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Effective Field Theories for Heavy Quarks:

Heavy Quark Effective Theory, Heavy Quark Expansion, Non-Relativistic QCD

Lectures at Les Houches 2017

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Abstract

DISCLAIMER: These notes are still incomplete and will be completed after the lectures were given, including the feedback of the students. Since this has been just written, it is also not free on typos, which will be removed upon polishing this document. This is in particular true for the references, most of which will be added upon completion of these notes. There is also not yet a detailed chack of signs and factors of π etc. ...

This version is not for distribution, it is exclusively for the students of the Les Houches School as teaching material to complement their notes.

Contents

1 Introduction

Effective field theory (EFT) and Renormalization Group (RG) Methods have developed into a quite universal tools that can be applied in various fields of physics. Most efficient use of EFT methods can be made in systems, in which vastly different mass scales appear and appropriate ratios of these mass scales define small parameters one aims to expand in.

In nuclear and particle physics obvious scales are the masses of the fundamental constituents. For the purpose of these lectures we will not discuss the Higgs mechanism which is assumed to give masses to the quarks and leptons, we rather assume that the masses are fundamental parameters. Another relevant scale in nuclear and particle physics is the scale $\Lambda_{\rm QCD} \sim 300$ MeV, which is generated by dimensional transmutation in QCD; this scale is the typical scale of the masses of light hadrons and also governs the running (i.e. the dependence on the renormalization scale μ) of the strong coupling "constant" $\alpha_s(\mu)$.

When considering weak interactions, the typical scale is set by the W boson mass M_W which at low energies manifests itself in the Fermi coupling constant $G_F \sim 1/M_W^2$ relevant for the four-fermion coupling in the weak interaction EFT. When studying a weak decay of a bottom hadron, the typical scale is set by the b-quark mass m_b. The elementary interaction is expressed in terms of quark currents, however, the observed states are hadrons.

To this end we have to deal with the effects of strong interactions, which are described in QCD. One important feature of QCD is its *asymptotic freedom*, which implies that its running coupling constant $\alpha_s(\mu) \to 0$ as $\mu \to \infty$. In practical terms this means that $\alpha_s(M_W)$ as well as $\alpha_s(m_b)$ is a small parameter which allows us to perform a perturbative expansion.

Thus a hadronic matrix element of a quark current evaluated at the scale $\mu \sim m_b$ still contains perturbatively computable pieces, which can be extracted by switching to an EFT description, which for the cases to be discussed below is the "Heavy Quark Effective Theory" (HQET). By applying HQET, the hadronic matrix elements are expressed as a combination of perturbatively computable coefficients and new, suitably defined hadronic matrix elements.

For some cases it is convenient to also treat the mass of the charm quark m_c still as a perturbative scale. This amounts to describe also the charm quark in HQET. However, once we arrive at scales $\mu \sim \Lambda_{\rm QCD}$, the strong coupling $\alpha_s(\mu)$ becomes order one or bigger, indicating that perturbation theory becomes useless.

Thus weak decays involve a sequence of vastly different mass scales. Assuming that the Standard Model (SM) itself is an effective theory and we have some physics beyond the SM (BSM) at some high scale $\Lambda_{\rm NP}$, we have $\Lambda_{\rm NP} \gg M_W \gg m_b \gg m_c \gg \Lambda_{\rm QCD}$ where - except for the last step - the QCD effects can be treated perturbatively by a tower of suitably constructed EFTs, until finally the nonperturbative QCD effects remain as matrix elements depending solely on the small scale $\Lambda_{\rm QCD}$.

Up to that point this describes the ubiquitous machinery of EFT in general. However, heavy quark methods are in one aspect quite special. In weak interactions at low scales all effects of e.g. the W boson appear only in the couplings (like the Fermi coupling), and at low scales this is the only remnant of the heavy W boson. However, consider now a bottom quark in QCD. In pure QCD, the bottom number is a conserved quantity, and this statement is independent of the scale. Thus a hadron with one unit of bottom quantum number will at low scales (i.e. below the bottom quark mass) still have the bottom quark inside, however, *this bottom quark will behave like a static source of a color field.* This is in full analogy to the hydrogen atom: Although it is a two particle problem, it is a very good approximation to treat the proton inside the H atom as a static source of a Coulomb field, in which the electron moves; any corrections to this picture will be of order $1/m_{\mathrm{Proton}}$.

This is a common feature of all EFTs with heavy quarks. The simplest type of such a theory is the already mentioned HQET which describes systems with a single heavy quark and where all light degrees of freedom are "soft", i.e. all components of their momenta are of the order Λ_{QCD}. In the first part of the lectures we will mainly discuss such systems.

The second part of these lectures is devoted to inclusive processes. Using the methods of OPE we will set up an expansion, called *Heavy Quark Expansion*, which has become the basis of many precision calculations in heavy quark physics.

Heavy quarks can decay weakly into light quarks, and hence there is also the kinematic situation, where light quarks acquire energies (in the rest frame of the decaying heavy quark) which scale with the heavy quark mass. For these situations an EFT has been developed, which is called *Soft Collinear Effective Theory* (SCET. This theory has also many applications in high-energy collider physics and thus has become a broad field of research. SCET will be covered by a different lecture at this school.

A second class are systems with a heavy quark and a heavy antiquark forming a bound system such as a charmonium, a bottomonium and also a B_c meson[1]. As we shall discuss below, the simple static limit will not be sufficient for these systems, rather a non-relativistic description is needed, which has been called Non-Relativistic QCD (NRCCD). We will look into this kind of effective theory in the last part of the lecture.

[1]One may also consider systems with two heavy quarks, such as "doubly heavy" baryons, but we will not include these here.

2 Heavy Quark Effective Theory (HQET)

We start with the simplest heavy quark expansion, which is the heavy quark effective theory for systems with a single heavy quark. We shall first construct its Lagrangian from integrating out heavy degrees of freedom. The remarkable and for phenomenology very relevant feature of HQET are the Heavy Quark Symmetries (HQS) which eventually yield constraints on the non-perturbative matrix elements at low scales, which are not evident in full QCD. Since $\alpha_s(m_b)$ is a perturbative scale, we will compute the one-loop matching of full QCD to HQET, which will give us some insight into the anatomy of HQET. Finally we will collect a few results that are used in current phenomenology.

2.1 Construction of the HQET Lagrangian

There are two ways to construct the Lagrangian of HQET. One follows straight the idea of EFT by identifying the heavy degree of freedom and integrating it out from the functional integral. This approach is quite instructive, since it can be explicitly performed at tree level and also at one loop. This approach leads to closed form for the HQET Lagrangian, at least at tree level.

A second approach follows usual non-relativistic reduction of the Dirac equation, leading finally to a recursive construction of the terms of higher order. This approach has the disadvantage that it does not exhibit the typical feature of an EFT. We will not discuss this in detail here.

The two approaches seem to have different results. Indeed, the Lagrangians derived in the two cases look differently, but we will show that they are related by a field redefinition, and that the results for physical quantities are the same in both case.

We will first consider the derivation of the HQET Lagrangian from the usual machinery of EFT. The starting point is the Lagrangian of QCD with a single heavy quark Q written as

$$\mathcal{L}_{\rm QCD} = \bar{Q}(i\not{D} - m_Q)Q + \mathcal{L}_{\rm light} \tag{2.1}$$

where m_Q is the mass of the heavy quark, $D_\mu = \partial_\mu + igA_\mu$ is the usual QCD covariant derivative including the interaction with the gluon A_μ, and $\mathcal{L}_{\rm light}$ is the Lagrangian for the light quarks and gluons[2].

To obtain the Green functions of the corresponding quantum field theory one may gather them in a generating functional, which is expressed as a functional integral over the field variables. Thus we write

$$Z(\eta,\bar{\eta},\lambda) = \int [dQ][d\bar{Q}][d\phi_\lambda] \exp\left\{ i\int d^4x\, \mathcal{L}_{\rm QCD} + i\int d^4x\, (\bar{\eta}Q + \bar{Q}\eta + \phi_\lambda\lambda) \right\}, \tag{2.2}$$

where $\phi_\lambda = q,\ A^a_\mu$ denotes the light degrees of freedom (light quarks q and gluons A_μ). Functional differentiation with respect to the source terms $\eta, \bar{\eta}$ and λ and subsequently setting the sources to zero yields the Green functions of QCD[3].

In order to derive the HQET Lagrangian we consider a system with a single heavy quark which is bound in a heavy hadron. This hadron has a mass m_H and moves with a certain

[2] $\mathcal{L}_{\rm light}$ also contains all terms relevant for the gauge fixing and possibly ghost fields needed for the quantization of QCD.

[3] We note that the funtional integral (2.2) is mathematically ill defined. For our purposes we take the practitioner's point of view and look at (2.2) as a short-hand notation for perturbation theory, which results from expanding in the strong coupling g_s.

momentum p_H. In case the hadron contains only a single heavy quark, its mass will scale with the heavy quark mass, likewise its momentum will scale with the heavy quark mass. To this end, it is convenient to define a four velocity

$$v = \frac{p_H}{m_H}, \quad v^2 = 1, \quad v_0 > 0\ . \tag{2.3}$$

which is independent of the heavy quark mass. This vector defines a specific frame, e.g. $v = (1, 0, 0, 0)$ is the rest frame of the heavy hadron. Eventually we want to consider the heavy quark inside the heavy hadron; the momentum of the heavy quark my be written as $p_Q = m_Q v + k$ where k is a small "residual" momentum satisfying $k \ll m_Q$.

To implement this idea on the technical side, we use this "external" velocity vector v to decompose the heavy-quark field Q into an "upper" (or "large") component ϕ and a "lower" (or "small") component χ

$$\phi_v = \frac{1}{2}(1 + \not{v})Q, \qquad \not{v}\phi_v = \phi, \tag{2.4}$$

$$\chi_v = \frac{1}{2}(1 - \not{v})Q, \qquad \not{v}\chi_v = -\chi, \tag{2.5}$$

and to define a decomposition of the covariant derivative into a "time" and a "spatial" ($\perp$) part

$$D_\mu = v_\mu (v \cdot D) + D^\perp_\mu\ , \quad D^\perp_\mu = (g_{\mu\nu} - v_\mu v_\nu) D^\nu, \quad \left\{\not{D}^\perp\ , \not{v}\right\} = 0. \tag{2.6}$$

In terms of these new fields (2.4) and using (2.6) the Lagrangian of the heavy quark field (i.e. the first term of (2.1) takes the form

$$\mathcal{L}_{\text{heavy}} = \bar{\phi}\{i(v \cdot D) - m_Q\}\phi - \bar{\chi}\{i(v \cdot D) + m_Q\}\chi + \bar{\phi} i\not{D}^\perp \chi + \bar{\chi} i\not{D}^\perp \phi \tag{2.7}$$

To proceed further, we now implement the decomposition of the heavy quark momentum into a "large" and a residual piece. This is achieved by multiplying the heavy quark field by a phase

$$\phi_v = e^{-im_Q(v\cdot x)} h_v\ , \qquad \chi_v = e^{-im_Q(v\cdot x)} H_v\ . \tag{2.8}$$

Note that the momentum of a field is the derivative acting on the field, i.e.

$$p_Q^\mu \sim i\partial^\mu Q(x) \quad \text{hence} \quad i\partial^\mu \phi_v(x) = e^{-im_Q(v\cdot x)}(m_Q v^\mu + i\partial^\mu) h_v(x)$$

which means that the derivative acting on the field h_v reproduces the residual momentum introduced above. This observation provides us with the power counting of HQET: once we have reformulated the theory in terms of h_v, we aim at an expansion in iD_μ/m_Q.

We express the Lagrangian of the heavy quark in term of the fields h_v and H_v and obtain

$$\mathcal{L}_{\text{heavy}} = \bar{h}_v i(v \cdot D) h_v - \bar{H}_v\{i(v \cdot D) + 2m_Q\}H_v + \bar{h}_v i\not{D}^\perp H_v + \bar{H}_v i\not{D}^\perp h_v\ . \tag{2.9}$$

With this form of the Lagrangian we can now easily identify the degrees of freedom. The field h_v does not have a mass term, while the field H_v has acquired a mass term $2m_Q$; the remaining terms are couplings between h_v and H_v. Thus in the sense of EFT the field H_v is the heavy degree of freedom, while the field h_v is light.

It is interesting to note, that the functional integral can be explicitly performed, at least for the tree-level Lagrangian (2.9), since there are only quadratic dependences on the relevant field. In order to integrate over the heavy field H_v, we first split also the source terms in (2.2) according to

$$\int d^4x\,(\bar{\eta}Q + \bar{Q}\eta) = \int d^4x\,(\bar{\rho}_v h_v + \bar{h}_v \rho_v + \bar{R}_v H_v + \bar{H}_v R_v), \tag{2.10}$$

where ρ_v and R_v are now source terms for the upper-component field h_v and the lower component part H_v, respectively. When studying processes at scales well below the scale $2m_Q$, no Green function involving the heavy field H_v will be relevant, hence we can put the corresponding sources to zero. Performing the Gaussian integral over the field H_v we obtain

$$\begin{aligned} Z(\rho_v, \bar{\rho}_v, \lambda) &= \int [dh_v][d\bar{h}_v][d\lambda]\,\Delta \\ &\times \exp\left\{ iS + S_\lambda + i\int d^4x\,(\bar{\rho}_v h_v + \bar{h}_v\rho_v + \phi_\lambda \lambda)\right\}\ , \end{aligned} \tag{2.11}$$

where now the action functional for the heavy quark becomes a non-local object

$$S = \int d^4x \left[\bar{h}_v i(v\cdot D)h_v - \bar{h}_v \not{D}^\perp \left(\frac{1}{i(v\cdot D) + 2m_Q - i\epsilon}\right) \not{D}^\perp h_v \right]. \tag{2.12}$$

depending solely on the field h_v and (via the covariant derivatives) on gluon fields.

The quantity Δ is the determinant resulting form the Gaussian integration, which may formally be written as

$$\begin{aligned} \Delta &= \exp\left(\frac{1}{2}\ln\left[i(v\cdot D) + 2m_Q\right]\right) \\ &= \text{const}\exp\left(\frac{1}{2}\ln\left[1 + \frac{1}{i(v\cdot\partial) - 2m_Q + i\epsilon} g_s(v\cdot A)\right]\right) \end{aligned} \tag{2.13}$$

However, unlike in other quantum field theories, this determinant is a constant (i.e.independent of the gluon fields). This can bee seen by either chosing the gauge $v\cdot A = 0$ or by expanding the logarithm which leads to expression that look like fermion bubble diagrams in ordinary QCD; however, here the particles propagate only in forward time-like directions and hence a closed loop always yieds a zero result.

Integrating our degrees of freedom in general yields non-local action functionals such as (2.12). However, if the degree of freedom that has been integrated out is heavy, it is in general possible to expand the result in inverse powers of the mass of the heavy scale. In our case this is quite evident, since we have $(v\cdot D) \ll 2m_Q$ as $(v\cdot D)$ is related to the residual momentum of the heavy quark. Consequently we expand and get

$$\frac{1}{i(v\cdot D) + 2m_Q - i\epsilon} = \frac{1}{2m_Q}\sum_{n=0}^{\infty}\left(\frac{-i(v\cdot D)}{2m_Q}\right)^n \tag{2.14}$$

which expresses the nonlocal distribution on the left-hand side by a series of local distributions.

Truncating at some order N yields a local action functional, and hence we get as the Lagranian

$$\mathcal{L}_{1/\mathrm{m_Q-Expansion}} = \bar{h}_v i(v\cdot D)h_v - \frac{1}{2m_Q}\bar{h}_v\not{D}^\perp \sum_{n=0}^{N}\left(\frac{-i(v\cdot D)}{2m_Q}\right)^n \not{D}^\perp h_v \tag{2.15}$$

This expression is the expansion of the QCD Lagrangian up to the order $1/m_Q^{N+1}$. The leading term

$$\mathcal{L}_{\rm HQET} = \bar{h}_v i(v \cdot D) h_v \tag{2.16}$$

is the Lagrangian for a static heavy quark moving with the four velocity v, i.e the Lagrangian on Heavy Quark Effective Theory (HQET).

By itself this Lagrangian is not very useful; by choosing an axial gauge $v \cdot A = 0$ the coupling to the gluons can even be made to vanish. However, this Lagrangain becomes useful as soon as additional interactions are implemented which are hard enough to change the velocity of the heavy quark. In the applications we are going to consider these are typically electroweak processes which insert large momentum transfers into the system.

To illustrate this in some more detail, let us consider the semileptonic decay $B \to D\ell\bar{\nu}$. The relevant hadronic matrix element is

$$\langle B(p)|\bar{b}\gamma_\mu c|D(p')\rangle$$

which may be obtained from inserting the weak transition current into the generating functional (2.2)

$$\begin{aligned} Z^{(b\to c)}(\eta_b, \bar\eta_b, \eta_c, \bar\eta_c, \lambda) &= \int [db][d\bar{b}][dc][d\bar{c}][d\phi_\lambda]\, \bar{b}(0)\gamma_\mu c(0) \\ &\times \exp\left\{ i \int d^4x\, \mathcal{L}_{\rm QCD} + i \int d^4x\, (\bar\eta_b b + \bar{b}\eta_b + \bar\eta_c c + \bar{c}\eta_c + \phi_\lambda \lambda) \right\}\ , \end{aligned} \tag{2.17}$$

corresponding to insertions the weak $b \to c$ current into the QCD Green functions.

At scales below m_c we may use the static limit for both the b and the c quark, however, the two mesons have different velocities $v = p/M_b$ and $v' = p'/M_D$, so we need to introduce two static quarks b_v and c_v with different velocities. Going through the same steps as before, now for two heavy quarks with different velocities, we get

$$\mathcal{L}_{\rm HQET}^{b\to c} = \bar{b}_v i(v \cdot D) b_v + \bar{c}_{v'} i(v' \cdot D) c_{v'} \tag{2.18}$$

Although this looks like a lagrangian with two heavy quarks, the weak current ensures that for $x_0 \leq 0$ we have only the bottom quark (moving with velocity v) , which at $x_0 = 0$ decays into a charm quark, moving with velocity v'.

This kind of approximation is well known since almost one century. It has been used already in the context of the infrared problem of QED, which for soft photons becomes "Heavy Electron Effective Theory". Furthermore, once there are two velocities v and v' in the game, there is no possibility to trivialize the theory by a choice of gauge.

Once one considers operator insertions in the Greens functions as in (2.17) one also needs to re-write the fields appearing in the current, which amounts to re-express the full QCD field by the static field h_v. We get

$$\begin{aligned} Q(x) &= e^{-im_Q vx}\,[h_v + H_v] = e^{-im_Q vx} \left[1 + \left(\frac{1}{2m + ivD} \right) i\not{D}_\perp \right] h_v \\ &= e^{-im_Q vx} \left[1 + \frac{1}{2m_Q}\not{D}_\perp + \left(\frac{1}{2m_Q} \right)^2 (-ivD)\not{D}_\perp + \cdots \right] h_v \end{aligned} \tag{2.19}$$

where we make use of the equations of motion for the field H_v.

