\mu\nu}_{L,R}$	$\mathcal{O}(p^2)$

$$\mathbf{F}^{\mu\nu}_L \equiv \partial^\mu \mathbf{l}^\nu - \partial^\nu \mathbf{l}^\mu - i\,[\mathbf{l}^\mu, \mathbf{l}^\nu]$$

$$\mathbf{F}^{\mu\nu}_R \equiv \partial^\mu \mathbf{r}^\nu - \partial^\nu \mathbf{r}^\mu - i\,[\mathbf{r}^\mu, \mathbf{r}^\nu]$$

General connected diagram with N_d vertices of $\mathcal{O}(p^d)$ and L loops:

$$D \,=\, 2\,L + 2 + \sum_d N_d\,(d-2)$$

Weinberg

Chiral Power Counting

$\mathbf{U}$	$\mathcal{O}(p^0)$
$D_\mu \mathbf{U}\, ,\ \mathbf{l}_\mu\, ,\ \mathbf{r}_\mu$	$\mathcal{O}(p^1)$
$\chi\, ,\ \mathbf{F}^{\mu\nu}_{L,R}$	$\mathcal{O}(p^2)$

$$\mathbf{F}_L^{\mu\nu} \equiv \partial^\mu \mathbf{l}^\nu - \partial^\nu \mathbf{l}^\mu - i\ [\mathbf{l}^\mu, \mathbf{l}^\nu]$$

$$\mathbf{F}_R^{\mu\nu} \equiv \partial^\mu \mathbf{r}^\nu - \partial^\nu \mathbf{r}^\mu - i\ [\mathbf{r}^\mu, \mathbf{r}^\nu]$$

General connected diagram with N_d vertices of $\mathcal{O}(p^d)$ and L loops:

$$D = 2L + 2 + \sum_d N_d\,(d-2) \qquad \text{Weinberg}$$

- $D = 2:$ $\quad L = 0\ ,\ \ d = 2$
- $D = 4:$ $\quad L = 0\ ,\ \ d = 4\ ,\ \ N_4 = 1$

  $L = 1\ ,\ \ d = 2$

$\mathcal{O}(\mathbf{p}^4)\quad \chi\mathbf{PT}$

i) $\mathcal{L}_4$ at tree level (Gasser–Leutwyler)

$$
\begin{aligned}
\mathcal{L}_4 \; = \;& L_1\, \langle D_\mu U^\dagger D^\mu U\rangle^2 \,+\, L_2\, \langle D_\mu U^\dagger D_\nu U\rangle \langle D^\mu U^\dagger D^\nu U\rangle \\
&+\; L_3\, \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle \,+\, L_4\, \langle D_\mu U^\dagger D^\mu U\rangle \langle U^\dagger \chi + \chi \dagger U\rangle \\
&+\; L_5\, \langle D_\mu U^\dagger D^\mu U \left(U^\dagger \chi + \chi^\dagger U\right)\rangle \,+\, L_6\, \langle U^\dagger \chi + \chi^\dagger U\rangle^2 \\
&+\; L_7\, \langle U^\dagger \chi - \chi^\dagger U\rangle^2 \,+\, L_8\, \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle \\
&-\; i\, L_9\, \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \,+\, L_{10}\, \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned}
$$

$\mathcal{O}(\mathbf{p^4})\quad \chi\mathbf{PT}$

i) $\mathcal{L}_4$ **at tree level** (Gasser–Leutwyler)

$$\begin{aligned}
\mathcal{L}_4 \;=\;& L_1 \langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U\rangle \langle D^\mu U^\dagger D^\nu U\rangle \\
&+ L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4 \langle D_\mu U^\dagger D^\mu U\rangle \langle U^\dagger \chi + \chi \dagger U\rangle \\
&+ L_5 \langle D_\mu U^\dagger D^\mu U \left(U^\dagger \chi + \chi^\dagger U\right)\rangle + L_6 \langle U^\dagger \chi + \chi^\dagger U\rangle^2 \\
&+ L_7 \langle U^\dagger \chi - \chi^\dagger U\rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle \\
&- i\, L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned}$$

ii) $\mathcal{L}_2$ **at one loop (unitarity):** $T_4 \sim p^4 \left\{ a \log(p^2/\mu^2) + b(\mu)\right\}$

