Exercise 1: Slow growth of fluctuations during radiation domination

The equation governing the growth of density fluctuations for non-relativistic matter in a spatially flat FRW geometry is

$$\ddot{\delta}_{\mathbf{k}} + 2H\,\dot{\delta}_{\mathbf{k}} + \left(\frac{c_s^2\mathbf{k}^2}{a^2} - 4\pi G\rho_{m0}\right)\delta_{\mathbf{k}} = 0\,, \tag{1}$$

where $\delta = \delta\rho_m/\rho_{m0}$ is the fractional fluctuation in the matter density, $\mathbf{k}$ is its Fourier label while a is the scale factor and $H = \dot{a}/a$ and so $H^2 = 8\pi G\rho_0/3$.

For a matter-dominated universe, for which $\rho_0 \simeq \rho_{m0} \propto 1/a^3$ and $a \propto t^{2/3}$ show that as $c_s\mathbf{k} \to 0$ eq (1) gives power-law solutions of the form $\delta_0 \propto t^n$ with $n = \frac{2}{3}$ or $n = -1$. (The growing mode verifies the claim in class that $\delta_0 \propto a$ during matter domination.)

Consider now the transition between radiation and matter domination, for which $\rho_0 = \rho_{m0} + \rho_{r0}$ and so

$$H^2(a) = \frac{8\pi G\rho_0}{3} = \frac{H_{\rm eq}^2}{2}\left[\left(\frac{a_{\rm eq}}{a}\right)^3 + \left(\frac{a_{\rm eq}}{a}\right)^4\right]\,, \tag{2}$$

where radiation-matter equality occurs when $a = a_{\rm eq}$, at which point $H(a = a_{\rm eq}) = H_{\rm eq}$. The matter part of this expansion comes from

$$H_m^2 := \frac{8\pi G\rho_{m0}}{3} = \frac{H_{\rm eq}^2}{2}\left(\frac{a_{\rm eq}}{a}\right)^3\,. \tag{3}$$

Verify that $\delta_0(x)$ satisfies

$$2x(1+x)\,\delta_0'' + (3x+2)\,\delta_0' - 3\,\delta_0 = 0\,, \tag{4}$$

where the scale factor, $x = a/a_{\rm eq}$, is used as a proxy for time and primes denote differentiation with respect to x. Show that this is solved by $\delta_0 \propto \left(x + \frac{2}{3}\right)$, and thereby show how the growing mode during matter domination does not grow during radiation domination. (*Bonus:* show that the linearly independent solutions to this one only grow logarithmically with x deep in the radiation-dominated era, for which $x \ll 1$.)

Exercise 2:
Calculation of vacuum energy for a scalar field in a static spacetime

There are a variety of ways commonly used to compute quantum corrections to the vacuum energy, and this tutorial is meant to show how they are related. For the purposes of the exercise a free real scalar field is used, with action $S = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L}$ and Lagrangian $L = \int \mathrm{d}^3x\, \mathcal{L}$. The Lagrangian density is

$$\mathcal{L} = -\sqrt{-g}\ \left[\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}\,m^2\,\phi^2\right]\,, \tag{5}$$

and the resulting field equation is the Klein-Gordon equation

$$(-\Box + m^2)\phi = (-g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2)\phi = 0\,. \tag{6}$$

But the relationship between the calculations described below is more general than just for this one example.

Canonical calculation

The simplest approach to calculating the vacuum energy is the same calculation that identifies all of the energy eigenstates and eigenvalues. This starts by assuming a static background spacetime with metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$, for which g_{ij} is time-independent and a conserved energy can be formulated. Using the above action the field's canonical momentum is

$$\pi(x) = \frac{\delta S}{\delta\dot\phi(x)} = \sqrt{-g}\ \dot\phi(x)\,, \tag{7}$$

(where an over-dot as in $\dot\phi$ denotes ∂_t) and so the Hamiltonian density is

$$\mathcal{H} = \pi\,\dot\phi - \mathcal{L} = \frac{\pi^2}{2\sqrt{-g}} + \frac{1}{2}\sqrt{-g}\,\left[g^{ij}\,\nabla_i\phi\,\nabla_j\phi + m^2\,\phi^2\right]. \tag{8}$$