The Hamiltonian which can be derived from (2.15) has the unusual property that it contains "time derivatives" (i.e. terms involving $(iv\partial)Q_v$). However, these can be removed by field redefinitions, resulting in an Hamiltonian without time derivatives. This Hamiltonian can also be constructed from the start by performing a transformation (a so-called Foldy-Wouthuysen transformation), which decouples the "large" and the "small" components of the spinor Q. This yields a Lagrangian which look different form the one derived above, starting at $1/m_Q^3$; likewise, the expansion of the field in terms if the static field also looks different. We shall use this Lagrangian in the contect of NRQCD later in these lectures.

For physical matrix elements both approaches eventually yield the same answer. To see how this works, we consider a matrix element with a heavy-to-light current of the form $\bar{q}\Gamma Q$ with heavy meson in the initial state $|M(v)\rangle$ and some final state $|A\rangle$. Computing to order $1/m_Q$ we get

$$\begin{aligned}\langle A|\bar{q}\Gamma Q|M(v)\rangle &= \langle A|\bar{q}\Gamma h_v|H(v)\rangle + \frac{1}{2m_Q}\langle A|\bar{q}\Gamma P_- i\not{D} h_v|H(v)\rangle \\ &-i\int d^4x \langle A|T\{L_1(x)\bar{q}\Gamma h_v\}|H(v)\rangle + \mathcal{O}(1/m^2)\ , \end{aligned} \tag{2.20}$$

where L_1 is the $1/m$ corrections to the Lagrangian as given in (2.15). In addition, $|M(v)\rangle$ is the state of the heavy meson in full QCD, including all of its mass dependence, while $|H(v)\rangle$ is the corresponding state in the infinite-mass limit.

A contribution in L_1 with a time derivative will become – upon insertion into the T product – a local operator which in turn means that it could be as well absorbed into the first term by a field redefinition. Using the a Hamiltonian without time derivative (such as the one derived from the Foldy-Wouthuysen transformation) will not have any local contributions in the second term, while the closed expression (2.15) and (2.19) will generate such terms.

2.2 Symmetries of HQET

Probably the most important property of HQET for phenomenology are the Heavy Quark Symmetries (HQS). These appear in the infinite mass mass limit and are not present in full QCD. These symmetries have very simple physical origins and are already manifest in the Lagrangians derived in the last section. In addition, there are two more symmetries which we shall briefly discuss. One is related to the fact, that the definition of the quark mass underlying the splitting of the heavy quark momentum could still be changed by an amount of the order $\Lambda_{\rm QCD}$; such a "residual mass term" is related to the question which definition of the quark mass is used. The second one is related to the fact, that the construction of HQET requires to introduce a four velocity vector, which is not present in full QCD. Thus a change of this velocity vector by an amount of the order $1/m_Q$ may not change the physics. This so-called "reparametrization invariance" has interesting consequences, since it relates different order in the $1/m_Q$ expansion.

2.2.1 Flavour Symmetry

The QCD Lagrangian is known to have flavour symmetries in the case where quarks become mass-degenerate: The approximate degeneracy of the up and the down quark leads to the isospin symmetry, in case all quarks are assumed to be massless, QCD has a chiral symmetry, of which the flavour $SU(3)$ is manifest. The underlying reason is that the interaction of the quarks with the gluons does not depend on the mass, it depends only on the color charge of the quarks which is defined by putting all quarks into the fundamental representation of color $SU(3)$.

This still remains true in the infinite mass limit. Once a heavy quark becomes a static source of color, its flavour becomes irrelevant. To make this explicit, we consider the $b \to c$ HQET Hamiltonian (2.18) for the case of two equal velocities

$$\mathcal{L}^{b,c}_{\text{HQET}} = \bar{b}_v i(v \cdot D) b_v + \bar{c}_v i(v \cdot D) c_v = (\bar{b}_v \,,\, \bar{c}_v) \begin{pmatrix} i(v \cdot D) & 0 \\ 0 & i(v \cdot D) \end{pmatrix} \begin{pmatrix} b_v \\ c_v \end{pmatrix} \tag{2.21}$$

which as a manifest $SU(2)$ symmetry: for any unitary 2×2 matrix U we define the transformation

$$\begin{pmatrix} b_v \\ c_v \end{pmatrix}' = U \begin{pmatrix} b_v \\ c_v \end{pmatrix} ,$$

under which the Lagrangian (2.21) remains invariant. Note that this symmetry relates only heavy quark moving with the same velocity v.

As a practical application, consider a semileptonic decay of a B meson into a D meson. Assuming both b and c to be heavy, we may look into the point of maximal momentum transfer to the leptons, which is $q^2_{\text{max}} = (m_B - m_D)^2 \approx (m_b - m_c)^2$. Looking at this decay in the rest frame of the B meson (which is also the rest frame of the b quark as $m_b \to \infty$), the final state D meson (as well as the c quark as $m_c \to \infty$) remains at rest at this kinematic point, while the two leptons carry away the energy difference $m_B - m_D \approx m_b - m_c$ in a back-to-back momentum configuration. As a consequence of heavy flavour symmetry, the light degrees of freedom (the light quark(s) and gluons forming the meson) cannot be affected by this transition (at this special kinematic point), which means that their state did not change! We will return to this example when discussing weak transition form factors.

2.2.2 Spin Symmetry

The second HQS is the so-called heavy quark spin symmetry. It originates form the fact that in gauge theories like QED and QCD the interaction of the spin of a particle is always of the form $\vec{\sigma} \cdot \vec{B}$, where $\vec{B}$ is the corresponding (chromo)magnetic field. However, this is a dimension-five operator, and its coupling constant is $g/(2m_Q)$, which is the QCD analogue of the Bohr magneton of the particle. As a consequence, the spin of a particle decouples in QCD and hence the rotations of the particles spin become a symmetry.

To make this explicit, we look at the HQET Lagrangian and decompose the heavy quark field into the two spin components. This is achieved by introducing a spin vector s with $s \cdot v = 0$ and $s^2 = -1$ such that we can define the projections

$$h_v^{\pm s} = \frac{1}{2}(1 \pm \gamma_5 \not{s}) h_v \qquad h_v = h_v^{+s} + h_v^{-s} \tag{2.22}$$

In terms of these projections we have

$$\mathcal{L} = \bar{h}_v^{+s}(ivD)h_v^{+s} + \bar{h}_v^{-s}(ivD)h_v^{-s} = (\bar{h}_v^{+s} \,,\, \bar{h}_v^{-s}) \begin{pmatrix} i(v \cdot D) & 0 \\ 0 & i(v \cdot D) \end{pmatrix} \begin{pmatrix} h_v^{+s} \\ h_v^{-s} \end{pmatrix} \tag{2.23}$$

Then, similarly as before we have an $SU(2)$ symmetry: for any unitary 2×2 matrix U we define the transformation

$$\begin{pmatrix} h_v^{+s} \\ h_v^{-s} \end{pmatrix}' = U \begin{pmatrix} h_v^{+s} \\ h_v^{-s} \end{pmatrix} ,$$

under which the HQET Lagrangian remains invariant. Note that this symmetry relates again only heavy quarks moving with the same velocity v.

2.2.3 Consequences of Heavy Quark Symmetries

These symmetries have a few interesting consequences which are important to make HQET a useful tool, since they constrain the non-perturbative matrix elements of HQET.

The heavy quark spin symmetry has the consequence that all the heavy-hadron states moving with the velocity v fall into spin-symmetry doublets as $m_Q \to \infty$. In Hilbert space, this symmetry is generated by operators $S_v(\epsilon)$ as

$$[h_v, S_v(\epsilon)] = i\not{\epsilon}\not{v}\gamma_5 h_v \ , \tag{2.24}$$

where ϵ, with $\epsilon^2 = -1$, is the rotation axis. The simplest spin-symmetry doublet in the mesonic case consists of the pseudoscalar meson $H(v)$ and the corresponding vector meson $H^*(v,\epsilon)$, since a spin rotation yields

$$\exp\left(iS_v(\epsilon)\frac{\pi}{2}\right)|H(v)\rangle = (-i)|H^*(v,\epsilon)\rangle \ , \tag{2.25}$$

where we have chosen an arbitrary phase to be $(-i)$.

Thus the pseudoscalar ground state meson forms a spin symmetry doublet with the vector ground state meson; assuming that the bottom is heavy we have the doublets

$$\begin{array}{ccc} |(b\bar{u})_{J=0}\rangle = |B^-\rangle & \longleftrightarrow & |(b\bar{u})_{J=1}\rangle = |B^{*-}\rangle \\ |(b\bar{d})_{J=0}\rangle = |\overline{B}^0\rangle & \longleftrightarrow & |(b\bar{d})_{J=1}\rangle = |\overline{B}^{*0}\rangle \\ |(b\bar{s})_{J=0}\rangle = |\overline{B}_s\rangle & \longleftrightarrow & |(b\bar{s})_{J=1}\rangle = |\overline{B}^*_s\rangle \end{array} \tag{2.26}$$

which become degenerate in the infinite mass limit.

For baryons, the situation is more complicated, since the two light quarks can have either spin 0 or spin 1. The doublets with u and d quarks are

$$\left|[(ud)_0 Q]_{1/2}\right\rangle = |\Lambda_Q\rangle \qquad |\Lambda_Q \Uparrow\rangle \longleftrightarrow |\Lambda_Q \Downarrow\rangle \tag{2.27}$$

$$\left|[(uu)_1 Q]_{1/2}\right\rangle, \left|[(ud)_1 Q]_{1/2}\right\rangle, \left|[(dd)_1 Q]_{1/2}\right\rangle = |\Sigma_Q\rangle \tag{2.28}$$

$$\left|[(uu)_1 Q]_{3/2}\right\rangle, \left|[(ud)_1 Q]_{3/2}\right\rangle, \left|[(dd)_1 Q]_{3/2}\right\rangle = |\Sigma^*_Q\rangle \qquad |\Sigma_Q\rangle \longleftrightarrow |\Sigma^*_Q\rangle$$

and similar relation for the strange baryons Ξ_b and Ω_b.

To leading order, the mass of a heavy Q hadron is the mass of the quark m_Q. However, we may expand the hadron mass in terms of the quark mass, which reads for the mesonic ground states

$$m_H = m_Q + \bar{\Lambda} - \frac{\lambda_1 - 3\lambda_2}{2m_Q} + ... \tag{2.29}$$

$$m_{H^*} = m_Q + \bar{\Lambda} - \frac{\lambda_1 + \lambda_2}{2m_Q} + ... \tag{2.30}$$

where we have introduced new parameters $\bar{\Lambda}$, λ_1 and λ_2. $\bar{\Lambda}$ is the binding-energy parameter for the heavy hadron

$$\bar{\Lambda} = \frac{\langle 0|q\ i v\overleftarrow{D}\ \gamma_5 h_v|\tilde{H}(v)\rangle}{\langle 0|q\gamma_5 h_v|\tilde{H}(v)\rangle} \ , \tag{2.31}$$

while λ_1 and λ_2 are defined by the HQET matrix elements

$$\begin{aligned} 2m_H\lambda_1 &= \langle\tilde{H}(v)|\bar{h}_v(iD^\perp)^2 h_v|\tilde{H}(v)\rangle && (2.32)\\ 2m_H\lambda_2 &= \langle\tilde{H}(v)|\bar{h}_v(iD^\perp_\mu)(iD^\perp_\nu)(i\sigma^{\mu\nu})h_v|\tilde{H}(v)\rangle && (2.33)\end{aligned}$$

where $|\tilde{H}(v)\rangle$ is the pseudoscalar Q meson ground state in the infinite mass limit.

These parameters have a simple physical interpretation: λ_1 is the kinetic energy induced by the residual motion of the heavy quark, λ_2 corresponds to the interaction of the chromomagnetic moment of the heavy quark induced by the interaction with the chromomagnetic field $\vec{\sigma}\cdot\vec{B}$ produced by the light degrees of freedom. This implies in particular that (taking the b and the c quark to be heavy)

$$m_{B^*}^2 - m_B^2 = m_{D^*}^2 - m_D^2 = 4\lambda_2 + \mathcal{O}(1/m_Q) \tag{2.34}$$

from which we get $\lambda_2 \approx 0.12$ GeV^2 which is indeed of the order of $\Lambda^2_{\rm QCD}$. Similar relations can be written for the Q baryons, in particular, the chromomagnetic parameter λ_2 vanishes for Λ_Q baryons, since the light degrees are in a spin-0 state and hence cannot induce a chromomagnetic field.

HQS also constrain hadronic matrix elements. In order to extract the corresponding relations, it is useful to write down a useful representation for the spins in the ground state mesons. Introducing a spinor $v(v,\pm)$ with spin direction $\pm$ for the light antiquarks and $u(v,\pm)$ for the heavy quark, we may couple the spins to get the total spin of the meson

$$|(b\bar{u})_{J=0}(v)\rangle \to \frac{1}{\sqrt{2}}\left[u_\alpha(v,+)\bar{v}_\beta(v,-) - u_\alpha(v,-)\bar{v}_\beta(v,+)\right] = \left(\gamma_5\frac{\not{v}-1}{2}\right)_{\alpha\beta} \tag{2.35}$$

$$|(b\bar{u})_{J=1,M=0}(v)\rangle \to \frac{1}{\sqrt{2}}\left[u_\alpha(v,+)\bar{v}_\beta(v,-) + u_\alpha(v,-)\bar{v}_\beta(v,+)\right] = \left(\not{\epsilon}_{\rm long}\frac{\not{v}-1}{2}\right)_{\alpha\beta} \tag{2.36}$$

where α and β are spinor indices. Including the proper normalization of the states, we define the representation matrices for these states

$$\begin{aligned} H(v) &= \frac{1}{2}\sqrt{m_H}\gamma_5(\not{v}-1) \qquad \text{for the pseudoscalar meson,} && (2.37)\\ H^*(v,\epsilon) &= \frac{1}{2}\sqrt{m_H}\not{\epsilon}(\not{v}-1) \qquad \text{for the vector meson,} && (2.38)\end{aligned}$$

where the two indices of the matrices correspond to the indices of the heavy quark and the light anti-quark, respectively, and ϵ is the polarization vector of the vector meson.

We may now use these representation matrices to exploit the consequences of spin symmetry in a very simple fashion. We look at a transition current of the form $\bar{h}_{v'}\Gamma h_v$ induced e.g. by a weak transition (such as a $b\to c$ semileptonic process). Heavy quark spin symmetry implies that the spin of the heavy quark in the current is the same as the one of the quark inside the meson, which means that the heavy-quark index of the representation matrix has to hook directly to the Dirac matrix Γ in the current. Thus for a $0^-\to 0^-$ transition we have

$$\langle M(v')|\bar{h}_{v'}\Gamma h_v|M(v)\rangle = \mathrm{Tr}\left[\overline{H}(v')\Gamma H(v)\mathcal{M}(v,v')\right] \tag{2.39}$$

where the two light-quark indices of $\overline{H}(v')\Gamma H(v)$ will be contracted with a Dirac-matrix valued function $\mathcal{M}(v,v')$ of v and v' which describes the dynamics of the light quarks in the transition. This matrix can be decomposed into the basis of the sixteen Dirac matrices, thus we can write

$$\mathcal{M}(v,v') = \mathbf{1}\,\xi_1(v\cdot v') + \not{v}\,\xi_2(v\cdot v') + \not{v}'\,\xi_3(v\cdot v') + \not{v}\not{v}'\,\xi_3(v\cdot v') \tag{2.40}$$

with scalar functions ξ_i. Inserting this into (2.39) we see that for any Γ this collapses into

$$\langle M(v')|\bar{h}_{v'}\Gamma h_v|M(v)\rangle = \mathrm{Tr}\left[\overline{H}(v')\Gamma H(v)\right]\ \xi(v\cdot v') \tag{2.41}$$

with $\xi(v\cdot v') = \xi_1(v\cdot v') + \xi_2(v\cdot v') - \xi_3(v\cdot v') - \xi_4(v\cdot v')$

Likewise we can discuss the transitions between the transitions $0^- \to 1^-$ and $1^- \to 1^-$ between ground state mesons. Spin symmetry tells us that the function $\mathcal{M}(v,v')$ for the light degrees of freedom is the same in all cases, and hence *any transition within the ground-state spin flavour multiplet $\mathcal{H}(v)$ to the ground-state multiplet $\mathcal{H}(v')$, where $\mathcal{H}(v)$ denotes either $H(v)$ or $H^*(v,\epsilon)$ is described by a single nonperturbative function $\xi(v\cdot v')$* . This function is called the Isgur Wise (IW) function. This relation is one of the "Wigner-Eckart Theorems" of Spin symmetry and can be written as

$$\langle\mathcal{H}(v')|\bar{h}_{v'}\Gamma h_v|\mathcal{H}(v)\rangle = \xi(v\cdot v')\ \mathrm{Tr}\ \left\{\overline{\mathcal{H}}(v')\Gamma\mathcal{H}(v)\right\}\ , \tag{2.42}$$

This relation has remarkable consequences. Assuming that both b and c are heavy, we find that the six form factors describing the semileptonic transitions $B\to D$ and $B\to D^*$ collaps into a single one, the IW function. Furthermore, the current $\bar{b}\gamma_\mu b$ is a conserved current in pure QCD, which translates into a normalization statement for the IW function

$$\xi(v\cdot v' = 1) = 1 \tag{2.43}$$

where the physical argument for this normalization has been given in the last paragraph of sec. 2.2.1. Note that the point $v\cdot v' = 1$ corresponds exactly to the point q^2_{max} of maximal recoil to the leptons discussed in sec. 2.2.1.

For phenomenological applications these symmetries are very useful, however, only once the corrections to the symmetry limit can be somehow handled. There are two sources of corrections, which are on the one hand the radiative corrections though hard gluons, on the other hand the ones induced by subleading terms in the $1/m_Q$ expansion.