- Chiral Logarithms unambiguously predicted

$\mathcal{O}(\mathbf{p}^4)$ χPT

i) $\mathcal{L}_4$ **at tree level** (Gasser–Leutwyler)

$$\begin{aligned}
\mathcal{L}_4 \;=\;& L_1 \,\langle D_\mu U^\dagger D^\mu U\rangle^2 \,+\, L_2 \,\langle D_\mu U^\dagger D_\nu U\rangle\,\langle D^\mu U^\dagger D^\nu U\rangle \\
&+\, L_3 \,\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle \,+\, L_4 \,\langle D_\mu U^\dagger D^\mu U\rangle\,\langle U^\dagger \chi + \chi \dagger U\rangle \\
&+\, L_5 \,\langle D_\mu U^\dagger D^\mu U \left(U^\dagger \chi + \chi^\dagger U\right)\rangle \,+\, L_6 \,\langle U^\dagger \chi + \chi^\dagger U\rangle^2 \\
&+\, L_7 \,\langle U^\dagger \chi - \chi^\dagger U\rangle^2 \,+\, L_8 \,\langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle \\
&-\, i\, L_9 \,\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \,+\, L_{10} \,\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned}$$

ii) $\mathcal{L}_2$ **at one loop (unitarity):** $T_4 \sim p^4 \left\{ a \,\log(p^2/\mu^2) + b(\mu) \right\}$

- Chiral Logarithms unambiguously predicted
- L_i's fixed by QCD dynamics. 1-loop divergences ➡ $L_i^r(\mu)$

$\mathcal{O}(\mathbf{p}^4)$ χPT

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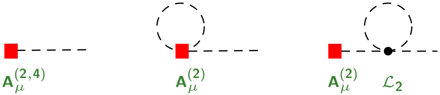
&+\; L_3\,\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle \,+\, L_4\,\langle D_\mu U^\dagger D^\mu U\rangle\,\langle U^\dagger\chi + \chi \dagger U\rangle \\
&+\; L_5\,\langle D_\mu U^\dagger D^\mu U\left(U^\dagger\chi + \chi^\dagger U\right)\rangle \,+\, L_6\,\langle U^\dagger\chi + \chi^\dagger U\rangle^2 \\
&+\; L_7\,\langle U^\dagger\chi - \chi^\dagger U\rangle^2 \,+\, L_8\,\langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle \\
&-\; i\, L_9\,\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \,+\, L_{10}\,\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned}$$

ii) $\mathcal{L}_2$ **at one loop (unitarity):** $T_4 \sim p^4\,\left\{a\,\log(p^2/\mu^2) + b(\mu)\right\}$

- Chiral Logarithms unambiguously predicted
- L_i's fixed by QCD dynamics. 1-loop divergences ➡ $L_i^r(\mu)$

iii) **Wess–Zumino–Witten term (chiral anomaly):** $\pi^0, \eta \to \gamma\gamma$

Meson Decay Constants:



$$\mu_P \equiv \frac{M_P^2}{32\pi^2 f^2} \log\left(\frac{M_P^2}{\mu^2}\right)$$

$$f_\pi = f \left\{1 - 2\mu_\pi - \mu_K + \frac{4M_\pi^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu)\right\}$$

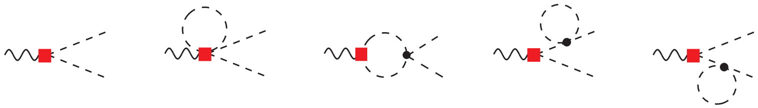
$$f_K = f \left\{1 - \frac{3}{4}\mu_\pi - \frac{3}{2}\mu_K - \frac{3}{4}\mu_{\eta_8} + \frac{4M_K^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu)\right\}$$

$$f_{\eta_8} = f \left\{1 - 3\mu_K + \frac{4M_{\eta_8}^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu)\right\}$$

$$\frac{f_K}{f_\pi} = 1.22 \pm 0.01 \;\longrightarrow\; L_5^r(M_\rho) = (1.4 \pm 0.5)\cdot 10^{-3} \;\longrightarrow\; \frac{f_{\eta_8}}{f_\pi} = 1.3 \pm 0.05$$

Vector Form Factor:

$$\langle \pi^+\pi^- | J^\mu_{\rm em} | 0\rangle = (p_+ - p_-)^\mu \, F^V_\pi(s)$$

