Evolution and factorization in high-energy QCD beyond leading order

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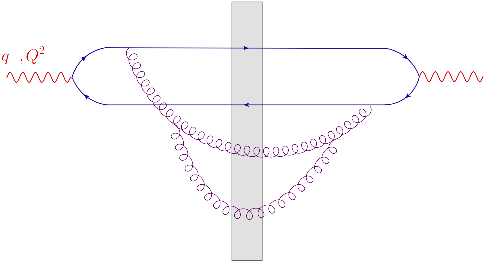
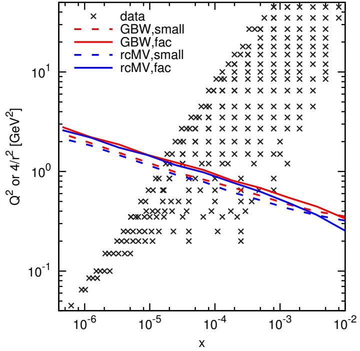
w/ J.D. Madrigal, A.H. Mueller, G.Soyez, and D.N. Triantafyllopoulos



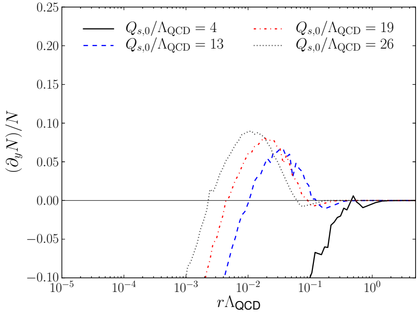


Introduction

- pQCD at high-energy, or 'small-x', is complicated by non-linear effects associated with the high gluon densities
    - gluon saturation, multiple scattering
    - resummations: Wilson lines, Color Glass Condensate
    - non-linear evolution equations: BK, B-JIMWLK
- Realistic phenomenology requires (at least) NLO accuracy
- The CGC formalism has recently been promoted to NLO
    - inclusion of running coupling corrections in BK
      (Kovchegov and Weigert, 2016; Balitsky, 2016)
    - NLO versions for the BK and B-JIMWLK equations
      (Balitsky and Chirilli, 2008, 2013; Kovner, Lublinsky, and Mulian, 2013)
    - NLO impact factor for particle production in pA collisions
      (Chirilli, Xiao, and Yuan, 2012; Mueller and Munier, 2012)
- But the NLO approximations turned out to be disappointing

NLO BK evolution

- "Negative growth" of the dipole scattering amplitude



Lappi, Mäntysaari, arXiv:1502.02400

- Hardly a surprise
  - similar problems for NLO BFKL
  - large transverse logarithms
  - collinear resummations
  - Mellin representation

(Salam, Ciafaloni, Colferai, Stasto, 98-03; Altarelli, Ball, Forte, 00-03)

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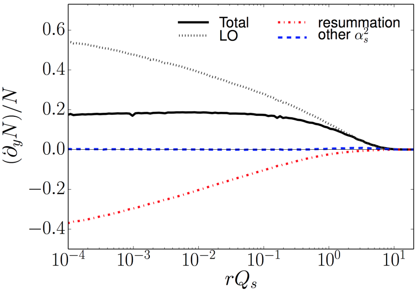

Lappi, Mäntysaari, arXiv:1601.06598

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(Salam, Ciafaloni, Colferai, Stasto, 98-03; Altarelli, Ball, Forte, 00-03)

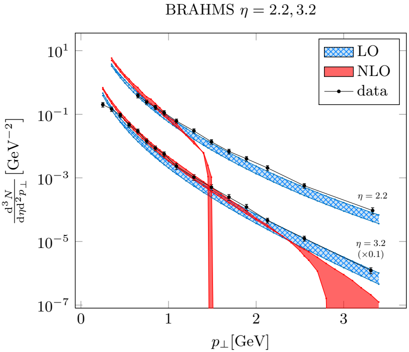
- Collinear improvement for NLO BK (transverse coordinates)

  (E.I., J. Madrigal, A. Mueller, G. Soyez, and D. Triantafyllopoulos, 2015)

- Evolution becomes stable with promising phenomenology
  - excellents fits to DIS *(Iancu et al, 2015; Albacete, 2015)*

d+Au collisions at RHIC

- Single inclusive hadron production at forward rapidities
- Very good agreement at low $p_\perp$ ☺ ... but negative at larger $p_\perp$ ☹