Background about quantization and mode functions

Because this is quadratic in the fields it is essentially a fancy harmonic oscillator. To diagonalize it we expand the fields in terms of creation and annihilation operators

$$\phi(x) = \sum_n \Big[a_n\,U_n(x) + a_n^\star\,U_n^*(x)\Big]\,, \tag{9}$$

where we choose the mode functions, $U_n(x)$, to be simultaneous eigenstates of $-g^{ij}\nabla_i\nabla_j$ and $i\partial_t$. That is they satisfy the Klein-Gordan equation, $(-\Box + m^2)U_n = 0$, in a basis that also satisfies

$$-g^{ij}\nabla_i\nabla_j U_n(x) = \omega_n^2 U_n(x) \qquad \text{and} \qquad i\dot U_n = \varepsilon_n\,U_n(x)\,, \tag{10}$$

for eigenvalues ω_n^2 and ε_n. The Klein-Gordon equation imposes a relation between these eigenvalues since $-\Box U_n = \ddot U_n - g^{ij}\nabla_i\nabla_j U_n = (-\varepsilon_n^2 - g^{ij}\nabla_i\nabla_j)U_n$ and so

$$\Big(-g^{ij}\nabla_i\nabla_j + m^2\Big)U_n = (\omega_n^2 + m^2)U_n = \varepsilon_n^2\,U_n\,. \tag{11}$$

This shows how $\varepsilon_n^2 = \omega_n^2 + m^2$ gets determined by the spectrum of $g^{ij}\nabla_i\nabla_j$ for the spacetime of interest.

So we may write

$$U_n(\mathbf{x}, t) = \frac{1}{\sqrt{2\varepsilon_n}}\, u_n(\mathbf{x})\, e^{-i\varepsilon_n t}\,, \tag{12}$$

where the prefactor is chosen for later convenience. Similarly

$$\pi(x) = \sqrt{-g}\;\dot\phi = -i\sqrt{-g}\;\sum_n \varepsilon_n \Big[a_n\, U_n(x) - a_n^\star\, U_n^*(x)\Big]\,. \tag{13}$$

The covariant normalization condition for the modes is defined using the Wronskian by

$$\begin{aligned} W_\Sigma(U_n, U_m) &:= -i\int_\Sigma \mathrm{d}^3\mathbf{x}\sqrt{-g}\;\Big[\dot U_n^*(x)\, U_m(x) - U_n^*(x)\, \dot U_m(x)\Big] \\ &= \int_\Sigma \mathrm{d}^3\mathbf{x}\sqrt{-g}\;\Big[\varepsilon_n U_n^*\, U_m + \varepsilon_m U_n^*\, U_m\Big] = \delta_{mn}\,, \end{aligned} \tag{14}$$

where Σ is a slice of fixed t. Similarly, because (10) tell us $i\dot U_n^* = -\varepsilon_n U_n^*$, we see that U_n^* and U_n are eigenstates for different energy eigenvalues (notice for $m \neq 0$ there are no zero eigenvalues), and so are also orthogonal

$$\begin{aligned} W_\Sigma(U_n^*, U_m) &:= -i\int_\Sigma \mathrm{d}^3\mathbf{x}\sqrt{-g}\;\Big[\dot U_n(x)\, U_m(x) - U_n(x)\, \dot U_m(x)\Big] \\ &= (\varepsilon_m - \varepsilon_n)\int_\Sigma \mathrm{d}^3\mathbf{x}\sqrt{-g}\; U_n\, U_m = 0\,. \end{aligned} \tag{15}$$

It doesn't matter which t we choose for W when evaluating these orthogonality conditions provided the falloff of U_n is sufficiently good at spatial infinity (if this exists), and this is the point of why W is defined the way it is. To see why notice $(-\Box + m^2)U_n = 0$ implies the following chain of equalities

$$\begin{aligned} 0 &= -i\int_\Sigma^{\Sigma'} \mathrm{d}^4x\;\sqrt{-g}\Big[[(-\Box + m^2)U_n]^* U_m - U_n^*[(-\Box + m^2)U_m]\Big] \\ &= i\int_\Sigma^{\Sigma'} \mathrm{d}^4x\;\sqrt{-g}\;\nabla_\mu\Big[(\nabla^\mu U_n)^* U_m - U_n^*(\nabla^\mu U_m)\Big] \\ &= i\oint_\Sigma^{\Sigma'} \mathrm{d}^3x\;\sqrt{-g}\;n_\mu\Big[(\nabla^\mu U_n)^* U_m - U_n^*(\nabla^\mu U_m)\Big] \\ &= W_\Sigma(U_n, U_m) - W_{\Sigma'}(U_n, U_m)\,. \end{aligned} \tag{16}$$

Here the integration in the first line is over a slab of spacetime lying between two constant-t slices, Σ and Σ'. The second line then integrates both terms by parts and the third line uses Gauss' theorem to write the result in terms of a surface integral over the boundaries of the spacetime region of interest, with n_μ being the outward-pointing normal. If there are no spatial boundaries (or if the boundary conditions are chosen at spatial infinity appropriately) then the only boundaries contributing to the integrals are Σ and Σ'. Then $n_\mu \mathrm{d}x^\mu = \pm\mathrm{d}t$ for the two surfaces, and the last line follows by recognizing that the surface integrals are precisely the Wronskians for each of the bounding constant-t surfaces. Comparing first and last lines shows that $W_\Sigma(U_n, U_m)$ does not depend on Σ.