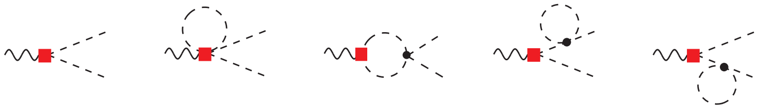


$$F^V_\pi(s) = 1 + \frac{2L_9^r(\mu)}{f^2}\, s - \frac{s}{96\pi^2 f^2}\left[A\left(\frac{m_\pi^2}{s}, \frac{m_\pi^2}{\mu^2}\right) + \frac{1}{2}\, A\left(\frac{m_K^2}{s}, \frac{m_K^2}{\mu^2}\right)\right]$$

$$= 1 + \frac{1}{6}\,\langle r^2\rangle^V_\pi\, s + \cdots$$

$$A\left(\frac{m_P^2}{s}, \frac{m_P^2}{\mu^2}\right) = \log\left(\frac{m_P^2}{\mu^2}\right) + \frac{8m_P^2}{s} - \frac{5}{3} + \sigma_P^3\, \log\left(\frac{\sigma_P+1}{\sigma_P-1}\right) \quad , \quad \sigma_P \equiv \sqrt{1 - \frac{4m_P^2}{s}}$$

Vector Form Factor: $\langle \pi^+\pi^- | J^\mu_{\rm em} | 0 \rangle \; = \; (p_+ - p_-)^\mu \; F^V_\pi(s)$



$$F^V_\pi(s) \;=\; 1 \;+\; \frac{2L_9^r(\mu)}{f^2}\, s \;-\; \frac{s}{96\pi^2 f^2}\left[A\left(\frac{m_\pi^2}{s},\frac{m_\pi^2}{\mu^2}\right) + \frac{1}{2}\, A\left(\frac{m_K^2}{s},\frac{m_K^2}{\mu^2}\right)\right]$$

$$= \; 1 \;+\; \frac{1}{6}\,\langle r^2\rangle^V_\pi \, s \;+\; \cdots$$

$$A\left(\frac{m_P^2}{s},\frac{m_P^2}{\mu^2}\right) \;=\; \log\left(\frac{m_P^2}{\mu^2}\right) + \frac{8 m_P^2}{s} - \frac{5}{3} + \sigma_P^3 \,\log\left(\frac{\sigma_P+1}{\sigma_P-1}\right) \qquad , \qquad \sigma_P \equiv \sqrt{1-\frac{4m_P^2}{s}}$$

$$\langle r^2\rangle^V_\pi \;=\; \frac{12\, L_9^r(\mu)}{f^2} \;-\; \frac{1}{32\pi^2 f^2}\left\{2\,\log\left(\frac{M_\pi^2}{\mu^2}\right) + \log\left(\frac{M_K^2}{\mu^2}\right) + 3\right\}$$

$\langle r^2\rangle^V_\pi \;=\; (0.439 \pm 0.008)\ \mathrm{fm}^2$ ➡ $L_9^r(M_\rho) = (6.9 \pm 0.7)\cdot 10^{-3}$

$\mathcal{O}(\mathbf{p}^4)$ χPT COUPLINGS

i	$L_i^r(M_\rho) \times 10^3$	Source	Γ_i
1	0.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	$3/32$
2	1.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	$3/16$
3	-3.5 ± 1.1	K_{e4}, $\pi\pi \to \pi\pi$	0
4	-0.3 ± 0.5	Zweig rule	$1/8$
5	1.4 ± 0.5	F_K/F_π	$3/8$
6	-0.2 ± 0.3	Zweig rule	$11/144$
7	-0.4 ± 0.2	GMO, $L_{5,8}$	0
8	0.9 ± 0.3	$M_{K^0} - M_{K^+}$, L_5, $(m_s - \hat{m})/(m_d - m_u)$	$5/48$
9	6.9 ± 0.7	$\langle r^2 \rangle_V^\pi$	$1/4$
10	-5.5 ± 0.7	$\pi \to e\nu\gamma$	$-1/4$

- $L_i = L_i^r(\mu) + \Gamma_i \, \frac{\mu^{D-4}}{32\pi^2} \left\{ \frac{2}{D-4} + \gamma_E - \log(4\pi) - 1 \right\}$