Stasto, Xiao, and Zaslavsky, arXiv:1307.4057

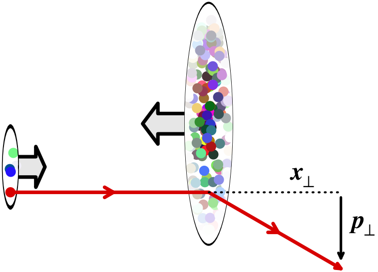
- Is this a real problem ?
  - "small-x resummations do not apply at large $p_\perp$"
  - but $p_\perp \sim Q_s$ is not that large !
  - and the turn-over is dramatic
- Are the 2 problems related ?
  - transverse logs are ubiquitous

- A fresh look at the NLO calculation of the cross-section

  (E.I., A. Mueller and D. Triantafyllopoulos, arXiv:1608.05293)

Quark production at forward rapidity

- A quark initially collinear with the proton acquires a transverse momentum $p_\perp$ via multiple scattering off the dense nucleus



$$\eta = \frac{1}{2}\ln\frac{p^+}{p^-}$$

$$x_p = \frac{p_\perp}{\sqrt{s}}\,\mathrm{e}^{\eta}$$

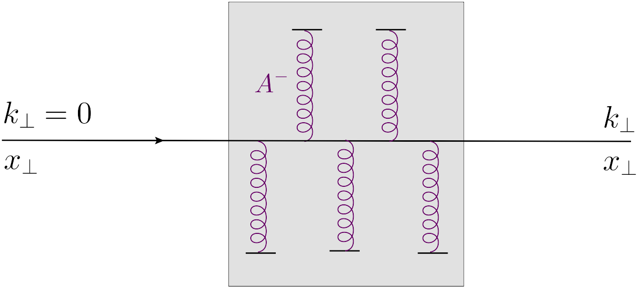
$$X_g = \frac{p_\perp}{\sqrt{s}}\,\mathrm{e}^{-\eta}$$

  - η : quark rapidity in the COM frame
  - x_p : longitudinal fraction of the quark in the proton
  - X_g : longitudinal fraction of the gluon in the target

- $\eta > 1$: 'forward rapidity' $\Longrightarrow$ $X_g \ll x_p$ ('dense-dilute')
  - RHIC: $p_\perp = 2$ GeV, $\eta = 3 \Longrightarrow x_p = 0.2$ & $X_g = 5 \times 10^{-4}$

Wilson lines

- Multiple scattering can be resummed in the eikonal approximation



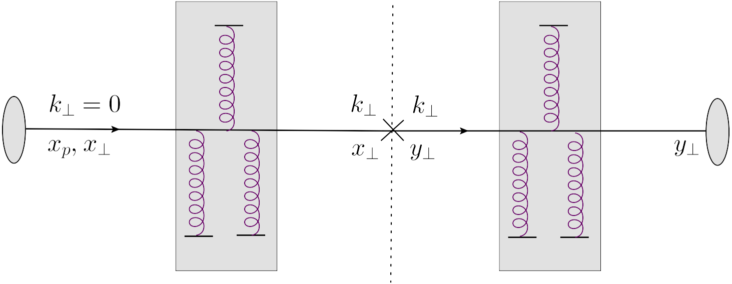
Amplitude: $$\mathcal{M}_{ij}(\boldsymbol{k}_\perp) \equiv \int \mathrm{d}^2\boldsymbol{x}_\perp\, \mathrm{e}^{-i\boldsymbol{x}_\perp\cdot\boldsymbol{k}_\perp}\, V_{ij}(\boldsymbol{x}_\perp)$$

Wilson line: $$V(\boldsymbol{x}_\perp) = \mathrm{P}\exp\left\{ig\int \mathrm{d}x^+ A_a^-(x^+,\boldsymbol{x}_\perp)t^a\right\}$$

- A_a^- : color field representing small-x gluons in the nucleus

Wilson lines

- Multiple scattering can be resummed in the eikonal approximation



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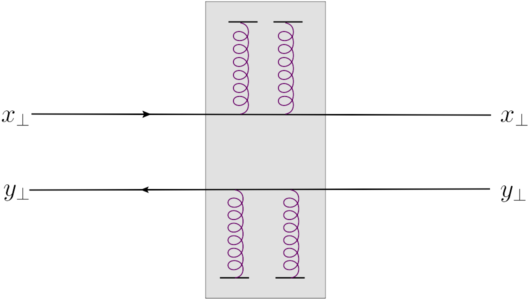
Cross-section: $$\frac{\mathrm{d}\sigma}{\mathrm{d}y\,\mathrm{d}^2\boldsymbol{k}_\perp} \simeq x_p q(x_p, Q^2)\,\frac{1}{N_c}\left\langle \sum_{ij} |\mathcal{M}_{ij}(\boldsymbol{k}_\perp)|^2 \right\rangle_{X_g}$$