Given the above conventions and normalization condition, completeness of the modes implies

$$\sum_n u_n(\mathbf{x})\, u_n^*(\mathbf{y}) = \frac{\delta^3(\mathbf{x},\mathbf{y})}{[-g(\mathbf{x})]^{1/4}[-g(\mathbf{y})]^{1/4}}\,, \tag{17}$$

where the delta function transforms as a bi-density distribution that vanishes when $\mathbf{x} \neq \mathbf{y}$ and satisfies the defining condition

$$f(\mathbf{x},t) = \int \mathrm{d}^3\mathbf{y}\; \delta^3(\mathbf{x},\mathbf{y})\, f(\mathbf{y},t) \tag{18}$$

for all f without any metrics. The completeness condition is related to the normalization condition because multiplying (17) by $\sqrt{-g(\mathbf{y})}\, u_m(\mathbf{y})$ and integrating over $\mathbf{y}$ must give the tautology $u_m(\mathbf{x}) = u_m(\mathbf{x})$, which it does but only because the u_m's are orthogonal and (14) implies each mode satisfies the normalization condition

$$\int_\Sigma \mathrm{d}^3\mathbf{x}\sqrt{-g}\; u_n^*(\mathbf{x}) u_n(\mathbf{x}) = 2\varepsilon_n \int_\Sigma \mathrm{d}^3\mathbf{x}\sqrt{-g}\; U_n^*(\mathbf{x}) U_n(\mathbf{x}) = 1\,. \tag{19}$$

Finally, the harmonic oscillator (or creation and annihilation) operator algebra is equivalent to the canonical quantization conditions because

$$[a_n, a_m] = 0 \qquad \text{and} \qquad [a_n, a_m^\star] = \delta_{nm}\,, \tag{20}$$

imply

$$\begin{aligned} [\phi(x), \phi(y)] &= \sum_{nm} \Big\{ U_n(x)\, U_m^*(y)[a_n, a_m^\star] + U_n^*(x) U_m(y)[a_n^\star, a_m] \Big\} \\ &= \sum_n \frac{1}{2\varepsilon_n} \Big\{ u_n(\mathbf{x})\, u_n^*(\mathbf{y}) - u_n^*(\mathbf{x}) u_n(\mathbf{y}) \Big\} = 0\,, \end{aligned} \tag{21}$$

and

$$\begin{aligned} [\pi(\mathbf{x},t), \phi(\mathbf{y},t)] &= -i\sqrt{-g(\mathbf{x})}\; \sum_{nm} \varepsilon_n \Big\{ U_n(x)\, U_m^*(y)[a_n, a_m^\star] - U_n^*(x) U_m(y)[a_n^\star, a_m] \Big\} \\ &= -\frac{i}{2}\sqrt{-g(\mathbf{x})}\; \sum_n \Big\{ u_n(\mathbf{x})\, u_n^*(\mathbf{y}) + u_n^*(\mathbf{x}) u_n(\mathbf{y}) \Big\} \qquad (22) \\ &= -i\frac{[-g(\mathbf{x})]^{1/4}}{[-g(\mathbf{y})]^{1/4}}\; \delta^3(\mathbf{x},\mathbf{y}) = -i\,\delta^3(\mathbf{x},\mathbf{y})\,. \qquad (23) \end{aligned}$$

Calculation of the energy eigenvalues and eigenstates

1. The point of the above is that the energy is diagonal when expressed in terms of the eigenstates of $a_n^\star a_n$, as we see by evaluating the Hamiltonian in terms of a_n and $a_n^\star$. To

this end write

$$H = \int \mathrm{d}^3\mathbf{x}\, \mathcal{H} = \int \mathrm{d}^3\mathbf{x} \left\{ \frac{\pi^2}{2\sqrt{-g}} + \frac{1}{2}\sqrt{-g}\, \Big[g^{ij}\, \nabla_i\phi\, \nabla_j\phi + m^2\, \phi^2 \Big] \right\}$$
$$= \int \mathrm{d}^3\mathbf{x} \left\{ \frac{\pi^2}{2\sqrt{-g}} + \frac{1}{2}\sqrt{-g}\, \phi \Big[-g^{ij}\, \nabla_i \nabla_j\phi + m^2\, \phi \Big] \right\} , \qquad (24)$$

and show that it can be written

$$H = \frac{1}{2} \sum_n \varepsilon_n \Big(a_n^\star a_n + a_n a_n^\star \Big) . \qquad (25)$$