We will discuss the latter and consider the $1/m_Q$ normalization statement (2.43) which originated from the conservation of the heavy quark current, which in turn is related to the fact that HQS can be traced back to conserved currents. In this case we can apply a theorem originally derived by Ademollo and Gatto to our case. The theorem states that *in the presence of explicit symmetry breaking, the matrix elements of the currents that generate the symmetry are still normalized up to terms which are second-order in the symmetry-breaking interaction.*

For the case at hand, the argument can be outlined in a simple way, taking as an expample the $b\to c$ case. The relevant symmetry is the heavy-flavor symmetry between b and c in the case $m_{b,c}\to\infty$. This symmetry is an $SU(2)$ symmetry and is generated by three operators $Q_\pm$ and Q_3, where

$$\begin{aligned}
&Q_+ = \int d^3x\,\bar{b}_v(x)c_v(x)\ , \quad Q_- = \int d^3x\,\bar{c}_v(x)b_v(x)\ ,\\
&Q_3 = \int d^3x\,(\bar{b}_v(x)b_v(x) - \bar{c}_v(x)c_v(x))\ ,\\
&[Q_+,Q_-] = Q_3\ , \qquad [Q_+,Q_3] = -2Q_+\ , \qquad (Q_+)^\dagger = Q_-\ .
\end{aligned} \tag{2.44}$$

Let us denote the ground-state flavour symmetry multiplet by $|B\rangle$ and $|D\rangle$. The operators then act in the following way:

$$\begin{aligned}
&Q_3|B\rangle = |B\rangle\ , \qquad Q_3|D\rangle = -|D\rangle\ ,\\
&Q_+|D\rangle = |B\rangle\ , \qquad Q_-|B\rangle = |D\rangle\ ,\\
&Q_+|B\rangle = Q_-|D\rangle = 0\ .
\end{aligned} \tag{2.45}$$

The Hamiltonian of this system has a $1/m_Q$ expansion which is decomposed into a symmetric and a symmetry breaking part

$$\begin{aligned} H &= H_0^{(b)} + H_0^{(c)} + \frac{1}{2m_b} H_1^{(b)} + \frac{1}{2m_c} H_1^{(c)} + \cdots \\ &= H_0^{(b)} + H_0^{(c)} + \frac{1}{2}\left(\frac{1}{2m_b} + \frac{1}{2m_c}\right)(H_1^{(b)} + H_1^{(c)}) \\ &\qquad + \frac{1}{2}\left(\frac{1}{2m_b} - \frac{1}{2m_c}\right)(H_1^{(b)} - H_1^{(c)}) + \cdots \\ &= H_{symm} + H_{break} \; . \end{aligned} \tag{2.46}$$

Note that the symmetry breaking term does not commute any more with $Q_\pm$ but it still commutes with Q_3 (which only means that we still can distinguish B and D). Thus to order $1/m_Q$ we still have common eigenstates of H and Q_3, which we shall denote by $|\tilde{B}\rangle$ and $|\tilde{D}\rangle$. Sandwiching the commutation relation, we obtain

$$\begin{aligned} 1 &= \langle \tilde{B}|Q_3|\tilde{B}\rangle = \langle \tilde{B}|[Q_+, Q_-]|\tilde{B}\rangle \\ &= \sum_n \left[\langle \tilde{B}|Q_+|\tilde{n}\rangle\langle \tilde{n}|Q_-|\tilde{B}\rangle - \langle \tilde{B}|Q_-|\tilde{n}\rangle\langle \tilde{n}|Q_+|\tilde{B}\rangle\right] \\ &= \sum_n \left[|\langle \tilde{B}|Q_+|\tilde{n}\rangle|^2 - |\langle \tilde{B}|Q_-|\tilde{n}\rangle|^2\right] \; , \end{aligned} \tag{2.47}$$

where the $|\tilde{n}\rangle$ form a complete set of states of the Hamiltonian $H_{symm} + H_{break}$. The matrix elements may be written as

$$\langle \tilde{B}|Q_\pm|\tilde{n}\rangle = \frac{1}{E_B - E_n}\langle \tilde{B}|[H_{break}, Q_\pm]|\tilde{n}\rangle \; , \tag{2.48}$$

where E_B and E_n are the energies of the states $|\tilde{B}\rangle$ and $|\tilde{n}\rangle$, respectively. In the case $|\tilde{n}\rangle = |\tilde{D}\rangle$ the matrix element on the left-hand side will be of order unity, since both the numerator and the energy difference in the denominator are of the order of the symmetry breaking. For all other states, the energy difference in the denominator is non-vanishing in the symmetry limit, and hence this difference is of order unity; thus the matrix element for these states will be of the order of the symmetry breaking. From this we conclude that

$$\langle \tilde{B}|Q_+|\tilde{D}\rangle = 1 + \mathcal{O}\left[\left(\frac{1}{2m_b} - \frac{1}{2m_c}\right)^2\right] . \tag{2.49}$$

For simplicity we have used states normalized to unity; writing (2.49) in terms of momentum eigenstates shows that the first term on the right-hand side is actually the form-factor normalization, see problem set.

2.2.4 Reparametrization Invariance

Finally there is another symmetry in HQET which originates from the fact that our starting point was full QCD which is a Lorentz invariant theory. Clearly, when introducing the velocity vector v we explicitly break Lorentz invariance by fixing a time like direction which, however, could be also varied sightly. To this end, a HQET constructed with v and a HQET constructed from $v + \delta v$ should give the same physical results. This invariance is called Reparametrization Invariance (RPI)

In order to study the consequences of this simple fact, we write down the variation $\delta_{\rm RPI}$ of the relevant quantities under a small change in the velocity

$$\begin{aligned}
&v \to v + \delta v \ , \qquad (v+\delta v)^2 = 1 \text{ and thus } v \cdot \delta v = 0 \ , \\
&h_v \to h_v + \frac{\delta \not{v}}{2} \left(1 + P_- \frac{1}{2m_Q + ivD} i\not{D} \right) h_v \ , \\
&iD \to iD - m_Q \, \delta v \ . \qquad (2.50)
\end{aligned}$$

In particular the last relation, which originates from the splitting of the heavy-quark momentum, leads to the observation that the transformation (2.50) relates different order in the $1/m_Q$ expansion.

This can be easily illustrated using the Lagrangian as an example. We start from the expression (2.12) for the action of the heavy quark after integrating out the the small-component field H_v. This (non-local) expression is invariant under (2.50). Expanding (2.12) in local operators according to (2.15) shows that (2.50) actually relates terms of subsequent orders such that

$$\delta_{\rm RPI} \mathcal{L}_{1/{\rm m_Q}-{\rm Expansion}} = \mathcal{O}(1/m_Q^{N+2}) \qquad (2.51)$$

since (2.15) includes all terms up to and including terms of order $1/m_Q^{N+1}$.

Looking at the leading term we find

$$\delta_{\rm RPI} \bar{h}_v (ivD) h_v = \bar{h}_v (i \, \delta v \, D) h_v$$

which is exactly cancelled by the variation of the first subleading term

$$\delta_{\rm RPI} \frac{1}{2m_Q} \bar{h}_v (i\not{D}^\perp)^2 h_v = - \bar{h}_v (i \, \delta v \, D) h_v$$

and as a consequence we have[4]

$$\delta_{\rm RPI} \left(\bar{h}_v (ivD) h_v + \frac{1}{2m_Q} \bar{h}_v (iD^\perp)^2 h_v \right) = \mathcal{O}(1/m_Q^2)$$

These relations are all tree level relations; however, RPI has to hold also including QCD corrections in HQET, which means the the relations derived from RPI should hold to all order in α_s. For the Lagangain this means that one may derive relations between the renormalization constants of the operators appearing in (2.15) which are true to any oder in α_s. In particular it means for renormalization constants of the first few terms

$$Z_h \bar{h}_v (ivD) h_v + (Z_h c_1) \frac{1}{2m_Q} \bar{h}_v (iD^\perp)^2 h_v = Z_h \left(\bar{h}_v (ivD) h_v + c_1 \frac{1}{2m_Q} \bar{h}_v (iD^\perp)^2 h_v \right) \qquad (2.52)$$

where Z_h is the renormalization constant of the the static heavy-quark field. Thus RPI fixes the renormalization constant of the kinetic energy term to be $c_1 \equiv 1$, which we will have to use later on.

[4]The antisymmetric combination $(i\sigma^{\mu\nu})(iD_\mu^\perp)(iD_\nu^\perp)$ is reparametrization invariant.

2.3 HQET at one loop

Up to now, all discussion refer to the tree-level expressions. We have set up an expansion in $\Lambda_{\mathrm{QCD}}/m_Q$ which is, however, only one of the small parameters we can expand in. To become a useful tool, also the perturbative QCD corrections have to be taken into account.

The strong coupling constant taken at the scale $\mu \geq 1$ GeV $\alpha_s(m_Q)$ constitutes another small parameter which may serve as an expansion parameter. In particular, the heavy quark-mass scale $\mu = m_Q$ is large enough to warrant a perturbative expansion. This has the advantage, that many contributions can be computed perturbatively, in particular the matching between HQET and full QCD. In the following we discuss the underlying technology and study the one loop diagrams.

2.3.1 The Feynman rules of HQET

I assume that the reader is to some extend familiar with the Feynman rules of QCD, including the discussion of gauge fixing, so I will not repeat here the standard technology of calculations within QCD.

However, to compute within HQET, we need to set up the Feynman rules of HQET. These are derived from the Lagrangian (2.16)

$$\mathcal{L}_{\mathrm{HQET}} = \bar{h}_v i(v \cdot D) h_v = \bar{h}_v i(v \cdot \partial) h_v + ig\bar{h}_v i(v \cdot A) h_v \tag{2.53}$$

The propagator can be read off from the first term, while the heavy quark-gluon coupling is encoded in the second term.

The recipe to obtain the propagator from the first term is to invert the distribution appearing between the two fields according to

$$(v \cdot \partial) P(x) = \delta^4(x) \tag{2.54}$$

Fourier transforming this relation yields

$$P(x) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{v \cdot k + i\epsilon} e^{-ikx}$$

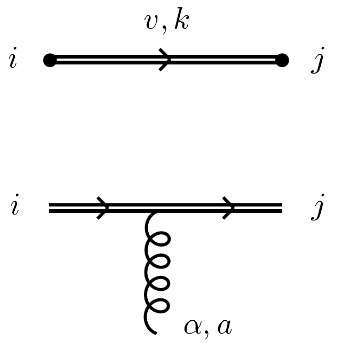
where we have already fixed the boundary conditions by adding a small imaginary part $i\epsilon$. The interpretation of this propagator becomes evident by performing the k integration in the rest frame $v = (1, 0, 0, 0)$; we get

$$P(x) = \theta(x_0)\delta^3(\vec{x}) \tag{2.55}$$

which is the propagator of a static, pointlike quark sitting at the origin. In order to insert this into a general Feynman diagram we have to still multiply this by the projector P_+ on the large components.

The second term in (2.53) yields the coupling of a static quark to the gluon field. The resulting Feynman rule has the same form as the usual one for the quark-gluon coupling, however, only the matrix γ_μ is replaced by v_μ, which reflects the heavy quark spin symmetry,

Fig. 1 shows the two resulting additional Feynman rules; here k denotes the residual momentum of the heavy quark moving with velocity v.



$$= \frac{i}{v \cdot k + i\eta} \frac{1 + \not{v}}{2} \delta_{ji}$$

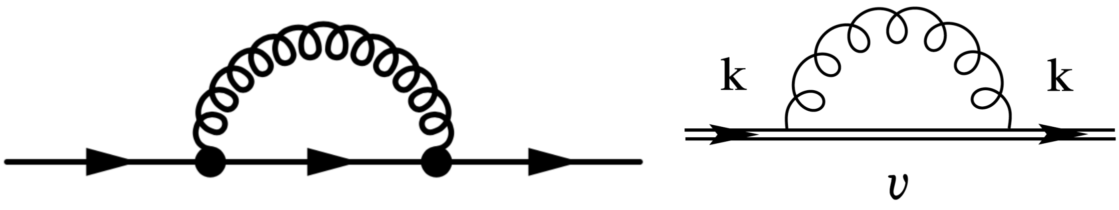
$$= i\ {}_s v^{\alpha} (t_a)_{ji}$$

Figure 1: Feynman rules of HQET. All other elements are the same as in full QCD. i and j are color indices, k is the residual momentum of the heavy quark moving with the velocity v.

2.3.2 One loop diagrams

We are now ready to compute Feynman diagrams. As in full QCD there is a set of divergent diagrams, and the handling of these divergencies requires renormalization. We shall discuss this here for a few examples at the one-loop level.

We start with the sample calculation of the self energy; fig. 2 (a) is the self energy in full QCD, while digram (b) shows the corresponding diagram in HQET. The expression in full QCD



(a) (b)

Figure 2: One-loop self energy diagram of a light and a heavy quark

.

(Diagram (a)) is well known and reads

$$\Sigma_{\rm QCD}(p) = -ig^2 T^a T^a \mu^{4-D} \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 + i\epsilon)} \frac{\gamma_\mu (\not{p} + \not{l} + m_Q)\gamma^\mu}{(p+l)^2 - m_Q^2 + i\epsilon} \tag{2.56}$$

Making use of the Feynman rules we get the expression corresponding to diagram (b)

$$\Sigma(v \cdot k) = -ig^2 T^a T^a \mu^{4-D} \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 + i\epsilon)(v \cdot k + v \cdot l + i\epsilon)} P_+ \tag{2.57}$$

where we have anticipated a divergence in $D = 4$ and regularize this diagram by dimensional regularization. As usual, the factor μ^{4-D} is introduced to keep the dimension of Σ fixed as D varies.

In order to evaluate (2.57), we quote a useful relation which we shall use to combine denominators of propagators

$$\frac{1}{A^n B^m} = 2^m \frac{\Gamma(m+n)}{\Gamma(n)\Gamma(m)} \int\limits_0^\infty d\lambda \frac{\lambda^{m-1}}{(A + 2\lambda B)^{m+n}} \tag{2.58}$$

where this relation also holds for non-integer m and m. Using this we can combine the denominators in (2.57) into

$$\Sigma(v \cdot k) = -2ig^2\mu^{4-D}C_F \int_0^\infty d\lambda \int \frac{d^Dl}{(2\pi)^D} \frac{1}{(l^2 + 2\lambda[v \cdot k + v \cdot l] + i\epsilon)^2} \tag{2.59}$$

where we inserted $T^aT^a = C_F\mathbf{I}$ where $C_F = 4/3$. In order to apply the one-loop master formula of dimensional regularization

$$\int \frac{d^Dl}{(2\pi)^D} \frac{(l^2)^\alpha}{(l^2 - M^2)^\beta} = (-1)^{\alpha+\beta} \frac{i}{2^D\pi^{D/2}} (M^2)^{\alpha-\beta+D/2} \frac{\Gamma(\alpha + D/2)\Gamma(\beta - \alpha - D/2)}{\Gamma(D/2)\Gamma(\beta)} \tag{2.60}$$

we need to shift the integration variable $l \to l - \lambda v$ which removes the term linear in l in the denominator, leaving us with

$$\Sigma(v \cdot k) = -ig^2\mu^{2\varepsilon}\frac{8}{3} \int_0^\infty d\lambda \int \frac{d^Dl}{(2\pi)^D} \frac{1}{(l^2 - \lambda^2 + \lambda v \cdot k + i\epsilon)^2} \tag{2.61}$$

where we have defined $D = 4 - 2\varepsilon$. Performing the integration over the loop momentum with the help of (2.60) we find

$$\Sigma(v \cdot k) = C_F \frac{\alpha_s}{2\pi} \Gamma(\varepsilon) \int_0^\infty d\lambda \left(\frac{4\pi\mu^2}{\lambda^2 - 2\lambda v \cdot k} \right)^\varepsilon \tag{2.62}$$

For the renormalization we are interested in the divergence as $D \to 4$ (or $\varepsilon \to 0$) , which manifests itself as a simple pole

$$\Sigma(v \cdot k) = C_F \frac{\alpha_s}{2\pi} \frac{1}{\varepsilon} (v \cdot k) + \text{finite Terms} \tag{2.63}$$

Renormalization proceeds in the usal way. We insert the self energy into the heavy quark popagator and get

$$\begin{aligned} S^{(1)}_{\mathrm{HQET}}(v \cdot k) &= \frac{i}{(v \cdot k)} + \frac{i}{(v \cdot k)}(-i\Sigma(v \cdot k))\frac{i}{(v \cdot k)} + \cdots \\ &= \left(1 + C_F \frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}\right) \frac{i}{(v \cdot k)} + \cdots \end{aligned} \tag{2.64}$$

from which we can read off the wave function renormalization constant of HQET (in the $\overline{MS}$ scheme)

$$Z_{\mathrm{HQET}} = 1 + C_F \frac{\alpha_s}{2\pi} \frac{1}{\varepsilon} \tag{2.65}$$

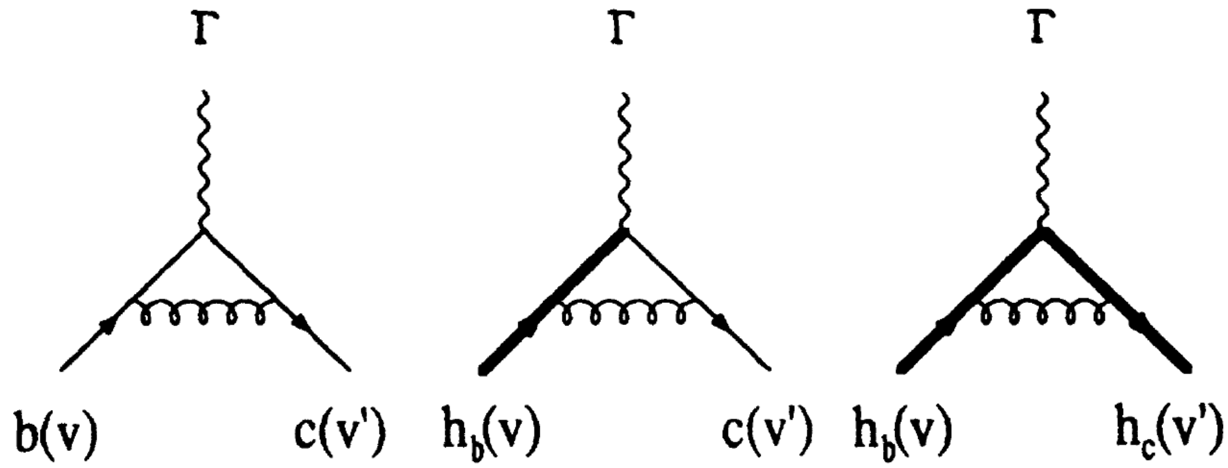
This cam be compared to the result in full QCD. A similar calculation yields for the wave function renormalization of the quark field in full QCD

$$Z_Q = 1 - C_F \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \tag{2.66}$$

We note that the two renormalization constants are different, which is not a surprise, since the UV behaviour of the two theories is different. In order to understand this in more detail, we look at the result of the quark self energy in full QCD in on-shell renormalization ... **to be added**

2.3.3 Example: The $b \to c$ current

As the next example we will discuss the $b \to c$ vector current and consider the QCD radiative corrections at one loop. Starting at a high scale above the b quark mass, we compute the one-loop diagrams shown in diagram (a) of fig. 3, together with the corresponding diagrams with self-energy insertions in the external legs. Adding the three contributions and including the proper renormalization, the one-loop result is UV finite, in other words, the current $\bar{b}\gamma_\mu c$ does not have an anomalous dimension, This actually is true at all orders, since this current is conserved in the limit of vanishing masses.



(a) (b) (c)

Figure 3: Feynman diagrams for the $b \to c$ vertex corrections in full QCD (a), in the theory with a static b quark (b) and in a theory with static b and c quarks (c). Thick lines denote HQET quarks.

