Les Houches: EFT for thermal systems, problems

1. (Thermodynamics) We computed $\langle T^{\mu\nu}\rangle$ in a free (massive) scalar field theory at finite temperature, to get the energy density and pressure:

$$\varepsilon = \int \frac{d^3p}{(2\pi)^3}\omega_p n_B(\omega_p), \qquad p = \frac{1}{3}\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{\omega_p}n_B(\omega_p). \tag{1}$$

   For an infinite system, argue that p is minus the free energy density, and deduce that $\varepsilon = -\partial_\beta(\beta p)$ (why?) where $\beta = 1/T$ is the inverse temperature. Check this explicitly for the above.

2. (3D scalar EFT) Consider the four-dimensional scalar theory with Euclidean Lagrangian density $\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2 + g^2\phi^4$, where g is small.[1] At finite temperature, the scalar acquires a thermal mass $\sim gT$. To deal with the hierarchy $gT \ll 2\pi T$, we argued that one can integrate out the scale $2\pi T$ (the nonzero Matsubara modes) and get a 3D effective theory:

$$S_{3d} = \int d^3x \left(\frac{(\partial\phi)^2}{2} + \frac{m_{th}^2\phi^2}{2} + \lambda_3\phi^4 + \dots\right) \tag{2}$$

   where $\lambda_3 = g^2T(1+O(g^2))$ (why?) and $m_{th}^2 = g^2T^2(1+O(g^2))$. In this problem you will clarify the dots "…", which stand for infinitely many terms generated by loops of nonzero Matsubara modes.

   a. Draw the simplest 4D graph which will generate a nonzero ϕ^6 term in 3d. How will its coefficient depend on g and T? What about the four-scalar interaction $\left[(\partial_i\phi\partial_i\phi)^2\right]$?

   b. The four-dimensional pressure is the sum of a UV (four-dimensional) contribution, plus T times (minus) the vacuum energy of the 3D theory. Use dimensional analysis, and that the only dynamical scale in the 3D theory is gT, to estimate the contribution to 4D pressure from the above two terms in the effective Lagrangian; recall that $[\phi] = \frac{1}{2}$ in 3D.

   c. Combining the powers of g from the Wilson coefficients and expectation values, argue that any operator not explicitly in eq. (2), with n scalar fields and k derivatives, contributes at most $g^{\frac{3n}{2}+k}T^4$ to the pressure (with some accidental cancelations for $n = 2$, why?) Enumerate all the operators needed to get the pressure to g^{10} accuracy (just the general structure, there aren't that many, assuming 3D rotation invariance).

   d. (optional) Recall that higher-dimensional operators in an effective Lagrangian are defined modulo total derivatives and modulo the lower equations of motion; for example, an operator $\varepsilon\phi(\partial^2)^2\phi$ can be removed by redefining $\phi \to \phi + \varepsilon\partial^2\phi$, and such field redefinitions can't change the physics. Use this to show that only *two* operators really need to be added to the Lagrangian in eq. (2) to accuracy g^{10} in the pressure (to what accuracy would one then need m_{th}^2, λ_3?)

[1] The Landau pole of this theory implies a UV cutoff $\Lambda \sim Te^{c/g^2(T)} \gg T$, which you should assume is so large as to have no practical implications for this discussion.

3. (Real-time formalism) The Schwinger-Keldysh contour has two time-like branches "1 and 2" which go from the initial density matrix and back, with action (dropping the part ϕ_0 which represented the initial density matrix in class):

$$S_{SK} = S[\phi_1] - S[\phi_2]. \tag{3}$$

We argued that it was much more effective to switch to the Keldysh basis of retarded/advanced fields, $\phi_r = \frac{\phi_1+\phi_2}{2}$, $\phi_a = \phi_1 - \phi_2$, where $D_{aa} = 0$ (why?) and:

$$D_{ra} = \frac{-i}{-(p_0 + i\varepsilon)^2 + \vec{p}^2 + m^2}, \qquad D_{rr} = \left(\frac{1}{2} + n_B(|p^0|)\right) 2\pi\delta(p_0^2 - \vec{p}^2 - m^2) \tag{4}$$

are the retarded propagator and anticommutator ('two outgoing arrows').

a. Compute the interactions in terms of ϕ_r, ϕ_a, for $S_{int}[\phi] = g\phi^3/3! + \lambda\phi^4/4!$, and draw the Feynman rules. Follow the arrow of time and draw: r=incoming arrow, a=outgoing. (Only 1 or 3 outgoing arrows should be possible.)

b. Check that the rules produce the claimed one-loop two-point function in ϕ^3:

$$G_{ra}(p) = D_{ra}(p) - \frac{g^2}{2}\int \frac{d^dq}{(2\pi)^d}\left(D_{ra}(q)D_{rr}(p-q) + D_{rr}(q)D_{ra}(p-q)\right) + O(g^4) \tag{5}$$

Feel free to drop a graph with a closed retarded loop (why?).

c. Define the retarded self-energy $\Pi_{ra}(p)$ as the sum of 1PI graphs with one incoming& one outgoing arrow. Show that the usual argument applies to the chain graphs for G_{ra}, which sum up to a geometric series:

$$G_{ra}(p) = \frac{-i}{-(p_0 + i\varepsilon)^2 + \vec{p}^2 + m^2 + \Pi_{ra}(p)} \tag{6}$$

d. (harder) According to the fluctuation-dissipation theorem, in equilibrium

$$G_{rr}(p) = \left(\frac{1}{2} + n_B(p^0)\right)(G_{ra}(p) - G_{ar}(p)). \tag{7}$$

Check this for D_{rr} above. Show that this relation is consistent with the Feynman rules, provided that Π_{rr}, defined as the sum of 1PI graphs with two outgoing arrows, satisfies the same relation.

(Order by order, the series for G_{rr} contains ill-defined terms $D_{rr}(p)D_{ra}(p) \propto \delta(p^2)/p^2$. Keldysh showed how to avoid such terms by systematically using the FDT relation (7).)

e. (Optional.) How to *not* do things. Show that correlators in the 1/2 basis are:

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} G_T & G^< \\ G^> & G_{\bar{T}} \end{pmatrix}, \tag{8}$$

where G_T and $G_{\bar{T}}$ stand for time-ordered and time-anti-ordered correlator, and $G^>$, $G^<$ are Witghtman (unordered) correlators. These can be derived from the fact the path integral computes *contour-ordered* correlators, for instance

$$G_{12}(x,y) \equiv \langle\phi_1(x)\phi_2(y)\rangle = \langle\phi(y)\phi(x)\rangle \equiv G^<(x-y). \tag{9}$$

Obtain the free propagator matrix (8) explicitly using the r/a results above and relations like $G_T = \frac{1}{2}(G_{ra} + G_{ar}) + G_{rr}$ (why?), which give "something like":

$$G_T(p) \simeq \frac{-i}{-p_0^2 + \vec{p}^2 + m^2 - i0} + n_B(|p_0|) 2\pi\delta(p_0^2 - \vec{p}^2 - m^2)\,, \quad \text{etc.} \tag{10}$$

If you feel brave, find the g^2 self-energy in ϕ^3 theory as a 2×2 matrix, and try to resum the chain graphs to reproduce the above results, simplifying the matrix multiplications as much as you can.