2. The previous question shows that H is diagonal in the basis for which the operators $a_n^\star a_n$ are diagonal for all n. Show this by using the commutation relation $[a_n, a_m^\star] = \delta_{nm}$ to rewrite H as

$$H = E_0 + \sum_n \varepsilon_n\, a_n^\star a_n , \qquad (26)$$

with the constant E_0 being formally written as

$$E_0 = \frac{1}{2} \sum_n \varepsilon_n . \qquad (27)$$

This expression is 'formal' because the sum typically diverges. It can be regularized in many ways (and you might reasonably wonder whether or not physical results depend on which way is used). One such is zeta-function regularization, which defines

$$\zeta(s) := \sum_n \varepsilon^{-s} , \qquad (28)$$

for complex s. This often converges where the real part of s is sufficiently large and positive, and one tries to analytically extend this result down to the desired result $E_0 = \zeta(-1)$. Another way to proceed is instead to differentiate E_0 sufficiently many times with respect to m^2 that the sum converges, and then integrate the sum again to get E_0,

The energy eigenvalues for H are clearly given by $H|\{N_k\}\rangle = E|\{N_k\}\rangle$ with

$$E = E_0 + \sum_n N_n \varepsilon_n , \qquad (29)$$

where the next exercise shows the allowed values for the N_n are $N_n = 0, 1, 2, \cdots$. The state $|0\rangle$ denotes the ground state (or vacuum) for which $N_n = 0$ for all n and has eigenvalue

$$H|0\rangle = E_0|0\rangle . \qquad (30)$$

3. The basis diagonalizing $a_n^\star a_n$ for all n is called the 'occupation-number' basis and denoted $|\{N_k\}\rangle = |N_{n_1}, N_{n_2}, N_{n_3}, \cdots\rangle$ where the labels N_n are the eigenvalues for $a_n^\star a_n$, for all possible values taken by n. That is, they satisfy

$$a_n^\star a_n |\{N_k\}\rangle = N_n |\{N_k\}\rangle . \qquad (31)$$

Prove that the $N_n = 0, 1, 2, \cdots$ are non-negative integers as follows. First prove $[a_n^\star a_n, a_m] = -\delta_{nm} a_n$ and $[a_n^\star a_n, a_m^\star] = +\delta_{nm} a_n^\star$. Show that these relations imply that if $|\{N_k\}\rangle$ is an eigenstate with eigenvector of $a_n^\star a_n$ with eigenvalue N_n then $a_n|\{N_k\}\rangle$ is also an eigenstate of $a_n^\star a_n$ but with eigenvalue $N_n - 1$ and $a_n^\star|\{N_k\}\rangle$ is an eigenvector with eigenvalue $N_n + 1$.

Next prove $N_n \geq 0$ by evaluating $\langle\{N_k\}|a_n^\star a_n|\{N_k\}\rangle = N_n\langle\{N_k\}|\{N_k\}\rangle = N_n$ and recognizing that the left-hand side is non-negative because it is the norm of the vector $a_n|\{N_k\}\rangle$. But this is inconsistent with the result that a_n always lowers the eigenvalue by one unit unless there exists an eigenstate for which $a_n|\Psi\rangle = 0$. Repeating this argument for all labels n shows there must be a state, $|0\rangle$, for which $a_n|0\rangle = 0$ for all n, and then all other eigenstates of $a_n^\star a_n$ are obtained by acting repeatedly on $|0\rangle$ with $a_n^\star$. (For example consider the particular state $|2_{n_5}, 6_{n_{20}}\rangle$, for which the particle state labelled by n_5 has eigenvalue $N_{n_5} = 2$ for $a_{n_5}^\star a_{n_5}$ and the state labelled by n_{20} has eigenvalue $N_{n_{20}} = 6$ for $a_{n_{20}}^\star a_{n_{20}}$. This is proportional to $(a_{n_5}^\star)^2\ (a_{n_{20}}^\star)^6|0\rangle$, and so on for any other choices for these eigenvalues.)