Since the anomalous dimension of this current vanishes, we will not encounter any large logarithms of the form $(\alpha_s/\pi)\ln(M_W^2/m_b^2)$ when we run down to the bottom mass scale. At $\mu = m_b$ we have to match the vector current $V_\mu^{(b\to c)}$ of full QCD to operators in a theory where we use HQET for the b quark. We have schematically

$$V_\mu^{(b\to c)} = \sum_i C_i^{(0)}(\mu) J_{i,\mu}^{(b\to c)} + \frac{1}{2m_b} \sum_k C_k^{(1)}(\mu) O_{k,\mu}^{(b\to c)} + \cdots \tag{2.67}$$

where the ellipses denote even higher orders in the $1/m_b$ expansion. In our example we will consider only the leading term which is expresses in terms of two operators

$$J_{1,\mu}^{(b\to c)} = \bar{c}\gamma_\mu h_v \tag{2.68}$$
$$J_{2,\mu}^{(b\to c)} = \bar{c} h_v \, v_\mu \tag{2.69}$$

The relations (2.67) are operator relations, and in order to compute the matching we may use any states we prefer. For the case at hand we want to compute the perturbative corrections, and thus it is convenient to use on shell states for the b and c quarks. Furthermore, since the mass scale of the charm quark is still irrelevant at the scale m_b, we compute with a massless charm quark.

Computing the one-loop diagrams in full QCD shown in diagram (a) of fig. 3 (together with the diagrams with self-energy insertions in the external legs) and expanding in the result in $1/m_b$ yields the result

$$\langle V_\mu^{(b\to c)} \rangle = \left(1 + \frac{\alpha_s}{2\pi} \left[\ln \frac{m_b^2}{\lambda^2} - \frac{11}{6} \right] \right) \gamma_\mu + \frac{2\alpha_s}{3\pi} v_\mu \tag{2.70}$$

As stated above, the result is UV finite, but we had to introduce an infrared regulator λ which is e.g. a small gluon mass. This is due to the fact, that we are using on-shell "free" quark states in the calculation; if we could compute the matrix element with hadronic states, these IR singularities would be absent.

The next step in the matching procedure is to compute the corresponding diagrams in an HQET where the b quark is replaced by a static quark. Computing the one-loop contribution shown in diagram (b) of fig. 3 (together with the diagrams with self-energy insertions in the external legs) and perform the proper renormalization of the heavy and light quark fields we obtain (using again a small gluon mass to regulate the infrared divergence of the amplitudes)[5]

$$\langle J_{1,\mu}^{(b\to c)} \rangle = \left(1 + \frac{\alpha_s}{2\pi} \left[\frac{1}{\bar\varepsilon} + \ln \frac{\mu^2}{\lambda^2} - \frac{5}{6} \right] \right) \gamma_\mu \tag{2.71}$$

Note that this result is UV divergent which is to be expected, since we changed the high-energy behaviour of the theory by switching to a static b quark. Consequently we need to renormalize the current operator, doing this in the $\overline{MS}$ scheme juts removes the $1/\bar\varepsilon$ pole.

We can now read off the coefficients in appearing in (2.67) by taking the corresponding matrix elements of (2.67), we obtain

$$C_1^{(0)}(\mu) = 1 + \frac{\alpha_s}{2\pi} \left[\ln \frac{m_b^2}{\mu^2} - \frac{8}{3} \right] \tag{2.72}$$

$$C_2^{(0)}(\mu) = \frac{2\alpha_s}{3\pi} \tag{2.73}$$

Note that the IR regulator λ has dropped out which is a general feature of a matching calculation. It is due to the fact that the IR behaviour in the full and the effective theory have to be the same, once the effective theory is properly constructed.

The UV divergence in the effective theory is related to the mass dependence in the full theory. This fact allows us to make use of renormalization-group (RG) methods in order to resum logarithms of the mass m_b. However, these logarithms will be of the form $\ln(m_b^2/m_c^2)$ once we scale down to the charm-quark mass m_c, and the RG methods will perform a resummation of terms of order $(\alpha_s/\pi)^n \ln^n(m_b^2/m_c^2)$, which makes sense as soon as the log is so large that it overwhelms the α_s supression. We shall assume this as we go on, although the constant term $-3/8$ in (2.72) is numerically comparable to the term with the logarithm.

In order to obtain an RG improvement, we note that the left hand side of (2.67) is μ independent. The μ dependence originates form the fact that we decided to shift the contributions of scales between m_b and $\mu \leq m_b$ into the wilson coefficient, while the pieces form scales belwo μ still are contained in the matrix element of the operator $J_{1,\mu}^{(b\to c)}$. This observation leads to

$$0 = \mu \frac{d}{d\mu} \langle V_\alpha^{(b\to c)} \rangle = \left(\mu \frac{d}{d\mu} C_1^{(0)}(\mu) \right) \langle J_{1,\alpha}^{(b\to c)} \rangle_\mu + C_1^{(0)}(\mu) \left(\mu \frac{d}{d\mu} \langle J_{1,\alpha}^{(b\to c)} \rangle_\mu \right) \tag{2.74}$$

[5] We use $1/\bar\varepsilon = 1/\varepsilon - \gamma + \ln 4\pi$.

The logarithmic derivative acting on the matrix element of the current is the anomalous dimension, which is computed from the divergence occurring in (2.71). In our one-loop case the anomalous dimension is given by

$$\left(\mu\frac{d}{d\mu}\langle J_{1,\alpha}^{(b\to c)}\rangle_\mu\right) = -\gamma_{h_b\to c}\,\langle J_{1,\alpha}^{(b\to c)}\rangle_\mu = \frac{\alpha_s}{\pi}\,\langle J_{1,\alpha}^{(b\to c)}\rangle_\mu \tag{2.75}$$

This translates into an evolution equation for the coeffiient $C_1^{(0)}(\mu)$

$$\left(\mu\frac{d}{d\mu} - \gamma_{h_b\to c}\right) C_1^{(0)}(\mu) = 0 \tag{2.76}$$

The coefficient $C_1^{(0)}(\mu)$ is evaluated in a power sereis in α_s, and hence the μ dependence has actually two sources: Aside form the explicit dependence (see (2.72)) there is also the μ dependence of α_s. To make this explicit, we write

$$C_1^{(0)}(\mu) = C_1^{(0)}(\alpha_s,\mu)$$

and write the total derivative as

$$\mu\frac{d}{d\mu}C_1^{(0)}(\mu) = \left(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right) C_1^{(0)}(\alpha_s,\mu)$$

where we have introduced the QCD β function as

$$\mu\frac{d}{d\mu}\alpha_s(\mu) = \beta(\alpha_s(\mu)) = -2\alpha_s(\mu)\frac{\alpha_s(\mu)}{4\pi}\left(11 - \frac{2}{3}n_f\right) + \cdots \tag{2.77}$$

where n_f is the number of active flavours. Inserting all this yields

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} - \gamma_{h_b\to c}\right) C_1^{(0)}(\alpha_s,\mu) = 0 \tag{2.78}$$

The general solution of this equation is given by

$$C_1^{(0)}(\alpha_s(\mu),\mu) = \exp\left(-\int\limits_{\alpha_s(\mu)}^{\alpha_s(m_b)} \frac{\gamma_{h_b\to c}(a)}{\beta(a)}da\right) C_1^{(0)}(\alpha_s(m_b),m_b) \tag{2.79}$$

which gives the coefficient $C_1^{(0)}(\alpha_s(\mu),\mu)$ in terms of the initial value $C_1^{(0)}(\alpha_s(m_b),m_b)$ obtained form the matching calculation.

In order to obtain the leading log result, we insert the one-loop results for $\gamma_{h_b\to c}$ and β and use the tree-level value $C_1^{(0)}(\alpha_s(m_b),m_b) = 1$ for the matching coefficient, which yields finally

$$C_1^{(0)}(\alpha_s(\mu),\mu) = \left(\frac{\alpha_s(m_b)}{\alpha_s(\mu)}\right)^{-6/25} \tag{2.80}$$

Expanding this result in $\alpha_s(\mu)$ using the one-loop result for the running coupling reproduces the logarithmic term in (2.67).

Scaling further down we eventually arrive at the charm quark mass m_c. Assuming that we can also treat the charm quarks as a heavy quark, we may again switch our description and replace the charm quark be a static quark. This allows us to scale further down to scales below m_c, however, at some point we arrive at $\mu = \Lambda_{\text{QCD}}$ where we cannot compute perturbatively any more.

We shall again look at the vector current, however, due to spin symmetry the results will also hold for other currents. At one-loop, we need to compute diagram (c) of fig. 3 and match it to the result obtained in the theory where only the b quark is taken to be static.

The diagram (c) of fig. 3 contains also an UV divergence, which is now related to the logarithmic m_c dependence of diargam (b), if we had included the charm mass in the calculation. Thus we need to include a renormalization of the heavy-to-heavy current, which due to this has an anomalous dimension, for which we find at one loop

$$\gamma_{h_b \to h_c}(v \cdot v') = \frac{4\alpha_s}{3\pi}[(v \cdot v')r(v \cdot v') - 1] \tag{2.81}$$

with

$$r(x) = \frac{1}{\sqrt{x^2 - 1}} \ln\left(x + \sqrt{x^2 - 1}\right)$$

This result is remarkable, since usually anomalous dimension do not depend on kinematic variables. However, the velocities in HQET are external variables and thus this is not a problem. We also note that at $v = v'$ the anomalous dimension vanishes, which is necessary, since this current is a generator of HQS at this kinematic point.

The running below m_c is governed by the RGE

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} - \gamma_{h_b \to h_c}(v \cdot v')\right) \tilde{C}^{(0)}(\alpha_s, \mu) = 0 \tag{2.82}$$

where the number of active flavours is now 3. In the effective theory with both b and c as heavy quarks the matrix element of the currnt is (up to trivial factors) the Isgur-Wise function, and thus we can write our result as a renormlization of this function

$$\xi(v \cdot v') = \zeta(v \cdot v', m_b, m_c, \mu)\xi_0(v \cdot v', \mu) \tag{2.83}$$

where $\xi_0(v \cdot v', \mu)$ is the "bare" Isgur-Wise function and

$$\zeta(v \cdot v', m_b, m_c, \mu) = \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{-6/25} \left(\frac{\alpha_s(m_c)}{\alpha_s(\mu)}\right)^{(8/27)[(v \cdot v')r(v \cdot v') - 1]} \tag{2.84}$$

where the first factor originates form the running from m_b to m_c, while the second one comes from the running from m_c to some small scale μ.

In full QCD, the amplitude for a $b \to c$ transition via the $\bar{c}\gamma_\mu b$ current can be analytically continued to values of $q^2 \geq (m_B + m_D)^2$ which correspond to a creation of a B and a D meson by the current. In terms of the velocities this is the region where $v \cdot v' \leq -1$, in which case the anomalous dimension (2.81) pick up an imaginary part. At the fist look this is puzzling, however, it is related to the coulombic phases which appear once the two particles are both in the final state and can re-scatter through soft gluons.

2.4 Some Results

Finally we discuss a few phenomenological results obtained from HQET. The most prominent result is the fact that due to HQS all transitions between ground-state heavy mesons mediated by a bilinear quark current are given in terms of a single form factor, the Isgur Wise function introduced in (2.42). Assuming that both b and c quarks are heavy, we can consider the decays $B \to D\ell\bar{\nu}$ and $B \to D^*\ell\bar{\nu}$ for which the hadronic matrix element of the current $\bar{c}\gamma_\mu(1-\gamma_5)b$ is exactly of the form of (2.42).

For heavy quarks it is convenient to use the four velocities of the hadrons v and v' as kinematic variables, which are at leading order the same as the velocities if the heavy quarks. The general parametrization of the matrix elements requires a total of six form factor which can be defined as

$$\langle D(v')|\bar{c}\gamma_\mu b|B(v)\rangle = \sqrt{m_B m_D}\left[\xi_+(y)(v_\mu + v'_\mu) + \xi_-(y)(v_\mu - v'_\mu)\right] \ , \tag{2.85}$$

$$\langle D^*(v',\epsilon)|\bar{c}\gamma_\mu b|B(v)\rangle = i\sqrt{m_B m_{D^*}}\xi_V(y)\varepsilon_{\mu\alpha\beta\rho}\epsilon^{*\alpha}v'^\beta v^\rho \ , \tag{2.86}$$

$$\begin{aligned}\langle D^*(v',\epsilon)|\bar{c}\gamma_\mu\gamma_5 b|B(v)\rangle = \sqrt{m_B m_{D^*}}\,&\left[\xi_{A1}(y)(vv'+1)\epsilon^*_\mu - \xi_{A2}(y)(\epsilon^* v)v_\mu\right.\\ &\left.-\xi_{A2}(y)(\epsilon^* v)v'_\mu\right] \ ,\end{aligned} \tag{2.87}$$

where ϵ is the polarization of the charmed vector meson and $y = v\cdot v'$. Applying now (refwet) to (refff1 - 2.87) we find five relations among the form factors ξ_i

$$\xi_i(y) = \xi(y) \quad \text{for } i = +, V, A1, A3 \ , \qquad \xi_i(y) = 0 \quad \text{for } i = -, A2 \ . \tag{2.88}$$

which eventually reduces the number of independent form factors to only one.

In addition, we may make use of Lukes Theorem derived in section 2.2.3 which yields a statement about the size of the corrections; one finds

$$\begin{aligned}\xi_i(1) &= 1 + \mathcal{O}\left(\left[\frac{1}{2m_c} - \frac{1}{2m_b}\right]^2\right) \quad \text{for } i = +, V, A1, A3 \ ,\\ \xi_i(1) &= \mathcal{O}\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right) \quad \text{for } i = -, A2 \ .\end{aligned} \tag{2.89}$$

This has interesting phenomenological applications. Computing the rates for the exclusive decays $B \to D\ell\bar{\nu}$ and $B \to D^*\ell\bar{\nu}$ in terms of the form factors ξ_i we get

$$\begin{aligned}\frac{d\Gamma}{dy}(B \to D\ell\nu_\ell) &= \frac{G_F^2}{48\pi^3}|V_{cb}|^2(m_B+m_D)^2\left(m_D\sqrt{y^2-1}\right)^3\\ &\qquad\times\left|\xi_+(y) - \frac{m_B - m_D}{m_B + m_D}\xi_-(y)\right|^2\end{aligned} \tag{2.90}$$

$$\begin{aligned}\frac{d\Gamma}{dy}(B \to D^*\ell\nu_\ell) &= \frac{G_F^2}{48\pi^3}|V_{cb}|^2(m_B-m_{D^*})^2 m_{D^*}^2\left(m_{D^*}\sqrt{y^2-1}\right)\\ &\qquad\times(y+1)^2|\xi_{A1}(y)|^2\sum_{i=0,\pm}|H_i(y)|^2\end{aligned} \tag{2.91}$$

with the squared helicity amplitudes

$$|H_\pm(y)|^2 = \frac{m_B^2 - m_{D^*}^2 - 2y m_B m_{D^*}}{(m_B - m_{D^*})^2}\left[1 \mp \sqrt{\frac{y-1}{y+1}}R_1(y)\right]^2 \ , \tag{2.92}$$

$$|H_0(y)|^2 = \left(1 + \frac{m_B(y-1)}{m_B - m_{D^*}}\left[1 - R_2(y)\right]\right)^2 \ . \tag{2.93}$$

Here we have defined the form factor ratios

$$R_1(y) = \frac{\xi_V(y)}{\xi_{A1}(y)} \; , \quad R_2(y) = \frac{\xi_{A3}(y) + \frac{m_B}{m_{D^*}}\xi_{A2}(y)}{\xi_{A1}(y)} \; . \tag{2.94}$$

These expression collaps in the limit $m_b, m_c \to \infty$ into

$$\frac{d\Gamma}{dy}(B \to D\ell\nu_\ell) \;\; \to \;\; \frac{G_F^2}{48\pi^3}|V_{cb}|^2(m_B+m_D)^2\left(m_D\sqrt{y^2-1}\right)^3|\xi(y)|^2 \; , \tag{2.95}$$

$$\begin{aligned}\frac{d\Gamma}{dy}(B \to D^*\ell\nu_\ell) \;\; \to \;\; & \frac{G_F^2}{48\pi^3}|V_{cb}|^2(m_B-m_{D^*})^2m_{D^*}^2\left(m_{D^*}\sqrt{y^2-1}\right)(y+1)^2 \\ & \times\left[1+\frac{4y}{y+1}\frac{m_B^2-m_{D^*}^2-2ym_Bm_{D^*}}{(m_B-m_{D^*})^2}\right]|\xi(y)|^2 \; .\end{aligned} \tag{2.96}$$

The impact of these relations is that the absolute normalization of the form factor is given by HQS, and hence a model independent extraction of the CKM matrix element V_{cb} becomes possible by extrapolating the measured differential rates to the kinematic point $y=1$.

For the decay $B \to D^*\ell\bar{\nu}$ we find

$$\lim_{y\to 1}\frac{1}{\sqrt{y^2-1}}\frac{d\Gamma}{dy}(B\to D^*\ell\nu_\ell) = \frac{G_F^2}{4\pi^3}(m_B-m_{D^*})^2m_{D^*}^3|V_{cb}|^2|\xi_{A1}(1)|^2 \; . \tag{2.97}$$

where the form factor $xi_{A1}(1) = \xi(1) = 1$ is normalized by HQS. In fact, ξ_{A1} is also protected against linear corrections in $1/m_c$ due to (2.89), and hence one expects a determination of V_{cb} from (2.97) with an uncertainty of about ten percent.

One can also use the process $B \to D\ell\bar{\nu}$ for a determination of V_{cb}, however, there is an additional factor of y^2-1 which makes the extrapolation more difficult, furthermore, due to the presence of $\xi_-(1) \sim 1/m_c$ we expect this to be not as precise as for $B \to D^*\ell\bar{\nu}$.

The state of the art goes far beyond this simple reasoning. First of all, QCD corrections have been computed or both the vector and the axial vector current.

$$\langle c(v)|\bar{c}\gamma^\mu b|b(v)\rangle = 1 + \frac{2\alpha_s}{3\pi}\left[\frac{3m_b^2+2m_cm_b+3m_c^2}{2(m_b^2-m_c^2)}\ln\left(\frac{m_b}{m_c}\right)-2\right] \tag{2.98}$$

$$\langle c(v)|\bar{c}\gamma^\mu\gamma_5 b|b(v)\rangle = 1 - \frac{\alpha_s}{\pi}\left[\frac{m_b+m_c}{m_b-m_c}\ln\left(\frac{m_c}{m_b}\right)+\frac{8}{3}\right] \tag{2.99}$$

Numerically (including also the known α_s^2 corrections):

$$\langle c(v)|\bar{c}\gamma^\mu b|b(v)\rangle = \eta_V = 1.022 \pm 0.004 \tag{2.100}$$

$$\langle c(v)|\bar{c}\gamma^\mu\gamma_5 b|b(v)\rangle = \eta_A = 0.960 \pm 0.007 \tag{2.101}$$

Furthermore, QED corrections have been compute as well and amount to an enhancement of the rates by a factor $\eta_{\mathrm{ew}} = 1.007$. In addition the recoil corrections have been estimated by using QCD sum rules which indicate a furhter decrease of the matrix element of the axial current by another 10%.