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1	0.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	$3/32$
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- $\Lambda_\chi \sim 1\,\mathrm{GeV}$ ➡ $L_i \sim \dfrac{f_\pi^2/4}{\Lambda_\chi^2} \sim 2 \times 10^{-3}$

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1	0.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	$3/32$
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- $\Lambda_\chi \sim 1\,\text{GeV}$ ➡ $L_i \sim \frac{f_\pi^2/4}{\Lambda_\chi^2} \sim 2 \times 10^{-3}$
- χPT Loops $\sim$ $1/(4\pi f_\pi)^2$

Chiral Anomaly: $\delta Z[v,a,s,p] = -\frac{N_C}{16\pi^2}\int d^4x\,\langle \delta\beta(x)\,\Omega(x)\rangle$

$g_{L,R} \approx 1 + i\delta\alpha \mp i\delta\beta$

$$\Omega(x) = \varepsilon^{\mu\nu\sigma\rho}\left[v_{\mu\nu} v_{\sigma\rho} + \frac{4}{3}\,\nabla_\mu a_\nu \nabla_\sigma a_\rho + \frac{2}{3} i\,\{v_{\mu\nu}, a_\sigma a_\rho\} + \frac{8}{3} i\, a_\sigma v_{\mu\nu} a_\rho + \frac{4}{3}\, a_\mu a_\nu a_\sigma a_\rho \right]$$

$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i\,[v_\mu, v_\nu] \quad , \qquad \nabla_\mu a_\nu = \partial_\mu a_\nu - i\,[v_\mu, a_\nu] \quad , \qquad \varepsilon_{0123} = 1$$

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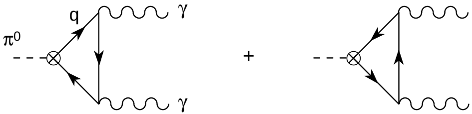
$$S[U,\ell,r]_{WZW} = -\frac{iN_C}{240\pi^2}\int d\sigma^{ijklm}\, \left\langle \Sigma^L_i \Sigma^L_j \Sigma^L_k \Sigma^L_l \Sigma^L_m \right\rangle - \frac{iN_C}{48\pi^2}\int d^4x\;\varepsilon_{\mu\nu\alpha\beta}\left(W(U,\ell,r)^{\mu\nu\alpha\beta} - W(\mathbf{1},\ell,r)^{\mu\nu\alpha\beta}\right)$$

Wess–Zumino–Witten

$$\begin{aligned} W(U,\ell,r)_{\mu\nu\alpha\beta} = \Big\langle & U\ell_\mu\ell_\nu\ell_\alpha U^\dagger r_\beta + \frac{1}{4} U\ell_\mu U^\dagger r_\nu U\ell_\alpha U^\dagger r_\beta + iU\partial_\mu\ell_\nu\ell_\alpha U^\dagger r_\beta \\ & + i\partial_\mu r_\nu U\ell_\alpha U^\dagger r_\beta - i\Sigma^L_\mu \ell_\nu U^\dagger r_\alpha U\ell_\beta + \Sigma^L_\mu U^\dagger \partial_\nu r_\alpha U\ell_\beta - \Sigma^L_\mu\Sigma^L_\nu U^\dagger r_\alpha U\ell_\beta \\ & + \Sigma^L_\mu \ell_\nu \partial_\alpha \ell_\beta + \Sigma^L_\mu \partial_\nu \ell_\alpha \ell_\beta - i\Sigma^L_\mu \ell_\nu\ell_\alpha\ell_\beta + \frac{1}{2}\Sigma^L_\mu \ell_\nu \Sigma^L_\alpha \ell_\beta - i\Sigma^L_\mu\Sigma^L_\nu\Sigma^L_\alpha\ell_\beta \Big\rangle \\ & - (L \leftrightarrow R) \end{aligned}$$

$$\Sigma^L_\mu = U^\dagger \partial_\mu U \qquad , \qquad \Sigma^R_\mu = U\partial_\mu U^\dagger$$

$\pi^0 \to \gamma\gamma$:



$A_3^\mu \equiv \bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d$

$$\Gamma(\pi^0 \to \gamma\gamma) \; = \; \left(\frac{N_c}{3}\right)^2 \frac{\alpha^2 M_\pi^3}{64\,\pi^3 f_\pi^2} \; = \; 7.73 \;\, \mathrm{eV}$$