Path integral method of evaluating the vacuum energy

An alternate way to proceed instead uses the path integral formulation for the effective action

$$e^{i\Gamma[g]} = \int \mathcal{D}\phi\, e^{iS[\phi,g]}\,, \tag{32}$$

where the action $S[\phi, g]$ is given as the integral over (5), regarded as a function of the fields ϕ and $g_{\mu\nu}$. In this expression $\Gamma[g]$ is a contribution to the action for the metric, $g_{\mu\nu}$, obtained after integrating out the field ϕ. It is to be added to other terms (like the Einstein-Hilbert term), but our interest is in anything of the form $\Gamma = -\int \mathrm{d}^4x\, \sqrt{-g}\, \rho_v$, because this gravitates like a cosmological constant (or vacuum energy). For time-translational invariant systems the integral over t diverges proportional to $\int_{-T}^{T} \mathrm{d}t = T$ as $T \to \infty$ and it is the energy $E_0 = -\Gamma/T$ that should remain finite in this limit.

Because the functional integral is Gaussian it can be evaluated in terms of a functional determinant of the quadratic operator appearing in the action: $\Delta = (-\Box + m^2)\delta^4(x-y)$.

$$e^{i\Gamma} = \Big[\det\Big(-\Box + m^2 - i\epsilon\Big)\Big]^{-1/2}\,, \tag{33}$$

and so

$$\Gamma = \frac{i}{2}\ln\det\Big(-\Box + m^2 - i\epsilon\Big) = \frac{i}{2}\mathrm{Tr}\,\ln\Big(-\Box + m^2 - i\epsilon\Big)\,. \tag{34}$$

Here ϵ is a positive quantity that is taken to zero at the end, and imposes (as usual for a Feynman propagator) the right boundary conditions to describe matrix elements in the vacuum. We suppress the $i\epsilon$ in what follows, but recall it when needed by regarding m^2 as having a small negative imaginary part.

To evaluate this again choose eigenfunctions that diagonalize $-g^{ij}\nabla_i\nabla_j$ and $i\partial_t$. That is choose a basis of functions, $V_n(x)$, for which

$$-g^{ij}\nabla_i\nabla_j V_n = \omega_n^2 V_n \quad \text{and} \quad i\partial_t V_n = \varepsilon V_n\,, \tag{35}$$

and so

$$(-\Box + m^2)V_n = (-\varepsilon^2 + \omega_n^2 + m^2)V_n \tag{36}$$

is diagonalized with eigenvalues $\lambda_n = -\varepsilon^2 + \omega_n^2 + m^2 = -\varepsilon^2 + \varepsilon_n^2$. Notice that unlike the previous section we do not also have $(-\Box + m^2)U_n = 0$ and so we *cannot* identify ε^2 with $\varepsilon_n^2 := \omega_n^2 + m^2$. Instead ε is the Fourier transform variable for time, arising generically for time-translationally invariant systems.

In terms of this our operator in this basis is

$$\langle n\varepsilon|\Delta|r\varepsilon'\rangle = (-\varepsilon^2 + \omega_n^2 + m^2)\; 2\pi\delta(\varepsilon - \varepsilon')\delta_{nr}\,, \tag{37}$$

and so the trace may be given by taking diagonal elements and summing over their eigenvalues, with

$$\Gamma(m^2) = \frac{i}{2}\mathrm{Tr}\,\ln\Big(-\Box + m^2\Big) = \frac{i}{2}\sum_n \int_{-\infty}^{\infty} \frac{\mathrm{d}\varepsilon}{2\pi}\; \ln(-\varepsilon^2 + \omega_n^2 + m^2)\; 2\pi\delta(0)\,. \tag{38}$$

The factor of $\delta(0)$ arises due to time translation invariance, as may be seen by writing

$$2\pi\delta(E) = \lim_{T\to\infty}\int_{-T}^{T} \mathrm{d}t\; e^{-i\varepsilon t} \quad \text{and so} \quad 2\pi\delta(0) = \lim_{T\to\infty} T\,, \tag{39}$$

and so the well-behaved quantity is the energy

$$E_0 = -\lim_{T\to\infty}\frac{\Gamma}{T} = -\frac{i}{2}\sum_n \int_{-\infty}^{\infty} \frac{\mathrm{d}\varepsilon}{2\pi}\; \ln(-\varepsilon^2 + \omega_n^2 + m^2)\,. \tag{40}$$