More recently, lattice calculations of the form factors have become available at the non-recoil point as well as for $y \neq 1$, even for finite values of the quark masses. All this yields a quite consistent picture giving us a quite reliable value for V_{cb}, a recent analysis [5] yields

$$|V_{cb}| = \left(41.9^{+2.0}_{-1.9}\right)\times 10^{-3} \tag{2.102}$$

Problems

- Convince yourself, that the static heavy quark propagates only forward in time
- Derive the equations of motion for the fields h_v and H_v and show that once can obtain the HQET Lagrangian by using the equation of motion.
- Show that (2.12) is invariant under the reparametrization (2.50) .
- Show that for a conserved vector current j_μ the matrix element between pseudoscalar ground states is given in terms of a single form factor given by

  $$\langle M(p)|j_\mu|M(p')\rangle = f(q^2)(p+p')_\mu \qquad q = (p-p')$$

  which is normalized at $q^2 = 0$.
- Expand (2.80) in $\alpha_s(\mu)$ and verify that one obtains the leading log in (2.67)

3 Heavy Quark Expansion

In inclusive processes one may often make use of the so-called Operator Product Expansion (OPE) which is a standard tool in quantum field theory. In fact the OPE lies at the heart of the EFT approach, since it is actually this tool which allows us to separate scales.

The most prominent example is deep inelastic scattering $e + p \to e' + X$ (DIS) which is an inclusive process governed by a large scale set by the momentum transfer Q^2 of the electron. The nonperturbative input are eventually the parton distributions of the quarks inside the proton, which are determined by the binding effects of the quarks inside the proton. The expansion which is set up in this case in in powers of $\Lambda_{\rm QCD}^2/Q^2$ which is very small, such that usually only the leading term is considered.

In the case at hand we shall proceed along the same lines as in DIS. The expansion, the heavy quark Expansion (HQE) we will set up will be in powers of $\Lambda_{\rm QCD}/m_Q$; however, in our case we will take into account subleading terms.

3.1 Inclusive Decays

Inclusive decays are all processes, where a summation over a significant number of final states is performed. The most inclusive quantity is the total decay width of a particle, where a summation over all final states is performed.

A second, more interesting class are the inclusive semileptonic decays, such as

$$\Gamma(B \to X\ell\bar{\nu}) = \sum_f \Gamma(B \to f\ell\bar{\nu})$$

where over all possible exclusive hadronic final states f is summed, while the kinematic information on the final-state leptons is fully available. Through their spectra and their kinematic distribution one can obtain important information.

3.2 Operator Product Expansion (OPE)

We start with the total decay rate of a heavy hadron $H(p_H)$. Assuming that H is a ground-state hadron, it can only decay by a weak decay, which is mediatied by an effective hamiltonian density $\mathcal{H}_{eff}(x)$. To leading order in the weak interaction we obtain - up to trivial factors - for the total rate[6]

$$\Gamma \propto \sum_X (2\pi)^4 \delta^4(p_H - p_X) |\langle X|\mathcal{H}_{eff}(0)|H(p_H)\rangle|^2 \tag{3.1}$$

where X is the final state with momentum p_X, and we sum over all final states taking into account four-momentum conservation. We use the relation

$$\mathcal{H}_{eff}(x) = e^{-i\hat{P}x} \mathcal{H}_{eff}(0) e^{i\hat{P}x} \,,$$

[6] In the following we often drop the argument of field operators or of other space time dependent operators $\mathcal{O}(x)$, we define $\mathcal{O} \equiv \mathcal{O}(0)$. Likewise we write $\partial_\mu \mathcal{O} \equiv (\partial_\mu \mathcal{O}(x))|_{x=0}$.

where $\hat{P}_\mu$ is the (four) momentum operator, and write

$$\begin{aligned}
&\sum_X (2\pi)^4 \delta^4(p_H - p_X) |\langle X | \mathcal{H}_{eff}(0) | H(p_H)\rangle|^2 \\
= & \sum_X \int d^4y \, \exp\left(i(p_H - p_X)y\right) \langle H(p_H)|\mathcal{H}_{eff}(0)|X\rangle\langle X|\mathcal{H}_{eff}(0)|H(p_H)\rangle \\
= & \sum_X \int d^4y \, \langle H(p_H)|\mathcal{H}_{eff}(y)|X\rangle\langle X|\mathcal{H}_{eff}(0)|H(p_H)\rangle \\
= & \int d^4y \, \langle H(p_H)|\mathcal{H}_{eff}(y)\mathcal{H}_{eff}(0)|H(p_H)\rangle \qquad (3.2)
\end{aligned}$$

where in the final step we made use of the fact, that

$$\sum_X |X\rangle \, \langle X|$$

is the unit operator, since we sum over all states in the Hilbert space.

Finally we may use the optical theorem to relate the matrix element of the product of the Hamiltonian to the time-ordered product

$$\begin{aligned}
&\int d^4y \, \langle H(p_H)|\mathcal{H}_{eff}(y)\mathcal{H}_{eff}(0)|H(p_H)\rangle \\
&\qquad = 2 \, \mathrm{Im} \int d^4y \, \langle H(p_H)|T\{\mathcal{H}_{eff}(y)\mathcal{H}_{eff}(0)\}|H(p_H)\rangle \qquad (3.3)
\end{aligned}$$

This relation is the starting point of all further considerations. The matrix element in (3.3) still contains the heavy quark mass m_Q and our goal is to set up an expansion in invers powers of this mass. Since $\mathcal{H}_{eff}$ induces a decay of the heavy quark, we expect it to be of the form

$$\mathcal{H}_{eff} = \bar{Q}R + \text{h.c.} \qquad (3.4)$$

where R consists of light(er) quarks and possibly gluons. In order to make the dependence on the heavy mass explicit, we use (2.8) and write[7]

$$Q(x) = \exp(-im_Q(v \cdot x))Q_v(x) \, , \quad v = \frac{p_H}{m_H} \qquad (3.5)$$

corresponding to the splitting of the heavy-quark momentum into the large part $m_Q v$ and a residual part related to the derivative acting on Q_v. Note that we do not use here the static field introduced above, rather $Q_v(x)$ is still the field of full QCD, up to the above phase redefinition.

With this phase redefinition we get

$$\begin{aligned}
&\int d^4y \, \langle H(p_H)|T\{\mathcal{H}_{eff}(y)\mathcal{H}^\dagger_{eff}(0)\}|H(p_H)\rangle \\
&\qquad = \int d^4y \, \exp(im_Q(v \cdot y))\langle H(p_H)|T\{\tilde{\mathcal{H}}_{eff}(y)\tilde{\mathcal{H}}^\dagger_{eff}(0)\}|H(p_H)\rangle \qquad (3.6)
\end{aligned}$$

where $\tilde{\mathcal{H}}_{eff}$ is obtained from $\mathcal{H}_{eff}$ by the replacement (3.5)

$$\tilde{\mathcal{H}}_{eff} = \bar{Q}_v R + \text{h.c.} \, .$$

[7] We note that $Q_v(0) = Q(0)$; however once a derivative is acting on the field Q_v it corresponds to the residual momentum $i\partial_\mu Q_v(0) \sim k_\mu$.

Expression (3.6) is the starting point of an Operator Product Expansion (OPE), which is a standard method in quantum field theory. Without going into details, the main relation is

$$\int d^4y\, e^{-iqx}\, T[O_1(x)O_2(0)] = \sum_n C_n(q)\mathcal{O}_n(0) \tag{3.7}$$

where O_1 and O_2 are renormalized local operators and $\mathcal{O}_n$ are renormalized local operators which can be ordered by increasing dimension, and $C_n(q)$ are coefficients depending on the momentum transfer q. Note that each term on the right hand side maust have the same dimension, so the increasing dimension of the operators will be compensated by inverse powers of q. Thus for sufficiently large momentum transfer q one may truncate the series on the right hand side, and one obtains an approximation scheme in terms of powers of $1/q$.

Applying the OPE in the context of QCD one may make use of the fact that at large q QCD becomes perturbative. This means in particular that we may compute the coefficients in QCD perturbation theory, while the matrix elements of the operators contain the non-perturbative information. This scheme is at the heart of all applications of EFT's and has been used in many different context such as weak interactions and in DIS.

Inclusive differential rates can be computed for processes with leptons and / or photons in the final state. These rates are inclusive with respect to the final-state hadrons, but we may discuss the kinematic distributions of the final state photons and leptons. To be explicit, let us consider a semileptonic transition base on the quark decay $Q \to q + \ell + \bar{\nu}$. The effective Hamiltonian can be written as

$$\mathcal{H}_{eff} = \frac{4G_F V_{\rm CKM}}{\sqrt{2}} J_\mu L^\mu \tag{3.8}$$

where $J_\mu = \bar{Q}_L\gamma_\mu q_L$ is the left-handed hadronic current and $L_\mu = \bar{\ell}_L\gamma_\mu\nu_L$ is the leptonic current. Inserting this into (3.2), we get

$$\begin{aligned} & 8G_F^2|V_{\rm CKM}|^2 \sum_{X,\ell\bar{\nu}} (2\pi)^4\delta^4(p_H - p_X - k - k')|\langle X\ell\bar{\nu}|J_\mu L^\mu|H(p_H)\rangle|^2 \\ = \;& 8G_F^2|V_{\rm CKM}|^2 \sum_X \widetilde{dk}\widetilde{dk}'(2\pi)^4\delta^4(p_H - p_X - k - k')|\langle X|J_\mu|H(p_H)\rangle\,\langle\ell(k)\bar{\nu}(k')|L^\mu|0\rangle|^2 \end{aligned} \tag{3.9}$$

where $\widetilde{dk}$ and $\widetilde{dk}'$ denote the phase-space integrations over the leptons. Since the leptons do not have any strong interaction, we can decompose this expression into an hadronic and a leptonic part. We get

$$\begin{aligned} & \sum_X \widetilde{dk}\widetilde{dk}'(2\pi)^4\delta^4(p_H - p_X - k - k')|\langle X|J_\mu|H(p_H)\rangle\,\langle\ell(k)\bar{\nu}(k')|L^\mu|0\rangle|^2 \\ & = \int \frac{d^4q}{(2\pi)^4} \sum_X (2\pi)^4\delta^4(p_H - p_X - q)\langle H(p_H)|J_\alpha^\dagger|X\rangle\langle X|J_\beta|H(p_H)\rangle \\ & \qquad \times \widetilde{dk}\widetilde{dk}'(2\pi)^4\delta^4(q - k - k')\langle 0|L^{\alpha\dagger}|\ell(k)\bar{\nu}(k')\rangle\langle\ell(k)\bar{\nu}(k')|L^\beta|0\rangle \end{aligned} \tag{3.10}$$

The leptonic part can be evaluated separately and is taken usually to lowest order in perturbation theory; the hadronic part is encoded in the hadronic tensor, which can be decomposed into scalar functions W_i, $i = 1,..,5$

$$\begin{aligned} W^{\alpha\beta}(q) &= \sum_X (2\pi)^4\delta^4(p_H - p_X - q)\langle H(p_H)|J^{\alpha\dagger}|X\rangle\langle X|J^\beta|H(p_H)\rangle \\ &= -g^{\alpha\beta}W_1 + v^\alpha v^\beta W_2 - i\epsilon^{\alpha\beta\mu\nu}v_\mu q_\nu W_3 + q^\alpha q^\beta W_4 + (v^\alpha q^\beta + v^\beta q^\alpha)W_5 \end{aligned} \tag{3.11}$$

where we introduced $p_H = m_H v$. These scalar function depend on the two invariants q^2 and $v \cdot q$; in terms of these we get e.g. for the triply differential rate ($E_\ell = v \cdot k$, $E_\nu = v \cdot k'$)

$$\frac{d\Gamma}{dq^2\, dE_\ell\, dE_\nu} = \frac{G_F^2 |V_{\rm CKM}|^2}{2\pi^3} \left[W_1 q^2 + W_2 \left(2E_\ell E_\nu - \frac{1}{2} q^2 \right) + W_3 q^2 (E_\ell - E_\nu) \right] \tag{3.12}$$

where the phase space is restricted by $4E_\ell E_\nu - q^2 \geq 0$.

With the hadronic tensor we can go through the same steps (3.1, ... ,3.3), but we have to insert the phase factor $\exp(-iqy)$ into the y integration:

$$\begin{aligned} &\int d^4y \, \exp(-iqy) \, \langle H(p_H)| J_\mu^\dagger(y) J_\nu(0) |H(p_H)\rangle \\ &\qquad = 2 \, \mathrm{Im} \int d^4y \, \exp(-iqy) \, \langle H(p_H)| T\{ J_\mu^\dagger(y) J_\nu(0)\} |H(p_H)\rangle \end{aligned} \tag{3.13}$$

Performing the replacement (3.5) we end up with

$$\begin{aligned} &\int d^4y \, \exp(-iqy) \, \langle H(p_H)| T\{ J_\mu^\dagger(y) J_\nu(0)\} |H(p_H)\rangle = \\ &\qquad \int d^4y \, \exp(iy(m_Q v - q)) \, \langle H(p_H)| T\{ \widetilde{J}_\mu^\dagger(y) \widetilde{J}_\nu(0)\} |H(p_H)\rangle \end{aligned} \tag{3.14}$$

The time-ordered product of the two hadronic currents has the same decomposition (3.11) as the hadronic tensor with scalar functions T_i, $i = 1, ..., 5$. These functions has an analytic structure as depicted in fig. 4: for a fixed value of q^2 we have $p_H - q = p_X$ where p_X is the momentum of the final hadronic state, thus $m_H^2 + q^2 - 2m_H(v \cdot q) = m_X^2$. Thus the maximal value of $v \cdot q$ is given by

$$(v \cdot q)_{\rm max} = \frac{1}{2m_H} (m_H^2 + q^2 - m_{X\,\rm min}^2)$$

where $m_{X\,\rm min}$ is the mass of the lightest hadronic state with the correct quantum numbers. Thus for the states with a q quark in the final state, the T_i exhibit a cut

$$-\infty \leq (v \cdot q) \leq \frac{1}{2m_H} (m_H^2 + q^2 - m_{X\,\rm min}^2) \tag{3.15}$$

However, there can also be intermediate states with two Q quarks and a q antiquark, which yield a branch cut

$$\frac{1}{2m_H} (m_H^2 + q^2 - m_{X(QQ\bar{c})\,\rm min}^2) \leq (v \cdot q) \leq \infty \tag{3.16}$$

where $m_{X(QQ\bar{c})\,\rm min}$ denotes the mass of the lightest state with the quark content $QQ\bar{c}$. The relevant W_i are given by the discontinuity T_i of the left hand cut according to (3.13).

To compute a doubly differential rate one needs to integrate over one of the variables. This involves an integration over the variable (possibly with some weight function). Using (3.13) this integration can be replaced by the contour integration depicted in (4). Note that there is a gap between the two cuts such that the contour does not get close to the singularity, which indicates that a perturbative calculation is possible for sufficiently "smeared" quantities.

Before closing the general set-up I need to point out some subtleties. The proof that an OPE exists, can strictly only be performed in the deep euclidean region, i.e. for $q^2 \to -\infty$ in (3.7). However, in all applications in heavy quark physics we are actually in the minkowskian region; the momentum in (3.6) is $m_Q v$ which is time-like, as well as the momentum $m_Q v - q$

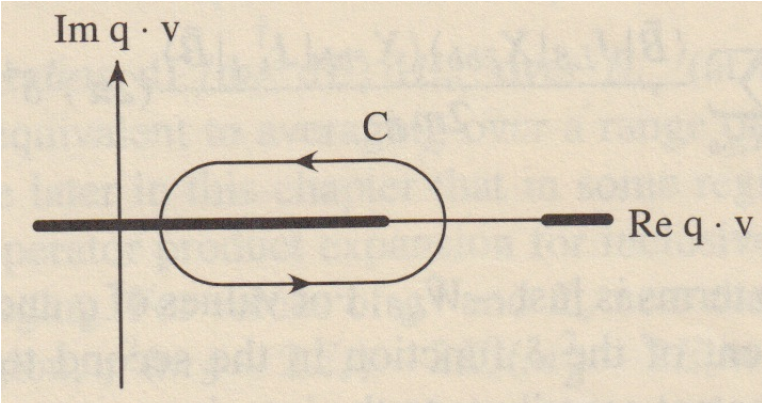


Figure 4: Sketch of the analytic structure of the T_i in the $v \cdot q$ plane for fixed q^2

for the differential rate. This innocent looking point of analytically continuing from euclidean to minkowskian region is, however, quite subtle; strictly speaking the OPE in the minkowskian region is not proven.

Annother, to some extent related issue is the issue of duality. As we shall see below, the leaading term of the "Heavy Quark Expansion" (HQE) for inclusive decays is the parton model, i.e. the decay of a "free" quark. Form a naive notion of quark-hadron duality one would assume this to be true, even for suitable "smeared" differential quantities. A more quantitative definition of duality can be given by linking this to the existence of an HQE, which means on the one hand the convergence of the expansion itself as well as the absence (or at least smallness) of non-analytic terms in the expansion parameters.

These points have to be taken as caveats, but as physicists we fearlessly proceed ....

3.3 Tree level Results

To be specific, we start out by constructing the OPE for an inclusive semileptonic $b \to c$ decay. The starting point is the expression (3.14) in the form (now we have $J_\mu = \bar{c}\gamma_\mu(1-\gamma_5)b$)

$$\int d^4y \, \exp(-iqy) \, T\{J_\mu^\dagger(y) J_\nu(0)\} = \int d^4y \, \exp(iy(m_Q v - q)) \, T\{\widetilde{J}_\mu^\dagger(y)\widetilde{J}_\nu(0)\} \tag{3.17}$$

for which we want to perform an OPE according to (3.7)

$$\int d^4y \, \exp(iy(m_Q v - q)) \, T\{\widetilde{J}_\mu^\dagger(y)\widetilde{J}_\nu(0)\} = \sum_n C_{\mu\nu}^{(n)} \mathcal{O}_n \tag{3.18}$$

The key point of the OPE is that it is an operator relation, which means that we can take any matrix element of this relation to compute the coefficients $C_{\mu\nu}^{(n)}$. Hence the simplest way to proceed (having discussed above that we may compute the coefficients in perturbation theory) is to take a matrix element with free quark and gluon states.