Exp: $(7.7 \pm 0.6)\ \mathrm{eV}$

There are no QCD corrections

The chiral anomaly contributes to: $\pi^0 \to \gamma\gamma \;\; , \;\; \eta \to \gamma\gamma$

$\gamma\, 3\pi \;\; , \;\; \gamma\, \pi^+\pi^-\eta \;\; , \;\; K\bar{K}3\pi \;\; , \;\; \cdots$

$\mathcal{O}(p^6)$ χPT

i) $\mathcal{L}_6 = \sum_i C_i \, O_i^{p^6}$ **at tree level** Bijnens-Colangelo-Ecker, Fearing-Scherer

$90+4$ $[53+4]$ terms in SU(3) [SU(2)] χPT (even-intrinsic parity only)

ii) $\mathcal{L}_4$ **at one loop,** $\mathcal{L}_2$ **at two loops** Bijnens-Colangelo-Ecker

Double chiral logarithms

Many Calculations: M_ϕ, f_ϕ, $\gamma\gamma \to \pi\pi$, $\pi\pi \to \pi\pi$, $\pi K \to \pi K$, K_{l4}, $\pi \to e\,\bar{\nu}_e\gamma$, $F_V(s)$, $F_S(s)$, $\Pi_{V,A}(s)$, $\cdots$

Amoros-Bijnens-Dhonte-Talavera, Ananthanarayan-Colangelo-Gasser-Leutwyler, Bellucci-Gasser-Sainio, Bürgui, Bijnens et al, Descotes-Genon et al, Golowich-Kambor, Post-Schilcher…

Theoretical Challenge: **QCD calculation of the χPT couplings**

$K^+ \to \pi^0 \ell^+ \nu_\ell$, $K^0 \to \pi^- \ell^+ \nu_\ell$:

$C_{K^+\pi^0} = \frac{1}{\sqrt{2}}$, $C_{K^0\pi^-} = 1$

$$\langle \pi | \bar{s} \gamma^\mu u | K \rangle = C_{K\pi} \left[(P_K + P_\pi)^\mu \, f_+^{K\pi}(t) + (P_K - P_\pi)^\mu \, f_-^{K\pi}(t) \right]$$

- **Lowest order** $[\mathcal{O}(p^2)]$**:** $f_+^{K\pi}(t) = 1$, $f_-^{K\pi}(t) = 0$

$K^+ \to \pi^0 \ell^+ \nu_\ell\ ,\ K^0 \to \pi^- \ell^+ \nu_\ell$:

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- **Lowest order** $[\mathcal{O}(p^2)]$**:** $\qquad f_+^{K\pi}(t) = 1 \qquad , \qquad f_-^{K\pi}(t) = 0$
- **Ademollo-Gatto Theorem:** $\qquad f_+^{K^0\pi^-}(0) \; = \; 1 \, + \, \mathcal{O}[(m_s - m_u)^2]$

$K^+ \to \pi^0 \ell^+ \nu_\ell$, $K^0 \to \pi^- \ell^+ \nu_\ell$:

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- π^0**–**η **mixing:** $f_+^{K^+\pi^0}(0) \, = \, 1 + \frac{3}{4} \, \frac{m_d - m_u}{m_s - \hat{m}} \, = \, 1.017$

$K^+ \to \pi^0 \ell^+ \nu_\ell\ ,\ K^0 \to \pi^- \ell^+ \nu_\ell$:

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- $\mathcal{O}(\mathbf{p}^4)$**:** $\quad f_+^{K^0\pi^-}(0) \; = \; 0.977 \quad , \quad \frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} \; = \; 1.022$

Gasser-Leutwyler '85

$K^+ \to \pi^0 \ell^+ \nu_\ell$, $K^0 \to \pi^- \ell^+ \nu_\ell$:

$C_{K^+\pi^0} = \frac{1}{\sqrt{2}}$, $C_{K^0\pi^-} = 1$

$$\langle \pi | \bar{s}\gamma^\mu u | K \rangle \; = \; C_{K\pi} \; \left[(P_K + P_\pi)^\mu \; f_+^{K\pi}(t) \, + \, (P_K - P_\pi)^\mu \; f_-^{K\pi}(t) \right]$$

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- **Ademollo-Gatto Theorem:** $f_+^{K^0\pi^-}(0) \, = \, 1 \, + \, \mathcal{O}[(m_s - m_u)^2]$
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- $\mathcal{O}(\mathbf{p}^4)$**:** $f_+^{K^0\pi^-}(0) \, = \, 0.977$, $\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} \, = \, 1.022$