Again the remaining sums and integrals diverge. The integration over ε passes through singularities at $\pm\varepsilon_n$, which we should navigate by Wick rotating. That is, keeping in mind (as usual) that $m^2 \to m^2 - i\epsilon$ is required for the Feynman propagator, we can rotate our contour of integration counter-clockwise by 90 degrees in the complex ε plane by writing $\varepsilon \to i\,\varepsilon_E$ with ε_E also running from $-\infty$ to ∞. The integral converges if we first differentiate with respect to m^2, so show that

$$\begin{aligned} \frac{\partial E_0}{\partial m^2} &= \frac{1}{2}\sum_n \int_{-\infty}^{\infty} \frac{\mathrm{d}\varepsilon_E}{2\pi}\left(\frac{1}{\varepsilon_E^2 + \omega_n^2 + m^2}\right) \\ &= \frac{1}{4\pi}\sum_n \left[\frac{1}{\varepsilon_n}\,\arctan\left(\frac{\varepsilon}{\varepsilon_n}\right)\right]_{-\infty}^{\infty} = \frac{1}{4}\sum_n \frac{1}{\varepsilon_n}\,, \end{aligned} \tag{41}$$

where, as above, $\varepsilon_n = \sqrt{\omega_n^2 + m^2}$. Integrating again with respect to m^2 then gives

$$E_0(m^2) = \frac{1}{2}\sum_n \varepsilon_n\,, \tag{42}$$

up to an arbitrary m^2-independent constant. This is the same sum as was obtained in the canonical calculation earlier.

Flat space evaluation

As a particularly simple case consider the case of a flat geometry, for which $-g^{ij}\nabla_i\nabla_j = -\nabla^2$ can be diagonalized in Fourier space, with eigenfunctions $\exp(i\mathbf{p}\cdot\mathbf{x})$ and eigenvalues $\mathbf{p}^2$.

In terms of this the required operator in this basis is

$$\langle p|\Delta|q\rangle = (p_\mu p^\mu + m^2)\,(2\pi)^4\delta^4(p-q)\,, \tag{43}$$

and so the trace may be given by taking diagonal elements and summing over their eigenvalues, with

$$\Gamma(m^2) = \frac{i}{2}\mathrm{Tr}\,\ln\Big(-\Box + m^2\Big) = \frac{i}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4 p}{(2\pi)^4}\;\ln(p_\mu p^\mu + m^2)\,(2\pi)^4\delta^4(0)\,. \tag{44}$$

The additional factor of $\delta^3(0)$ arises due to spatial translation invariance, as may be seen by writing (as we did before for time)

$$(2\pi)^3\delta^3(\mathbf{p}) = \lim_{L\to\infty}\int_{-L}^{L}\mathrm{d}^3\mathbf{x}\;e^{i\mathbf{p}\cdot\mathbf{x}} \quad\text{and so}\quad (2\pi)^3\delta^3(0) = \lim_{L\to\infty} L^3\,, \tag{45}$$

and so is proportional to the volume of space (as well as the previous proportionality to T). The well-behaved quantity for infinite translationally invariant systems is therefore the energy *density*,

$$\rho_v = \lim_{L\to\infty}\frac{E_0}{L^3} = -\lim_{L,T\to\infty}\frac{\Gamma}{TL^3} = -\frac{i}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4 p}{(2\pi)^4}\;\ln(p_\mu p^\mu + m^2)\,. \tag{46}$$

To avoid the singularities at $p^0 = \pm\sqrt{\mathbf{p}^2+m^2}$, we again Wick rotate. In the resulting euclidean integral the angular integrals can be done once and for all, giving a factor of the volume of the unit 3-sphere: $2\pi^2$. The remaining integral converges if we first differentiate with respect to m^2 thrice, so show that

$$\left(\frac{\partial}{\partial m^2}\right)^3\rho_v = \frac{2\pi^2}{2}\int_0^{\infty}\frac{p_E^3\mathrm{d}p_E}{(2\pi)^4}\;\frac{2}{(p_E^2+m^2)^3} = \frac{1}{32\pi^2 m^2}\,, \tag{47}$$

and so integrating three times with respect to m^2 then gives

$$\rho_v = \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{\mu^2}\right) + Am^4 + Bm^2 + C\,, \tag{48}$$

where μ, A, B and C are arbitrary m^2-independent constants. Although the values of A, B and C can depend on how the integrals were regulated, the logarithmic term cannot.

Exercise 3:
Quantum fluctuations of a scalar field in a class of inflationary spacetimes

For a change of pace we work in the Schrödinger picture, rather than the Heisenberg picture, and so compute the vacuum wavefunctional, $\Psi[\varphi,\, t]$, for a scalar field.

Action and hamiltonian

Our starting point is the lagrangian density for a spectator scalar

$$L = \int \mathrm{d}^3x\, a(t)^3 \left[\frac{1}{2}\,\dot{\phi}^2 - \frac{1}{2\,a^2(t)}\,(\nabla\phi)^2 - \frac{m^2(t)}{2}\,\phi^2\right] , \tag{49}$$

in an FRW spacetime with metric

$$\mathrm{d}s^2 \;=\; -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2 \tag{50}$$

and Hubble paramer $H(t) \;=\; \dot{a}/a$. Here $m(t)$ denotes the (possibly slowly time-dependent) mass.