We start with a free b quark with momentum $p_b = m_b v + k$ corresponding to the splitting of the quarks momentum into a large part $m_b v$ and a residual part k. The leading tree-lecel diagram is simply given by the propagator of the free charm quark, leaving us with

$$R_{\mu\nu}^{(0)} = \bar{u}(p_b)\gamma_\mu(1-\gamma_5)\left[\frac{1}{\not{Q} + \not{k} - m_c}\right]\gamma_\nu(1-\gamma_5)u(p_b) \tag{3.19}$$

where we introduce $Q = m_b v - q$. At tree level, the construction of the OPE proceeds by expanding in the residual momentum k which is assumed to be small in all its components. For the propagator we get

$$\frac{1}{\not{Q} + \not{k} - m_c} = \frac{1}{\not{Q} - m_c} - \frac{1}{\not{Q} - m_c}\not{k}\frac{1}{\not{Q} - m_c} + \frac{1}{\not{Q} - m_c}\not{k}\frac{1}{\not{Q} - m_c}\not{k}\frac{1}{\not{Q} - m_c} + \cdots \tag{3.20}$$

Starting with the leading term, we get

$$\begin{aligned} R^{(0,0)}_{\mu\nu} &= \frac{1}{Q^2 - m_c^2}\bar{u}(p_b)\gamma_\mu(1-\gamma_5)(\not{Q} + m_c)\gamma_\nu(1-\gamma_5)u(p) \\ &= \frac{2}{Q^2 - m_c^2}\bar{u}(p)\gamma_\mu\not{Q}\gamma_\nu(1-\gamma_5)u(p_b) \end{aligned} \tag{3.21}$$

Before we continue, a subtlety should be mentioned. We did not expand the spinors $u(p_b)$ in powers of k, which looks a bit inconsistent on first sight. The matching procedure we employ is to compare the matrix element between fixed states of the right and the left hand side of (3.18). Consequently, we also would need to compute the matrix element between free quark states with the same momentum $p_b = m_b v + k$ on the right hand side. Thus the same spinors $u(p_b)$ will appear on the right hand side, and thus the expansion of the spinors would cancel. Thus we can as well drop the expansion of the spinors in both the left and the right hand side.

Integrating over the leptonic phase space, neglecting the lepton mass, one finds that we have to contract this expression with the tensor

$$L^{\mu\nu} = q^2 g^{\mu\nu} - q^\mu q^\nu \tag{3.22}$$

For illustrative reasons we only discuss the first term (even without the factor q^2, so we contract only with the metric tensor) and leave it as an exercise to do the full contraction. We thus get

$$\begin{aligned} R^{(0,0)} &= \frac{2}{Q^2 - m_c^2}\bar{u}(p_b)\gamma_\mu\not{Q}\gamma^\mu(1-\gamma_5)u(p-B) \\ &= \frac{-4}{Q^2 - m_c^2}\bar{u}(p_b)\not{Q}(1-\gamma_5)u(p_b) \end{aligned} \tag{3.23}$$

Thus at leading order (with this particular contraction of the indices) we see that the leading order expression for the OPE is

$$\int d^4y\, \exp(iyQ)\, T\{\widetilde{J}^\dagger_\mu(y)\widetilde{J}^\mu(0)\} = \frac{-4}{Q^2 - m_c^2}\,(b_v(0)\not{Q}(1-\gamma_5)b_v(0)) + \mathcal{O}(k) \tag{3.24}$$

This is the simplest example of a matching calculation, i.e. the comparison of the right-hand side of (3.18) to the expansion of the left hand side. The leading terms turns out to be a dimension-three operator; as we shall see, the higher order terms involve higher-dimensional operators.

The next step is to take the matrix elements with the real B meson states. In our mini-example we need to discuss the matrix element of a dimension-three operator

$$\langle B(v)|\bar{b}_v(0)\gamma_\lambda(1-\gamma_5)b_v(0)|B(v)\rangle = \langle B(v)|\bar{b}(0)\gamma_\lambda(1-\gamma_5)b(0)|B(v)\rangle = 2m_B v_\lambda \tag{3.25}$$

which does not contain any unknown parameter, since the vector current $\bar{b}_v(0)\gamma_\lambda b_v(0)$ is a conserved current, while the axial current vanishes.

In fact, in other applications different dimension-six matrix elements can appear, which differ from the case at hand only by the Dirac matrix between the heavy quark operators Q_v,

$$\bar{Q}_v \Gamma Q_v = \bar{Q} \Gamma Q \ : \quad \text{General dimension three operator} \tag{3.26}$$

where Γ is an arbitrary Dirac matrix. However, taking a forward matrix element between the pseudoscalar ground state meson, only $\Gamma = 1$ and $\Gamma = \gamma_\mu$ are non-vanishing.

As pointed out above, the vector current of the heavy quark Q is conserved, with the consequence that it does not induce an unknown hadronic matrix element:

$$\langle H(p_H)|\bar{Q}_v \gamma_\lambda Q_v|H(p_H)\rangle = \langle H(p_H)|\bar{Q}\gamma_\lambda Q|H(p_H)\rangle = 2p_{H\lambda} \tag{3.27}$$

The matrix element of $\bar{Q}_v Q_v = \bar{Q}Q$ can also be related to the vector current by the equations of motion

$$\begin{aligned} \not{v} Q_v &= Q_v - \frac{i\not{D}}{m_Q} Q_v & (3.28) \\ (ivD) Q_v &= \frac{1}{2m_Q} (i\not{D})(i\not{D}) Q_v & (3.29) \end{aligned}$$

Using (3.28) we get

$$\begin{aligned} \langle H(p_H)|\bar{Q}_v Q_v|H(p_H)\rangle &= \langle H(p_H)|\bar{Q}_v \not{v} Q_v|H(p_H)\rangle + \frac{1}{m_Q}\langle H(p_H)|\bar{Q}_v (i\not{D}) Q_v|H(p_H)\rangle \\ &= 2m_H + \frac{1}{m_Q}\langle H(p_H)|\bar{Q}_v (i\not{D}) Q_v|H(p_H)\rangle \\ &= 2m_H + \frac{1}{m_Q}\langle H(p_H)|\bar{Q}_v \not{v}(ivD) Q_v|H(p_H)\rangle \\ &= 2m_H + \frac{1}{2m_Q^2}\langle H(p_H)|\bar{Q}_v \not{v}(i\not{D})^2 Q_v|H(p_H)\rangle & (3.30) \end{aligned}$$

which means that to leading order in the HQE no unknown hadronic parameter is induced in general. Note also that in (3.30) no contribution of order $1/m_Q$ appears.

In order to obtain the total rate, we have to take the imaginary part of (3.35). To this end, we re-install the $i\epsilon$ prescription into the propagator and use the relation

$$2\,\mathrm{Im}\,\frac{1}{x + i\epsilon} = (2\pi)\delta(x) \tag{3.31}$$

from which we finally obtain

$$\begin{aligned} \Gamma &\sim 2\,\mathrm{Im}\int d^4y\, \exp(iyQ)\, \langle B(v)|T\{\widetilde{J}^\dagger_\mu(y)\widetilde{J}^\mu(0)\}|B(v)\rangle & (3.32) \\ &= -8(2\pi)\delta(Q^2 - m_c^2) m_B (v \cdot Q) + \cdots & (3.33) \end{aligned}$$

This result is in fact a general statement: in combination with (3.30) we get

The leading term in the HQE is the partonic result, i.e. the decay of a “free” heavy quark.

Next we look at the first term in the k expansion

$$
\begin{aligned}
R^{(0,1)} &= 2\left(\frac{1}{Q^2 - m_c^2}\right)^2 \bar{u}(p_b)\gamma_\mu \not{Q} \not{k} \not{Q} \gamma^\mu (1-\gamma_5) u(p_b) \\
&= -4\left(\frac{1}{Q^2 - m_c^2}\right)^2 \bar{u}(p_b)[Q^2 \not{k} - 2(Q\cdot k)\not{Q}](1-\gamma_5)u(p_b) \qquad (3.34)
\end{aligned}
$$

Comparing this to the OPE (3.18) we find ($k_\mu \to iD_\mu$)

$$
\int d^4y \, \exp(iyQ)\, T\{\widetilde{J}_\mu^\dagger(y)\widetilde{J}^\mu(0)\} = \text{leading term} \qquad (3.35)
$$

$$
+2\left(\frac{1}{Q^2 - m_c^2}\right)^2 \left[Q^2(b_v(0)(i\not{D})(1-\gamma_5)b_v(0)) - 2Q^\mu Q^\nu (b_v(0)(iD_\mu)\gamma_\nu(1-\gamma_5)b_v(0))\right]
$$

Again we have to take the forward matrix element of this expression. We use the equations of motion (3.28) and (3.29) and get for the general case (the contribution with γ_5 vanish due to parity)

$$
\begin{aligned}
\langle H(p_H)|\bar{Q}_v(i\not{D})Q_v|H(p_H)\rangle &= \langle H(p_H)|\bar{Q}_v \not{v}(ivD)Q_v|H(p_H)\rangle \qquad (3.36)\\
&= \frac{1}{2m_Q}\langle H(p_H)|\bar{Q}_v \not{v}(i\not{D})(i\not{D})Q_v|H(p_H)\rangle \\
\langle H(p_H)|\bar{Q}_v(iD_\mu)\gamma_\nu Q_v|H(p_H)\rangle &= \langle H(p_H)|\bar{Q}_v v_\mu \gamma_\nu (ivD)Q_v|H(p_H)\rangle \qquad (3.37)\\
&= \frac{1}{2m_Q}\langle H(p_H)|\bar{Q}_v v_\mu \gamma_\mu (i\not{D})(i\not{D})Q_v|H(p_H)\rangle
\end{aligned}
$$

which shows that these contributions are actually higher order in the $1/m_Q$ expansion. This is in fact a general statement

$\boxed{\text{There are no contributions of order } 1/m_Q \text{ in the HQE}}$

This does not mean that the first term in the k expansion vanishes, rather the dimension-four matrix elements are $1/m_Q$ suppressed.

In order to obtain the corresponding contribution to the rate, we have to take the imaginary part by re-inserting the $i\epsilon$ prescription into the propagator. Using

$$
2\,\mathrm{Im}\left(\frac{1}{x+i\epsilon}\right)^2 = -2\,\mathrm{Im}\frac{d}{dx}\left(\frac{1}{x+i\epsilon}\right) = -(2\pi)\delta'(x) \qquad (3.38)
$$

we obtain a contribution to the differential rate proportional to the derivative of the "on-shell" δ function, which, however is of order $1/m_b^2$.

In order to obtain the full $1/m^2$ contributions, one needs to expand $R_{\mu\nu}$ to second order in k_μ, which yields for our toy example

$$
R^{(0,1)} = 2\left(\frac{1}{Q^2 - m_c^2}\right)^3 \bar{u}(p_b)\gamma_\mu \not{Q}\not{k}\not{Q}\not{k}\not{Q}\gamma^\mu(1-\gamma_5)u(p_b) \qquad (3.39)
$$

which eventually matches on operators with two derivatives:

$$
\bar{Q}_v\Gamma(iD_\mu)(iD_\nu)Q_v \;:\quad \text{General dimension five operator.} \qquad (3.40)
$$

However, at that order an obvious problem arises: While $k_\mu k_\nu$ is obviously a symmetric tensor, the product of two covariant derivatives contains an antisymmetric part, since the covariant derivatives do not commute, their commutator is the field-strength tensor

$$\bar{Q}_v \Gamma [(iD_\mu), (iD_\nu)] Q_v = -ig_s \bar{Q}_v \Gamma G_{\mu\nu} Q_v \tag{3.41}$$

Obviously the expansion in k cannot give us this antisymmetric piece. However, the antisymmetric part is related to the field strength, i.e. with the emission of a gluon. In order to pin this down we thus have to compute a matrix element of (3.18) between a quark state and a state with a quark and a gluon. For the left hand side of (3.6) the leading order result is

$$S^{(0)}_{\mu\nu} = \bar{u}(p_b)\gamma_\mu(1-\gamma_5)\left[\frac{1}{\not{Q} + \not{k} - m_c}\right] T^a \not{\epsilon}(q) \left[\frac{1}{\not{Q} + \not{k} + \not{q} - m_c}\right] \gamma_\nu (1-\gamma_5) u(p_b) \tag{3.42}$$

where q is the momentum of the gluon with color a and polarization ϵ. Note that also the momentum of the gluon is soft, so we have to perform a combined expansion in k and q. Furthermore, the gluon appears as a part of the covariant derivative, so also the polarization ϵ counts as one power in the $1/m_b$ expansion; this means that in order to arrive at the second order, we have to expand (3.42) only to first order in k and q.

To obtain the coefficient of the antisymmetric combination (3.41) we thus have to find the coefficient in front of the combination

$$G_{\alpha\beta} \longleftrightarrow q_\alpha \epsilon_\beta - q_\beta \epsilon_\alpha$$

This concludes the sketch of the practical aspects of the matching procedure to obtain the coefficients in (3.18). The procedure remains the same even once α_s corrections are included, which means that the expansions in k and gluon momenta and polarization has to be performed for the expression including α_s corrections. By comparison between the two sides of (3.6) one thus obtains the perturbative expansion of the coefficients.

Finally it is worthwhile to point out, that the tree level expressions for the case of semileptonic decays can be obtained systematically, since the ordering of the covariant derivatives can be traced by using for the charm propagator an external field propagator of the form

$$\left[\frac{1}{\not{Q} + i\not{D} - m_c}\right]$$

Expanding this under the assumption that the components of iD do not commute yields formally the same expression as (3.20)

$$\frac{1}{\not{Q} + i\not{D} - m_c} = \frac{1}{\not{Q} - m_c} - \frac{1}{\not{Q} - m_c} i\not{D} \frac{1}{\not{Q} - m_c} + \frac{1}{\not{Q} - m_c} i\not{D} \frac{1}{\not{Q} - m_c} i\not{D} \frac{1}{\not{Q} - m_c} + \cdots \tag{3.43}$$

but now the ordering of the covariant derivatives in the correct one, i.e. for the second order term we have

$$\frac{1}{\not{Q} - m_c} i\not{D} \frac{1}{\not{Q} - m_c} i\not{D} \frac{1}{\not{Q} - m_c} = (iD_\mu)(iD_\nu) \left[\frac{1}{\not{Q} - m_c}\gamma^\mu \frac{1}{\not{Q} - m_c}\gamma^\nu \frac{1}{\not{Q} - m_c}\right]$$

where the term in the bracket has the correct antisymmetric piece in the indices μ and ν. However, this unfortunately only works at tree level.

3.4 HQE parameters

The non-perturbative input in the HQE is given in terms of the hadronic matrix elements of operators, which have generically the form[8]

$$\langle H(p_H)|\bar{Q}_v(iD_{\mu_1})(iD_{\mu_2})\cdots(iD_{\mu_n})\Gamma Q_v|H(p_H)\rangle\,,$$

where Γ is some Dirac matrix. Note that these operators have dimension $n+3$ and are defined in full QCD, which implies that they still depend on the mass. In principle one can perform an expansion in $1/m_Q$ and the leading term will be just

$$\langle\tilde{H}(v)|\bar{h}_v(iD_{\mu_1})(iD_{\mu_2})\cdots(iD_{\mu_n})\Gamma h_v|\tilde{H}(v)\rangle\,,$$

with the static field h_v and the meson state $|\tilde{H}(v)\rangle$ in the infinite mass limit.

We have already discussed the dimension-three operators and have shown, that there is no unknown matrix element at dimension three, since all matrix elements can be related to the conserved Q-quark vector current, up to terms of order $1/m_Q^2$. We also saw already that all the matrix elements of dimension-four operators are suppressed by one power of $1/m_Q$, and thus the first nontrivial contribution appear at dimension five, i.e. for $n=2$.

Before going into the technicalities a historic remark is in order. The idea that the decay of a ground-state hadron with a single heavy quark can be approximately described by the decay of the "free" quark inside the hadron is quite old. However, the HQE proves this to be the leading term of a systematic expansion, where the leading non-perturbative corrections turn out to be of the order $\Lambda_{\rm QCD}^2/m_Q^2$. In the early days of the HQE this was seen as an embarrassment: As a consequence the lifetimes of all ground state hadrons of a specific heavy flavour should be identical to leading order, the corrections should be of order $\Lambda_{\rm QCD}^2/m_Q^2$, and, as we shall see below, the lifetime differences should even be of the order $\Lambda_{\rm QCD}^3/m_Q^3$. Before the precise measurement of bottom-hadron lifetimes the lifetimes of charmed hadrons were available; with $m_D\approx m_c\sim 1.8$ GeV and $\Lambda_{\rm QCD}\sim 0.3$ GeV we naively expect lifetime differences to be below one percent. However, the lifetimes of ground-state charmed hadrons vary by a factor of five, which is hard to explain as a $\Lambda_{\rm QCD}^3/m_Q^3$ effect. Nevertheless, in the meantime we have a qualitative understanding why such large lifetime differences can emerge.

At each order in the $1/m_Q$ expansion we need to identify, how many independent parameters actually appear. At dimension five the two independent parameters can be defined as

$$\begin{aligned}-2m_H\hat{\mu}_\pi^2 &= \langle H(p_H)|\bar{Q}_v(iD)^2Q_v|H(p_H)\rangle &(3.44)\\ -2m_H\hat{\mu}_G^2 &= \langle H(p_H)|\bar{Q}_v(iD_\mu)(iD_\nu)(i\sigma^{\mu\nu})Q_v|H(p_H)\rangle &(3.45)\end{aligned}$$

and any general matrix element can be related to these two through the "trace formula"

$$\begin{aligned}\langle H(p_H)|\bar{Q}_v(iD_\mu)(iD_\nu)\Gamma Q_v|H(p_H)\rangle &= -2m_H\frac{\hat{\mu}_\pi^2}{6}\,\mathrm{Tr}\left(\frac{1+\not{v}}{2}\Gamma\right)(g_{\mu\nu}-v_\mu v_\nu) &(3.46)\\ &\quad -2m_H\frac{\hat{\mu}_G^2}{12}\,\mathrm{Tr}\left(\frac{1+\not{v}}{2}(-i\sigma_{\mu\nu})\frac{1+\not{v}}{2}\Gamma\right)+\mathcal{O}\left(\frac{1}{m_Q}\right)\end{aligned}$$

From this we get our third statement on the HQE

$\boxed{\text{The first subleading corrections in the HQE are given by } \hat{\mu}_\pi \text{ and } \hat{\mu}_G.}$

[8]In fact, this does not cover all possible operators; there can also be operators with light quarks, which will be discussed separately.

The parameter $\hat{\mu}_\pi^2$ is called the kinetic energy parameter, since it is related to the term $\vec{p}^2/(2m_Q)$ appearing in the Schrödinger equation, the parameter $\hat{\mu}_G$ is called the chromomagnetic moment, since it describes the coupling $\vec{\sigma}\cdot\vec{B}$ of the heavy-quark spin to the chromomagnetic field $\vec{B}$.