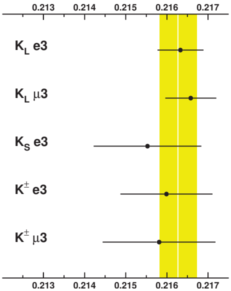
Gasser-Leutwyler '85

Needed to determine $\mathbf{V_{us}}$

$K \to \pi \, \ell \, \nu_\ell$

$$|V_{us} \, f_+(0)| = 0.2165 \pm 0.0004$$

Flavianet Kaon WG, arXiv:1005.2323; Moulson arXiv:1411.5252



$$\langle \pi^- | \bar{s} \gamma_\mu u | K^0 \rangle = (p_\pi + p_K)_\mu \, f_+(t) + (p_K - p_\pi)_\mu \, f_-(t)$$

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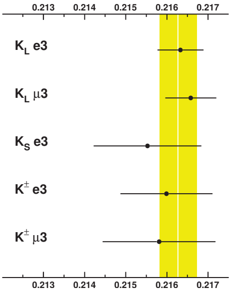

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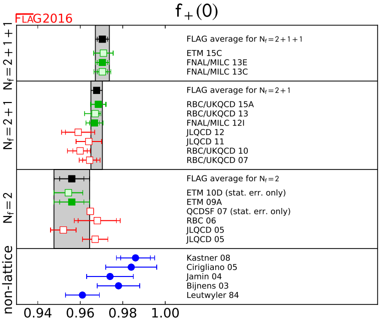

$K \to \pi \ell \nu_\ell$

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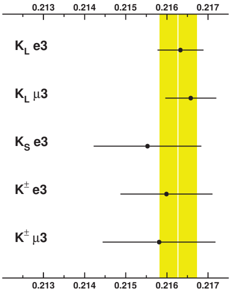
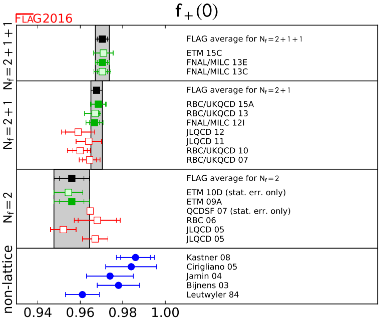


$$\langle \pi^- | \bar{s}\gamma_\mu u | K^0 \rangle = (p_\pi + p_K)_\mu\ f_+(t) + (p_K - p_\pi)_\mu\ f_-(t)$$



$$f_+(0) = 0.970\,(3)$$

$$\Rightarrow \quad |V_{us}| = 0.2232\,(8)$$

$$f_+(0) = 1 + f_2 + f_4 + \cdots$$

Large $\mathcal{O}(p^6)$ χPT correction

Backup Slides

Goldstones and Coset-Space Coordinates: $\mathbf{G} \xrightarrow{\text{SSB}} \mathbf{H}$

Goldstone fields: $\vec{\phi} \equiv (\phi_1, \ldots, \phi_N) \quad \longrightarrow \quad \vec{\phi}\,' = \vec{\mathcal{F}}(g, \vec{\phi}) \quad , \quad g \in G$

$$N = \dim(G) - \dim(H) \qquad , \qquad \vec{\mathcal{F}}(e, \vec{\phi}) = \vec{\phi} \qquad , \qquad \vec{\mathcal{F}}(g_1 g_2, \vec{\phi}) = \vec{\mathcal{F}}\left(g_1, \vec{\mathcal{F}}(g_2, \vec{\phi})\right)$$

Goldstones and Coset-Space Coordinates: $G \xrightarrow{\text{SSB}} H$

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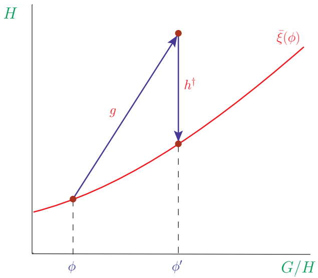
$\tilde{\mathcal{F}}$: invertible mapping between Goldstone fields and G/H