Find the canonical momentum, π_k, for each Fourier mode, φ_k, of the scalar field. Given the quantization condition $\pi_k = -i\delta/\delta\varphi_k$, show that the Hamiltonian density in Schrödinger representation can be expressed in Fourier space as

$$\mathcal{H} = \mathcal{H}_0 + \sum_k \mathcal{H}_k\,, \tag{51}$$

with $\mathcal{H}_k$ for $k \neq 0$ given by

$$\mathcal{H}_k \;=\; -\frac{1}{a^3}\,\frac{\delta^2}{\delta\varphi_k\,\delta\varphi_{-k}} + a^3\,\left[\frac{c_s^2\,k^2}{a^2} + m^2\right]\,\varphi_k\varphi_{-k} \tag{52}$$

where $\varphi^*_k \;=\; \varphi_{-k}$.

Ground state wave functional

Use this Hamiltonian to evolve the state wave-functional, $\Psi \;=\; \prod_k \Psi_k$, according to the Schrödinger equation,

$$i\,\frac{\partial\Psi_k}{\partial t} \;=\; \mathcal{H}_k\,\Psi_k\,, \tag{53}$$

and for free fields seek solutions subject to a gaussian ansatz,

$$\Psi[\varphi] = \prod_k \Psi_k[\varphi] \;=\; \prod_k \mathcal{N}_k(t)\,\exp\Big\{-a^3(t)\,\Big[\,\alpha_k(t)\,\varphi_k\,\varphi_{-k}\Big]\Big\} \tag{54}$$

and show that the variance of φ_k, $\langle|\varphi_k|^2\rangle$, is given by $[a^3(\alpha_k+\alpha^*_k)]^{-1}$. Determine the evolution equations for the functions $\mathcal{N}_k(t)$, $\alpha_k(t)$ by substituting into (53). Show that they imply α_k must satisfy

$$0 = \dot{\alpha}_k + i\,\alpha_k^2 + 3\,H\,\alpha_k - i\left(\frac{k^2}{a^2} + m^2\right) \qquad \text{for } k \geq 0 \tag{55}$$

where all quantities (including the Hubble parameter) can be time dependent, and the dot denotes derivative with respect to time. The additional equation for $\mathcal{N}_k$ ensures it evolves in a way that is consistent with normalization, but is not needed in what follows.

The solution for α_k can be made very explicit if we assume power-law expansion, $a = a_0(t/t_0)^p$ (so that $H = p/t$ and $\epsilon = -\dot{H}/H^2 = 1/p$) and a time-independent ratio m/H. (Show that de Sitter space can be obtained as the special case where $p \to \infty$ and so $\epsilon \to 0$.)

Equations of the form of (55) are integrated by changing variables from α_k to u_k where

$$\alpha_k = -i\left(\frac{\dot{u}_k}{u_k}\right) = i\,a\,H\left[\frac{\partial_a\,u_k(a)}{u_k(a)}\right]. \tag{56}$$

Show that (55) is then satisfied if u_k solves the Klein-Gordon equation,

$$\ddot{u}_k + 3H\,\dot{u}_k + \left(\frac{k^2}{a^2} + m^2\right)u_k = 0\,. \tag{57}$$

For constant ϵ and m^2/H^2 show that this is solved by

$$u_k(a) = \tilde{\mathcal{C}}_k\,y^q\,\sigma_k(y), \tag{58}$$

where $\tilde{\mathcal{C}}_k$ is a-independent, provided q and y are chosen as

$$q = \frac{3-\epsilon}{2\,(1-\epsilon)}\,, \tag{59}$$

and

$$y(a,k) := \frac{1}{(1-\epsilon)}\left(\frac{k}{aH}\right) = \frac{1}{(1-\epsilon)}\left(\frac{k}{a_0H_0}\right)\left(\frac{a_0}{a}\right)^{1-\epsilon}. \tag{60}$$

The point of these changes of variables is that they turn eq. (57) into the Bessel equation for σ_k:

$$y^2\,\sigma_k'' + y\,\sigma_k' + \left(y^2 - \nu^2\right)\,\sigma_k = 0\,, \tag{61}$$

where primes here denote derivatives with respect to y. Show that the order ν is given by

$$\nu^2 = \frac{1}{(1-\epsilon)^2}\left[\frac{(3-\epsilon)^2}{4} - \frac{m^2}{H^2}\right]. \tag{62}$$