The values of these parameters have to be taken from experimental information. This is particularly easy for $\hat{\mu}_G$ which can be obtained from hadron spectroscopy. Looking at the expansions of heavy hadron masses in inverse powers of the quark mass (2.29,2.30), we infer that we may use (2.34) to fix the value of the chomomagnetic moment, while the kinetic energy parameter cannot be obtained from spectroscopy.

Before we continue to higher orders, we point out a few details. In many applications it turns out to be useful to split the covariant derivative into a "time derivative" and a "spatial" part according to (2.6). To this end, one may as well use the definitions

$$-2m_H\mu_\pi^2 = \langle H(p_H)|\bar{Q}_v(iD^\perp)^2 Q_v|H(p_H)\rangle \tag{3.47}$$
$$-2m_H\mu_G^2 = \langle H(p_H)|\bar{Q}_v(iD^\perp_\mu)(iD^\perp_\nu)(i\sigma^{\mu\nu})Q_v|H(p_H)\rangle \tag{3.48}$$

and we have $\mu_\pi = \hat{\mu}_\pi + \mathcal{O}(1/m_Q)$ and $\mu_G = \hat{\mu}_G + \mathcal{O}(1/m_Q)$. Furthermore, all these parameters still depend on the heavy quark mass; expanding also this mass dependence yields mass independent parameters λ_1 and λ_2 defined in HQET by

$$2m_H\lambda_1 = \langle \tilde{H}(v)|\bar{h}_v(iD^\perp)^2 h_v|\tilde{H}(v)\rangle \tag{3.49}$$
$$2m_H\lambda_2 = \langle \tilde{H}(v)|\bar{h}_v(iD^\perp_\mu)(iD^\perp_\nu)(i\sigma^{\mu\nu})h_v|\tilde{H}(v)\rangle \tag{3.50}$$

The advantage of expanding any mass dependence and to define "static" quantities λ_1 and λ_2 is that these will be the same for *any* heavy quark, thus one might compare inclusive bottom with inclusive charm decays. The advantage of using μ_π and μ_G, or $\hat{\mu}_\pi$ and $\hat{\mu}_G$ becomes clear only when going to higher orders: Starting at $1/m_Q$ one also need to take into account the expansion of the state, since we have $|H(p_H)\rangle = |\tilde{H}(v)\rangle + \mathcal{O}(1/m_Q)$, which in general leads to non-local matrix elements involving the subleading terms in the Lagrangian (2.15).

At dimension six we will have three derivatives

$$\langle H(p_H)|\bar{Q}_v(iD_\mu)(iD_\alpha)(iD_\nu)Q_v|H(p_H)\rangle$$

together with four-quark operators. At tree level, we can express all dimension six matrix elements in terms of two parameters which are given by

$$2m_H\hat{\rho}_D^3 = \langle H(p_H)|\bar{Q}_v(iD_\mu)(ivD)(iD^\mu)Q_v|H(p_H)\rangle \tag{3.51}$$
$$2m_H\hat{\rho}_{LS}^3 = \langle H(p_H)|\bar{Q}_v(iD_\mu)(ivD)(iD_\nu)(-i\sigma^{\mu\nu})Q_v|H(p_H)\rangle \tag{3.52}$$

where we have again used the covariant definitions. For these parameters, the same remarks apply as for $\hat{\mu}_\pi$ vs μ_π vs. λ_1 etc., which will have differences appearing as terms of subleading order in the $1/m_Q$ expansion.

In a similar fashion as for the the terms of dimension five we can write a trace formula, which reads in this case

$$\begin{aligned}\langle H(p_H)|\bar{Q}_v(iD_\mu)(iD_\alpha)(iD_\nu)\Gamma Q_v|H(p_H)\rangle &= 2m_H\frac{\hat{\rho}_D^3}{6}\,\mathrm{Tr}\left(\frac{1+\not{v}}{2}\Gamma\right)(g_{\mu\nu}-v_\mu v_\nu)v_\alpha \\ &+2m_H\frac{\hat{\rho}_{LS}^3}{12}\,\mathrm{Tr}\left(\frac{1+\not{v}}{2}(-i\sigma_{\mu\nu})\frac{1+\not{v}}{2}\Gamma\right)v_\alpha + \mathcal{O}\left(\frac{1}{m_Q}\right)\end{aligned} \tag{3.53}$$

Note that a consistent calculation of higher order terms requires to also take into account the subleading terms in the trace formula (3.46).

One may continue in the same fashion to higher orders, however, the number of independent parameters will grow strongly as one proceeds to orders higher than $1/m_Q^3$. At order $1/m_Q^4$ there is a total of 11 independent parameters, at $1/m_Q^5$ there are already 25 new parameters. While the four parameters up to $1/m_Q^3$ can be extracted from the data, the large number of parameters appearing at even higher orders have to be modeled or may one day be taken from lattice calculations.

3.5 QCD Corrections

The HQE has the potential to compute total and specific differential rates with extremely high precision. However, as pointed out above, the leading term is always the decay of the heavy quark inside the heavy hadron, where the result is the same as if we were discussing a "free" quark. If we ignore for the moment the mass of the final state particles, the decay width will be

$$d\Gamma \sim G_F^2 |V_{\mathrm{CKM}}|^2 m_Q^5 \tag{3.54}$$

which induces an enormously strong dependence on the heavy quark mass. In the early days of the HQE this was considered to be problem, since the heavy-quark mass is not a straightforward observable. Unlike for an electron, this mass cannot be just measured as a pole in the propagator, since there are no asymptotic states of out going quarks. The quark mass is thus just a parameter in the QCD Lagrangian and, in fact, depends on the scheme one chooses to define it.

Thus it seems that any ambiguity or uncertainty related to the heavy quark mass enters into the predictions of HQET enhanced by a factor of five, however, as we shall discuss below, this problem can be controlled and is related to a suitable choice of a scheme in which the mass is actually defined.

3.5.1 Why do we need a mass scheme?

When computing Feynman diagrams we insert a quark mass into the propagators of quarks. This mass is defined by the location of the pole of the propagator, which is the usual definition of what is called the pole mass m_Q^{Pole}. When constructing HQET we redefine the heavy quark momentum by $p_Q = m_Q v + k$, using some mass definition, which we choose to be also the pole mass. However, due to

$$m_H = m_Q + \bar{\Lambda} + \mathcal{O}(1/m_Q)$$

we may compensate any redefinition of the mass by a corresponding shift in the parameter $\bar{\Lambda}$.

The mass renomalization is related to the quark propagator. Including the (one particle irreducible) self energy contributions $\Sigma(p)$, the renormalized quark propagator becomes

$$S(p) = \frac{-iZ_2^{\mathrm{OS}}}{\not{p} - m_0 + \Sigma(p, m_Q^{\mathrm{Pole}})} \longrightarrow \frac{-i}{\not{p} - m_Q^{\mathrm{Pole}}} \quad \text{as } p^2 \to (m_Q^{\mathrm{Pole}})^2 \tag{3.55}$$

where the pole mass is related to the bare mass by

$$m_0 = Z_m^{\mathrm{OS}} m_Q^{\mathrm{Pole}} = \left(1 + \sum_{n=1}^{\infty} c_n \left(\frac{\alpha_s}{\pi}\right)^n\right) m_Q^{\mathrm{Pole}} \tag{3.56}$$

The coeffiicients c_n are divergent and need to be regularized. The standard way to regularize QCD is dimensional regularization (DREG) where the loop integrals over momenta are computed in $D = 4 - 2\epsilon$ space-time dimensions. At one loop one obtains

$$c_1 = -C_F\left(\left[\frac{1}{\epsilon} + \gamma_E - 4\pi\right]\frac{3}{4} + 1 + \frac{3}{4}\ln\frac{\mu^2}{(m_Q^{\rm Pole})^2} + \mathcal{O}(\epsilon)\right)$$

where in the case of the pole mass the scale μ is fixed by the on-shell condition (3.55) and $C_F = 4/3$ is the value of the $SU(3)$ Casimir operator in the fundamental representation.

Alternatively one may also use another definition of the quark mass, such as the the $\overline{\rm MS}$ definition, for which we have a relation similar to (3.56)

$$m_0 = Z_m^{\overline{\rm MS}} m_Q^{\overline{\rm MS}} = \left(1 + \sum_{n=1}^\infty b_n\left(\frac{\alpha_s}{\pi}\right)^n\right) m_Q^{\overline{\rm MS}} \tag{3.57}$$

where the $\overline{\rm MS}$ scheme is defined by removing only the $1/\epsilon + \gamma_E - 4\pi$ term, which means at one loop order

$$b_1 = -C_F\left[\frac{1}{\epsilon} + \gamma_E - 4\pi\right]\frac{3}{4} \tag{3.58}$$

Note that the $\overline{\rm MS}$ mass depends on the scale μ and is a running parameter.

The key point relevant for our discussion is that different mass definitions can be related by pertubation theory with finite coefficients. Thus we have

$$m_Q^{\rm Pole} = z^{{\rm Pole}\to\overline{\rm MS}}\, m_Q^{\overline{\rm MS}} = \frac{Z_m^{\overline{\rm MS}}}{Z_m^{\rm OS}} m_Q^{\overline{\rm MS}} \tag{3.59}$$

and

$$z^{{\rm Pole}\to\overline{\rm MS}} = 1 + \sum_{n=1}^\infty a_n\left(\frac{\alpha_s}{\pi}\right)^n \quad \text{and} \quad a_1 = -C_F\left(\frac{3}{4}\ln\frac{\mu^2}{(m_Q^{\rm Pole})^2} + 1\right) \tag{3.60}$$

Consider now a rate of the form (3.54) and assume that we have fixed the mass scheme to be e.g. the pole mass. Computing radiative corrections to (3.54) takes the schematic form

$$d\Gamma \sim G_F^2|V_{\rm CKM}|^2(m_Q^{\rm Pole})^5\left(1 + \frac{\alpha_s}{\pi}r_1 + \left(\frac{\alpha_s}{\pi}\right)^2 r_2 + \cdots\right) \tag{3.61}$$

with (after proper renormalzation) finite coefficients r_i. Switching now to another mass definition such as e.g. we find

$$\begin{aligned} d\Gamma &\sim G_F^2|V_{\rm CKM}|^2(m_Q^{\overline{\rm MS}})^5(z^{{\rm Pole}\to\overline{\rm MS}})^5\left(1 + \frac{\alpha_s}{\pi}r_1 + \left(\frac{\alpha_s}{\pi}\right)^2 r_2 + \cdots\right) \\ &= G_F^2|V_{\rm CKM}|^2(m_Q^{\overline{\rm MS}})^5\left(1 + \frac{\alpha_s}{\pi}(r_1 + 5a_1) + \cdots\right) \end{aligned} \tag{3.62}$$

Thus we conclude that the choice of a mass scheme determines the size of the radiative corrections. In other words, with a clever choice of the mass definition one can absorb radiative corrections into the definition of the mass. Clearly such a mass definition must also satisfy that there are ways to extract this mass from independent data as precisely as possible, since the dependence on the fifth power is still present.

It turns out that the pole mass is a particularly bad choice as a mass scheme, since the coefficients r_1 are large and do not seem to converge well. Related to the bad convergence is another problem with the pole mass, since it has an intrinsic uncertainty of the order of $\Lambda_{\rm QCD}$ related to an infrared renormalon. Better definitions are so-called short distance masses (e.g. the $\overline{\rm MS}$ mass) which do not have tis problem and can thus be determined in principle with arbitrary precision. For most of these short-distance masses the QCD corrections converge much better; there are even mass definitions especially designed for the HQE.

3.5.2 Short Distance Masses

We use again the pole mass as a starting point. In terms of the $\overline{\rm MS}$ mass we have

$$m_Q^{\rm Pole} = z^{\rm Pole \to \overline{MS}}\; m_Q^{\overline{\rm MS}} = \left(1 + \sum_{n=1}^{\infty} a_n \left(\frac{\alpha_s}{\pi}\right)^n\right) m_Q^{\overline{\rm MS}} \tag{3.63}$$

The main point of the following discussion is the fact that this perturbative relation is not converging, rather it is an asymptotic series. This is due to factorially growing contributions in the coefficients $a_n \sim n!$. In fact, one can show that the asymptotic behavior of the perturbative series is

$$z^{\rm Pole \to \overline{MS}} = 1 + \frac{C_F e^{5/6}}{\pi} \frac{\mu}{m_Q^{\overline{\rm MS}}} \alpha_s \sum_n (-2\beta_0 \alpha_s)^n \, n! \tag{3.64}$$

where

$$\beta_0 = \frac{1}{4\pi}\left(11 - \frac{2n_f}{3}\right)$$

is the leading term of the β function of QCD and n_f is the number of active flavors.

In order to consider the consequences of this observation, we study the Borel transform of the perturbative series, defined by

$$z(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^{n+1} \quad \longrightarrow \quad B[z](t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \tag{3.65}$$

If all the series existed, one could define the reverse operation by

$$z(\alpha) = \int_0^\infty dt \, \exp\left(-\frac{t}{\alpha}\right) B[z](t) \tag{3.66}$$

which at least has the same series expansion as the original z. However, the terms shown in (3.64) lead to poles on the positive real axis in the Borel transform. For the case at hand, the leading term originates from a singularity at $t = 1/2$ and hence the integral in (3.66) cannot be computed without a prescription of how to avoid this pole. This leads to an ambiguity which can be expressed by shifting the singularity in the complex t plane by a small amount ϵ either upwards of downwards, hence we use

$$\frac{1}{t - 1/2 + i\epsilon} - \frac{1}{t - 1/2 - i\epsilon} = 2\pi\delta(t - 1/2)$$

leaving us with an ambiguity of the form

$$\Delta z(\alpha) \propto \frac{\mu}{m_Q^{\overline{\rm MS}}} \exp\left(-\frac{1}{2\alpha(\mu)}\right) \sim \frac{\Lambda_{\rm QCD}}{m_Q^{\overline{\rm MS}}} \tag{3.67}$$

where we have inserted the running coupling of QCD in terms of $\Lambda_{\rm QCD}$.

Although these arguments can still be made more stringent, we have at least seen the essence of the reasoning which leads to the conclusion that the pole mass has an intrinsic uncertainty of the order of $\Lambda_{\rm QCD}$, related to infrared contributions, which can be related to the coulombic self interactions of a heavy quark.[9]

To this end, it means that the pole mass cannot be used for precise predictions. In particular, inserting the pole mass into (3.61) yields large QCD corrections which are mainly due to this particular choice of the mass. In other words, a more clever choice of the mass definition can minimize the size of the QCD corrections and lead to a much better convergence.

Another problem induced by this becomes apparent once power corrections are included. Given an intrinsic uncertainty of the order $\Lambda_{\rm QCD}$ in the mass in (3.61) renders the power corrections, which are by themselves $\mathcal{O}(\Lambda_{\rm QCD}^2/m_Q^2)$, completely meaningless.

Thus it is obvious that we have to switch to a "short-distance" mass such as the $\overline{\rm MS}$ this mass depends on the scale μ which is usually taken to be $\mu \geq m_Q^{\overline{\rm MS}}$; below this scale one should switch to HQET, and thus it becomes clear that one should use mass definitions which are "designed" t go to scales as low as 1 GeV. There are two mass schemes which are frequently used in the context of the HQE which are the kinetic mass scheme and the $1S$ mass scheme.

3.5.3 Kinetic Mass Scheme

As we discussed above, the pole mass contains a renormalon ambiguity of the order of $\Lambda_{\rm QCD}$. However, when we look at the expansion of the heavy hadron mass of the pseudoscalar ground state meson (2.29) we have a physical quantity (the hadron mass) on the left hand side, which cannot suffer from such an ambiguity. However, on the right hand side, we have not yet specified, what mass definition is used. If we use the pole mass, we find that this ambiguity has to cancel between $m_Q^{\rm Pole}$ and the binding-energy parameter $\bar{\Lambda}$ defined in (2.31).

The parameter $\bar{\Lambda}$ is nonperturbative and can be obtained from a "Small-Velocity" sum rule. To construct such a sum rule we make use of the discussion of inclusive decays and consider a current of the form

$$J = \bar{Q}\Gamma q$$

of a heavy and a light(er) quark. We define the correlator by

$$T = ... \tag{3.68}$$

$$m_Q^{\rm kin}(\mu) = m_Q^{\rm Pole} - [\bar{\Lambda}(\mu)]_{\rm pert} - \frac{1}{2m_Q^{\rm kin}(\mu)}[\mu_\pi^2(\mu)]_{\rm pert} \tag{3.69}$$

$$[\bar{\Lambda}(\mu)]_{\rm pert} = \frac{16}{9}\frac{\alpha_s(\mu)}{\pi}\mu \tag{3.70}$$

$$[\mu_\pi^2(\mu)]_{\rm pert} = \frac{4}{3}\frac{\alpha_s(\mu)}{\pi}\mu^2 \tag{3.71}$$

3.5.4 1S Mass Scheme

The $1S$ mass is defined as half of the perturbatively calculated mass of the lowest lying $J^{PC} = 1^{--}$, ${}^3S_1$ bottomonium state. ... **to be completed**

[9] We note that the mass $m_Q^{\overline{\rm MS}}$ does not suffer from this problem. This can be seen from its relation to the bare mass, where only the ultraviolet $1/\epsilon$ poles are removed.

3.6 End-Point Regions

When studying the spectra of photons and leptons one finds in some regions of phase space some pathological curves which prevents us to interpret the spectra point by point. These regions are related to endpoints of the spectra where the HQE breaks down. As an example, let us consider the endpoint of the electron spectrum in semileptonic B decays. The maximal lepton energy is given by

$$E_{\max} = \frac{m_B^2 - m^2}{2m_B}$$

where m is the mass of the lightest final state that can be produced. Close to this energy the possible final states are very few, in the extreme case only the single state with mass m. Clearly one cannot expect an inclusive calculation to be correct here, in other words, the HQE breaks down in this region.

Neglecting the mass of the final-state quark (which we expect to be a good approximation for the $b \to u$ case) already the partonic result behaves pathological in the endpoint region, since it is a θ function. In fact, one finds for the charged lepton spectrum up to $1/m_b^2$

$$\begin{aligned} \frac{d\Gamma}{dy} &= \frac{G_F^2 m_b^5 |V_{ub}|^2}{192\pi^3} \left[\theta(2E - m_b) y \left\{ (3-2y)y - \frac{5y^2}{3}\frac{\mu_\pi^2}{m_b^2} + \frac{y}{3}(6+5y)\frac{\mu_G^2}{m_b^2} \right\} \right. \\ &\qquad \left. + \frac{\mu_\pi^2 - 11\mu_G^2}{6m_b^2}\delta(1-y) + \frac{\mu_\pi^2}{6m_b^2}\delta'(1-y) \right] \end{aligned} \tag{3.72}$$

with $y = 2E/m_b$. Nevertheless, the integrated inclusive rate exists and can be compute in a $1/m_b$ expansion as shown above, for the case at hand we get

$$\Gamma = \frac{G_F^2 m_b^5 |V_{ub}|^2}{192\pi^3} \left[1 - \frac{\mu_\pi^2 + 3\mu_G^2}{2m_b^2} \right] \tag{3.73}$$

In addition, one can show by the same steps as for the total rate also moments of the spectra can be computed in the HQE.