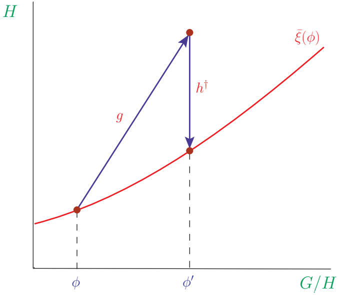
$$\vec{\mathcal{F}}(gh, \vec{0}) = \vec{\mathcal{F}}(g, \vec{0}) \qquad \forall g \in G\,,\ \forall h \in H$$

$$\vec{\mathcal{F}}(h, \vec{0}) = \vec{0} \quad , \quad h \in H \qquad \text{(vacuum invariant)}$$

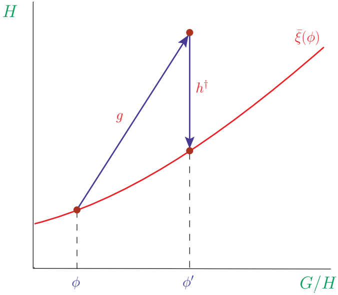
$$\vec{\mathcal{F}}(g_i, \vec{0}) = \vec{\mathcal{F}}(g_j, \vec{0}) \quad \longrightarrow \quad g_i^{-1} g_j \in H$$

Coset representative: $\bar{\xi}(\phi) \in G$

Coset Space Coordinates: $G \equiv SU(3)_L \otimes SU(3)_R \xrightarrow{SCSB} H \equiv SU(3)_V$



Coset Space Coordinates: $G \equiv SU(3)_L \otimes SU(3)_R \xrightarrow{SCSB} H \equiv SU(3)_V$

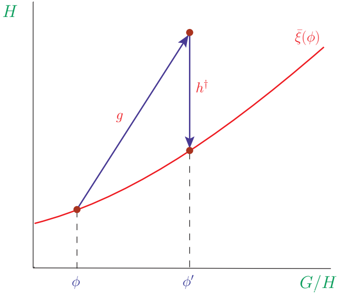


$$\bar{\xi}(\phi) \equiv (\xi_L(\phi), \xi_R(\phi)) \in G$$

$$\xi_L(\phi) \xrightarrow{G} g_L\, \xi_L(\phi)\, h^\dagger(\phi, g)$$

$$\xi_R(\phi) \xrightarrow{G} g_R\, \xi_R(\phi)\, h^\dagger(\phi, g)$$

Coset Space Coordinates: $G \equiv SU(3)_L \otimes SU(3)_R \stackrel{SCSB}{\longrightarrow} H \equiv SU(3)_V$



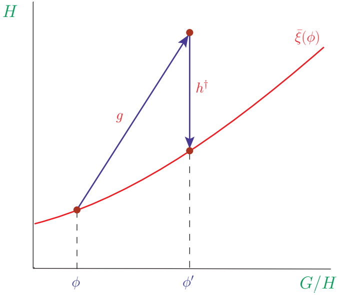
$$\bar{\xi}(\phi) \equiv (\xi_L(\phi), \xi_R(\phi)) \in G$$

$$\xi_L(\phi) \xrightarrow{G} g_L\, \xi_L(\phi)\, h^\dagger(\phi, g)$$

$$\xi_R(\phi) \xrightarrow{G} g_R\, \xi_R(\phi)\, h^\dagger(\phi, g)$$

$$\mathbf{U}(\phi) \equiv \xi_R(\phi)\, \xi_L^\dagger(\phi) \xrightarrow{G} g_R\, \mathbf{U}(\phi)\, g_L^\dagger$$

Coset Space Coordinates: $G \equiv SU(3)_L \otimes SU(3)_R \xrightarrow{SCSB} H \equiv SU(3)_V$



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$$\mathbf{U}(\phi) \equiv \xi_R(\phi)\, \xi_L^\dagger(\phi) \xrightarrow{G} g_R\, \mathbf{U}(\phi)\, g_L^\dagger$$

Canonical choice:

$$\xi_R(\phi) = \xi_L(\phi)^\dagger \equiv \mathbf{u}(\phi) \quad \xrightarrow{G} \quad g_R\, \mathbf{u}(\phi)\, h^\dagger(\phi, g) = h(\phi, g)\, \mathbf{u}(\phi)\, g_L^\dagger$$

$$\mathbf{U}(\phi) = \mathbf{u}(\phi)^2 = \exp\left\{ i\, \frac{\sqrt{2}}{f}\, \mathbf{\Phi} \right\}$$