The solutions for σ_k are (naturally) Bessel functions, and demanding agreement with the adiabatic vacuum before horizon exit tells us

$$u_k \propto \exp\left[\mp i\int \mathrm{d}t\left(\frac{k}{a}\right)\right] \propto e^{\pm iy} \qquad \text{for } k/a \gg H\,, \tag{63}$$

of which we choose the lower sign since this turns out below to ensure the real part of α_k is positive (as required to ensure Ψ_k can be normalized). Show that this fixes the mode functions to be

$$u_k(a) = \tilde{\mathcal{C}}_k\,y^q(a,k)\,H_\nu^{(2)}\left[y(a,k)\right] = \frac{\mathcal{C}_k}{\sqrt{a^3H}}\,H_\nu^{(2)}\left[y(a,k)\right] \tag{64}$$

where $\mathcal{C}_k \propto k^q \tilde{\mathcal{C}}_k$ relabels the integration constants and $H_\nu^{(2)}$ is the Hankel function of the second kind. The second equality in (64) follows from eq. (59), which implies $a^3 H y^{2q}$ is time-independent. Notice this reduces to the solution for a massive field in de Sitter space in the limit $\epsilon \to 0$.

Although $\mathcal{C}_k$ drops out of (56) and (so does not contribute directly to α_k), some later formulae are simpler if we choose $\mathcal{C}_k$ so that the Wronskian,

$$\mathcal{W}(u,v) := a^3(u^* \dot{v} - v^* \dot{u}) \,, \tag{65}$$

satisfies $\mathcal{W}(u,u) = i$. Prove that in this case is the expression for the real and imaginary parts of α_k become

$$\alpha_k + \alpha_k^* = -i \left(\frac{u_k^* \dot{u}_k - u_k \dot{u}_k^*}{|u_k|^2} \right) = \frac{1}{a^3 \, |u_k|^2} \tag{66}$$

$$\text{and} \quad \alpha_k - \alpha_k^* = -i \, a \, H \left[\frac{\partial_a \left(|u_k|^2 \right)}{|u_k|^2} \right] . \tag{67}$$

What does the first of these imply for the variance of φ_k in terms of u_k?

Because $\mathcal{W}$ is independent of time (when evaluated with solutions to (57)) it is convenient to compute the implications for $\mathcal{C}_k$ in the remote past, where $k \gg aH$, in which case the Hankel function has the asymptotic form

$$H_\nu^{(2)}(y) \to \sqrt{\frac{2}{\pi y}} \, e^{-iy + \frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)} \quad \text{for } y \to \infty \,. \tag{68}$$

Use this to show

$$|\mathcal{C}_k|^2 = \frac{\pi}{4(1-\epsilon)} \,, \tag{69}$$

for all k and ν.

Consequently the quantity relevant to fluctuations in the lectures is

$$|u_k|^2 = \frac{\pi}{4(1-\epsilon) a^3 H} \, |H_\nu^{(2)}(y)|^2 \,. \tag{70}$$

Use the asymptotic expression

$$H_\nu^{(2)}(y) \to \frac{i\Gamma(\nu)}{\pi} \left(\frac{y}{2} \right)^{-\nu} \quad \text{for } y \to 0 \,, \tag{71}$$

to derive the small-k limit

$$|u_k|^2 \to \frac{2^{2\nu - 2} |\Gamma(\nu)|^2 (1-\epsilon)^{2\nu - 1}}{\pi a^3 H} \left(\frac{aH}{k} \right)^{2\nu} . \tag{72}$$

Evaluate this for the case $\nu = \frac{3}{2}$ (which for the de Sitter case $\epsilon = 0$ is a massless scalar field) and show that it agrees with the result obtained using the mode function directly, which in the case $\nu = \frac{3}{2}$ is very simple:

$$u_k = -(1-\epsilon) \frac{H}{\sqrt{2k^3}} \, (y - i) e^{-iy} \qquad \text{for } \nu = \tfrac{3}{2} \tag{73}$$

up to an irrelevant phase. Prove that this does solve the Klein Gordon equation in the case $\nu = \frac{3}{2}$.

The power spectrum $\Delta^2(k)$ is proportional to $k^3|u_k|^2$ evaluated for $k \ll aH$. For de Sitter space H is constant, and in this case what is the predicted k-dependence for $k^3|u_k|^2$ when $\nu = \frac{3}{2}$? When $\epsilon \neq 0$ H is time dependent and we are supposed to evaluate H at the moment where $aH = k$. If this were the whole story (and it is not quite), and if $\Delta^2(k) \propto A(k/k_0)^{n_s-1}$, what is the prediction for n_s as a function of ϵ?