Obviously the spectrum cannot be interpreted point by point, in particular close to the endpoint, since the true expansion parameter is $\Lambda_{\rm QCD}/(m_b - 2E)$, which becomes large close to the endpoint. Very close to the endpoint we have a region which is dominated by single states or resonances, where a description in terms of (a sum over a few) exclusive states is appropriate, and this region is defined by $0 \le (m_b - 2E)^2 \le \Lambda_{\rm QCD}^2$.

This particularly means that such a fine "resolution" of the spectrum in the endpoint region is impossible within the HQE. However, if we look at the structure of the terms of the HQE, we see that the "most singular" term (i.e. the term with the highest derivative of the δ-function) is the last term in (3.72); in fact, proceeding to $1/m_b^3$ exhibits a term with $\delta''(1-x)$ etc. These terms can be summed by a technique analogous to to what is done in Deep Inelastic Scattering (DIS), leading to nonperturbative functions instead of nonperturbative parameters.

In order to illustrate this technique, we will (instead of $B \to X_u \ell \bar\nu$) consider $B \to X_s \gamma$. The leading contribution to this process is mediated by the operator

$$O_7 = \frac{e^2}{16\pi^2} m_b \, \bar s_L \sigma^{\mu\nu} b_R \, F_{\mu\nu} \qquad H_{\rm eff} = \frac{4G_F}{\sqrt{2}} V_{tb} V_{ts}^* C_7(\mu) O_7(\mu) \tag{3.74}$$

Computing the inclusive rate for $B \to X_s \gamma$ using only this operator yields up to order $1/m_b^2$

$$\Gamma(B \to X_s \gamma) = \frac{\alpha G_F^2 m_b^5}{16\pi^4} |V_{tb} V_{ts}^*|^2 |C_7(m_b)|^2 \left[1 - \frac{\mu_\pi^2 + 3\mu_G^2}{2m_b^2} \right] \tag{3.75}$$

However, one may also compute the photon spectrum for this decay, which at tree level and to leading order is a δ function, fixing the photon energy to the value $E_\gamma = m_b/2$ determined by the two-particle kinematics of the partonic process. This persists also for the tree-level expressions at higher orders in the HQE, leading to derivatives of δ functions.

Up to terms of order $1/m_b$ one finds for the spectrum

$$\begin{aligned}\frac{d\Gamma}{dy} &= \frac{\alpha G_F^2 m_b^5}{32\pi^4}|V_{tb}V_{ts}^*|^2|C_7(m_b)|^2 \\ &\times \left(\delta(1-y) - \frac{\mu_\pi^2 + 3\mu_G^2}{2m_b^2}\delta(1-y) + \frac{\mu_\pi^2 - \mu_G^2}{2m_b^2}\delta'(1-y) + \frac{\mu_\pi^2}{6m_b^2}\delta''(1-y)\right)\end{aligned} \tag{3.76}$$

Gluon emission will eventually lead to a nontrivial spectrum, however, a perturbative calculation is possible only in the region where the gluon and the final-state strange quark have a sizable invariant mass to warrant a perturbative treatment. Close to the endpoint we face the same situation as in $B \to X_u \ell\bar{\nu}$: The spectrum computed for the HQE cannot be interpreted point by point.

However, instead of studying the spectrum point by point, one may take moments of the spectrum. In fact, one may interpret the result (3.76) in terms of an expansion in singular functions, i.e. a moment expansion if the form

$$\frac{1}{\Gamma}\frac{d\Gamma}{dy} = \sum_{n=0}^{\infty} \frac{M_n}{n!}\delta^{(n)}(1-y) \tag{3.77}$$

where $\delta^{(n)}$ denotes the n^{th} derivative of the δ function, and the moments are defined as

$$M_n = \int_0^\infty (y-1)^n \left(\frac{1}{\Gamma}\frac{d\Gamma}{dy}\right) \tag{3.78}$$

From the structure of the HQE we infer that the moments M_n have a $1/m_b$ expansion, the leading term of which is of order $1/m_b^n$. For the case of $B \to X_s' gamma$ here we get

$$\begin{aligned} M_1 &= \mathcal{O}(1/m_b^2) = \frac{\mu_\pi^2 - \mu_G^2}{2m_b^2} \\ M_2 &= \frac{\mu_\pi^2}{6m_b^2} + \mathcal{O}(1/m_b^3) \\ M_3 &= -\frac{\rho_D^3}{18m_b^3} + \mathcal{O}(1/m_b^4) \end{aligned} \tag{3.79}$$

From this structure it is evident that a re-summation scheme would be desirable in which the leading contribution to each moment is re-summed. In order to set this up we take a look at the tree-level calculation of $B \to X_s\gamma$. Taking the time ordered product of two effective Hamiltonians from (3.74) and using the external-field propagator as in (3.20) (in this case of the massless s quark) we have

$$\frac{1}{\not{Q} + i\not{D}} = \frac{\not{Q} + i\not{D}}{Q^2 + 2(Q\cdot iD) + (i\not{D})^2} \tag{3.80}$$

where $Q = m_b v - q$, and q is the photon momentum. In the case where Q^2 is large compared to the terms with the covariant derivatives, one obtains the usual power counting and we may perform the expansion as in (3.20) with $m_c \to 0$. However, we have $Q^2 = m_b^2(1-y)$ and thus this quantity is not large compared to the other terms in the denominator, in which case cannot expand as in (refPropExp). Instead we are in the kinematic region, where Q^2 is small and $v \cdot Q$ is of the order m_b^2.

The region we are interested in is the one where Q^2 and $(Q \cdot iD)$ are of the same order, which is $m_b \Lambda_{\rm QCD}$. Note that this is not the resonance region, where - as discussed above - Q^2 is actually of order $\Lambda_{\rm QCD}^2$. Thus in the endpoint region $m_b(1-y) \sim \Lambda_{\rm QCD}$ we can re-sum the leading contributions to the moments by approximating

$$\frac{1}{\not{Q} + i\not{D}} = \frac{\not{Q}}{Q^2 + 2(Q \cdot iD)} + \cdots \tag{3.81}$$

Since Q is (almost) a light-like vector, it is convenient to introduce the shape function (or light-come distribution function) f according to

$$2M_B f(\omega) = \langle B(v) | \bar{b}_v \delta(\omega + i(n \cdot D)) | B(v) \rangle$$

which has the moment expansion

$$f(\omega) = \delta(\omega) + \frac{\mu_\pi^2}{6} \delta''(\omega) - \frac{\rho_D^3}{18} \delta'''(\omega) + \cdots \tag{3.82}$$

In terms of this function and using (3.81) one obtains for the spectrum of $B \to X_s \gamma$

$$\begin{aligned} \frac{d\Gamma}{dy} &= \frac{\alpha G_F^2 m_b^6}{32\pi^4} |V_{tb} V_{ts}^*|^2 |C_7(m_b)|^2 f(m_b(y-1)) \qquad (3.83) \\ &= \frac{\alpha G_F^2 m_b^5}{32\pi^4} |V_{tb} V_{ts}^*|^2 |C_7(m_b)|^2 \left(\delta(1-y) + \frac{\mu_\pi^2}{6m_b^2} \delta''(1-y) - \frac{\rho_D^3}{18m_b^3} \delta'''(1-y) + \cdots \right) \end{aligned}$$

The shape function f plays the same role as the parton distributions of DIS. They are genuinely non-perturbative, however, they are also universal. For the case of B decays, this means that this shape function appears in the end point regions $m_b(1-y) \sim \Lambda_{\rm QCD}$ of any inclusive heavy-to-light transition. In other words, it also appears in the description of the end-point region of $B \to X_u \ell \bar{\nu}$. This leads to a relation between this decay and $B \to X_s \gamma$ which is exploited in phenomenological analyses.

The shape function has a few interesting properties. First of all, we note the absence of a term with the first derivative of the δ function, which is a consequence of the equations of motion. Since this moment is taken with respect to the

This means that the shape function has to extend beyond the partonic endpoint $y = 1$ corresponding to the photon energy $E_\gamma = m_b/2$ and $\omega = 0$. The support of the shape function is $-\infty \le \omega \le \bar{\Lambda}$ where the region $0 \le \omega \le \bar{\Lambda}$ is entirely non-perturbative. The parameter $\bar{\Lambda}$ is exactly the same as the one appearing in the expansion of the heavy hadron masses (2.29,2.30), since the true phase space (ignoring the masses of the final-state hadrons) has a maximal photon energy $E_\gamma^{\rm max} = m_B/2$. Thus the shape function ensures the correct phase-space boundary.

The vanishing of the first moment corresponds to a definition of the quark mass. In fact, a measurement of the photon spectrum of $B \to X_s \gamma$ yields directly the shape function, the reference point for which the first moment vanishes yields a measurement of $\bar{\Lambda}$ and hence a definition of the quark mass.

All further discussion, including the way to include radiative corrections, requires more heavy machinery. Since the end-point region in heavy-to-light decays is related to (in the restframe of the B meson) energetic light degrees of freedom, the proper tool is in this case "Sort Collinear Effective Theory" which is beyond the scope of these lectures, and I refer the reader to the lectures of Thomas Becher at the same school.

Problems

- Construct the OPE for
$$\int d^4y\, e^{-iqx}\, T[\phi(x)\phi(0)]$$
where $\phi(x)$ is a free scalar field. Which operators appear, and what are their coefficients?

- Prove relation (3.25). Hint: write the matrix element in terms of form factors and make use of current conservation.

- Prove (3.28) and (3.29) by inserting the field Q_v into the Dirac equation. Hint for (3.29): Use $(i\not{D} + m_Q)(i\not{D} - m_Q) = (i\not{D})^2 - m_Q^2$

- List the independent HQE parameters appearing at dimension seven and give their physical interpretation.

4 Non-Relativistic QCD (NRQCD)

Up to this point we only discussed systems with a single heavy quark and otherwise only light quarks and gluons. However, there are a couple of systems that contain two heavy quarks. The most prominent systems are the quarkonia consisting of $c\bar{c}$ or $b\bar{b}$. These systems can decay into light hadrons by annihilation of the two heavy quarks into light quarks and gluons. Another class of "doubly heavy" systems are the $B-C$ states, consisting of $b\bar{c}$ or baryons with two (or event three) heavy quarks. In this last chapter we will discuss such systems by setting up an effective field theory approach.

4.1 Introduction: Why not just (HQET)^2?

The starting point of our discussion is analogous to the one in HQET. The rest frame of the heavy hadron defines again a velocity vector v, and both heavy quarks move with this velocity, up to small residual momenta. Considering quarkonia-like systems, the most naive way to implement this is to write down a theory with a heavy quark moving with velocity v and a heavy anti-quark with the same velocity. The binding between these two particles will be generated by multiple gluon exchange, and as an example we consider the diagram with two gluons depicted in fig. **??**. This diagram corresponds to the expression (leaving out all factors in the numerators)

$$B = \int \frac{d^D l}{(2\pi)^D} \left(\frac{1}{v \cdot l + i\epsilon} \right) \left(\frac{1}{-v \cdot l + i\epsilon} \right) \left(\frac{1}{(l+k)^2 + i\epsilon} \right) \left(\frac{1}{(l-k)^2 + i\epsilon} \right) \tag{4.1}$$

We note that this generates a "pinch singularity" at $v \cdot l = \pm i\epsilon$ which has to be removed by adding something to the denominators of the heavy (anti) quark.

An obvious candidate is to include some of the subleading terms of the $1/m_Q$ expansion into the particle propagator, such as the kinetic energy term. This would solve the problem with the pinch singularity, however, it will ruin the power counting of $1/m_Q$ powers, since the kinetic energy term is $1/m_Q$ suppressed. Thus in order to set up an effective field theory the power counting scheme has to be modified.

4.2 The NRQCD Lagrangian

The construction of the NRQCD Lagrangian proceeds along the same lines as the one for the HQET Lagrangian, which is a $1/m_Q$ expansion of QCD. Furthermore, the matching to QCD is performed in the same way as it is done in HQET. For the purpose of our subsequent discussion we will write down the Lagrangian in the rest frame $v = (1,0,0,0)$ and use the form obtained from the Foldy Woythuisen transformation. This has the advantage that all time-derivatives are eliminated from the higher-order terms, and the Lagrangain becomes

$$\begin{aligned} \mathcal{L}_{\mathrm{NRQCD}} \;\; = \;\; & \psi^\dagger \Bigg[(iD_0)\psi + c_2 \frac{(i\vec{D})^2}{2m_Q} + c_4 \frac{((i\vec{D})^2)^2}{8m_Q^2} + c_F g_s \frac{\vec{\sigma}\cdot\vec{B}}{2m_Q} + c_D \frac{\vec{D}\cdot\vec{E}}{8m_Q^2} \\ & + c_S g_s \frac{\vec{\sigma}\cdot(i\vec{D}\times\vec{E} - \vec{E}\times i\vec{D})}{8m_Q^2} + \cdots \Bigg] \psi \end{aligned} \tag{4.2}$$

where the field ψ corresponds to the field h_v in HQET taken in the rest frame $v = (1,0,0,0)$. The subsequent terms in (4.2) are known from the non-relativistic reduction of the Dirac equation as the kinetic energy, the chromomagnetic moment, the Darwin term and the spin-orbit term. We also introduced coefficients c_i which at tree level are unity, but can be computed in perturbative QCD as a power series in $\alpha_s(m_Q)$.

Reparametization invariance induces relations among these coefficients which hold to all order in perturbation theory

$$c_2 = c_4 = 1 \qquad c_S = 2c_F - 1 \tag{4.3}$$

Up to this point everything looks exactly as it looks like in HQET written in the rest frame. The key difference enters through the interpretation of the terms. As we discussed before, the leading term $\psi^\dagger(iD_0)\psi$ alone cannot be used to define the propagator of the field due to the problem of pinch singularities. Thus in NRQCD we use the first two terms in (4.2) to set up the propagator resulting in

$$\text{Propagator} \sim \frac{1}{k_0 - \vec{k}^2/(2m_Q) + i\epsilon} = \frac{1}{k_0 + i\epsilon} + \frac{1}{k_0 + i\epsilon}\left(\frac{\vec{k}^2}{2m_Q}\right)\frac{1}{k_0 + i\epsilon} + \cdots$$

and thus the kinetic energy in the denominator fixes the problem of pinch singularities.

Keeping the kinetic energy term in the denominator amounts to a re-interpretation of terms in (4.2). While in HQET only the first term containing the time derivative is kept, in NRQCD we keep the first two terms to define the "free" propagation of the quark. While this fixes the problem with the pinch singularities, it will force us to change our power counting, since the simple $1/m$ counting will not work any more. However, the matching between QCD and NRQCD remains the same as for HQET.

It is also interesting to note that the reparametrization invariance fixes the renormalization of the kinetic energy term to be the same as the one of the leading term. This also guarantees that the two terms used to define the NRQCD propagator renormalize in the same way.

4.3 Dynamically Generated Scales

Taking the the first to terms of the Lagrangian (4.2) for the "unperturbed" dynamics means to have instead of the static equation of motion $i\partial_0 h_v = 0$ the Schrödinger equation

$$i\partial_0\psi = \frac{(i\vec{\partial})^2}{2m_Q}\psi\,. \tag{4.4}$$

In order to set up a consistent power counting, the terms on both sides of the equation have to have to be of the same order in our power counting scheme. In non-relativistic quantum mechanics we have $i\vec{\partial} = \vec{p} = m_Q\vec{v}$ where $\vec{v}$ is the velocity of the particle (relative to the rest frame). Note that this is not to be confused with the four velocity used in HQET which defines the rest frame of the hadron. Applying this to the above Schrödinger equation (4.4) we find that the time derivative counts as

$$(iD_0) \sim m_Q\vec{v}^2 \qquad \text{and} \qquad (i\vec{D}) \sim m_Q|\vec{v}| \tag{4.5}$$

This eventually leads us to use the non-relativistic velocity $\vec{v}$ as the relevant parameter for the power counting.

In order to describe a quarkonium-like system we have to add to (4.2) another set of terms for the antiquark, χ which look exactly the same except for the sign of g_s. A quarkonium state is then obtained as a bound state of the quark ψ and the antiquark χ. Assuming for the moment that a potential constructed form a on-gluon exchange is sufficient for our purpose, the quarkonium state is given by a positronium-like configuration bound via a color coulombic potential.

Such a state will have various scales. The largest is the heavy quark mass m_Q which is a parameter appearing in the Lagrangian. In addition there are scales which are generated by the dynamics of the system. The first is the (inverse) size of the system (the Bohr radius) which is $m_Q\alpha_s \sim m_Q|\vec{v}|$ since the velocity of the quarks is $|\vec{v}| \sim \alpha_s$ in analogy to a non-relativistic system bound by a Coulomb potential. The second one is the binding energy of the system, which is $m_Q\alpha_s^2 \sim m_Q\vec{v}^2$ which is smaller by a factor of $|\vec{v}|$. Finally, there is still the scale $\Lambda_{\rm QCD}$ at which QCD becomes non-perturbative.

Assuming a very large mass m_Q one can imagine an ideal situation where the scale $\Lambda_{\rm QCD}$ is still small compared to the binding-energy scale $m_Q\vec{v}^2$. In such situation $\alpha_s(\mu = m_Q\vec{v}^2)$ is still a perturbative scale in which case this "ideal" quarkonium behaves like a coulombic system; in particular one expects that the spectum of such a system shows the typical $1/n^2$ spectrum of a Coulombic system. Unfortunately none of the candidate quarks (b and c) is heavy enough to be treated in this limit, while the top quark decays too fast to form a quarkonium state.

Thus we have to face the situation in which the scale $\Lambda_{\rm QCD}$ is somewhere in between $m_Q\vec{v}^2$ and m_Q and thus the quarkonia states are away from being ideal. To this end, a more realistic picture cen be obtained by assuming $m_Q \gg m_Q|\vec{v}| \gg \Lambda_{\rm QCD}$.

4.3.1 The case $mv^2 \leq mv \ll \Lambda_{\rm QCD}$

In the case where $m|\vec{v}|$ is still a perturbative scale, we may consider an effective field theory where also the scale $m|\vec{v}|$ is integrated out. The resulting effective theory has been named "potential NRQCD" (pNRQCD). The starting point is an NRQCD Lagrangian for a heavy quark and a heavy antiquark, for which we assume that they form a bound state. for scales $m|\vec{v}| \gg \mu \gg m\vec{v}^2 \sim \Lambda_{\rm QCD}$ we want to match to a theory with degrees of freedom consisting of a bound state of the quark and the antiquark.

I an nonrelativistic picture one may ... defien S and O multipole expansion etc...

4.4 Results for Quarkonia Processes

To be added eventually

4.5 Exotic States ?

To be added eventually

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