- Average over the color fields A^- in the target (CGC)

- Equivalently: the elastic S-matrix for a $q\bar{q}$ color dipole



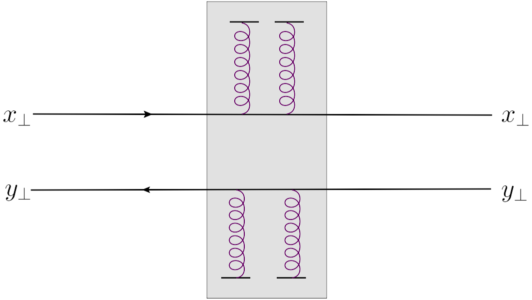
$$S(\boldsymbol{x},\boldsymbol{y};X_g) \equiv \frac{1}{N_c}\left\langle \mathrm{tr}\left[V(\boldsymbol{x})V^\dagger(\boldsymbol{y})\right]\right\rangle_{X_g}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y\,\mathrm{d}^2\boldsymbol{k}} \simeq x_p q(x_p)\int_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}}\; S(\boldsymbol{x},\boldsymbol{y};X_g)$$

- N.B. : The Fourier transform of the dipole S-matrix plays the role of the unintegrated gluon distribution in the nucleus.

Dipole picture

- Equivalently: the elastic S-matrix for a $q\bar{q}$ color dipole



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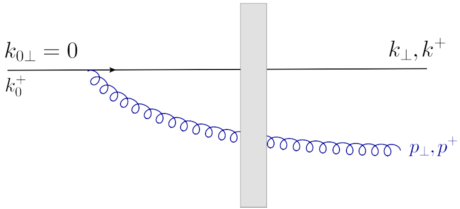
- This dipole picture is preserved by the high-energy evolution up to NLO *(Kovchegov and Tuchin, 2002; Mueller and Munier, 2012)*

High-energy evolution

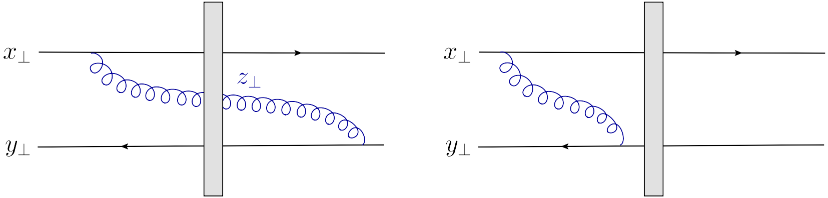
- Probability $\sim \alpha_s \ln \frac{1}{x}$ to radiate a soft gluon with $x \equiv \frac{p^+}{k_0^+} \ll 1$



- The gluon is not measured, but its emission modifies the cross-section for the production of the quark

High-energy evolution

- Probability $\sim \alpha_s \ln \frac{1}{x}$ to radiate a soft gluon with $x \equiv \frac{p^+}{k_0^+} \ll 1$



- Dipole picture: the gluon is emitted and reabsorbed within the dipole
- Evolution equation for the dipole S–matrix $S_{\boldsymbol{xy}}(Y)$ with $Y \equiv \ln(1/x)$

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}} \right]$$

  - dipole kernel: probability for the dipole to split *(Al Mueller, 1990)*
  - large-N_c approximation to the Balitsky-JIMWLK hierarchy

The BK equation *(Balitsky, '96; Kovchegov, '99)*

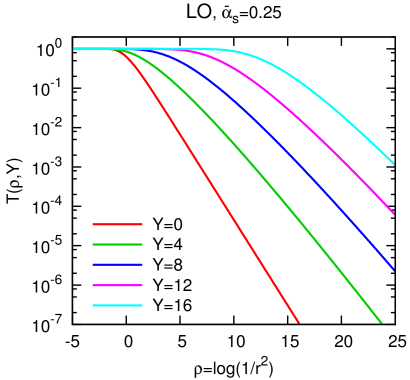
- Non-linear equation for the scattering amplitude $T_{\boldsymbol{xy}} \equiv 1 - S_{\boldsymbol{xy}}$

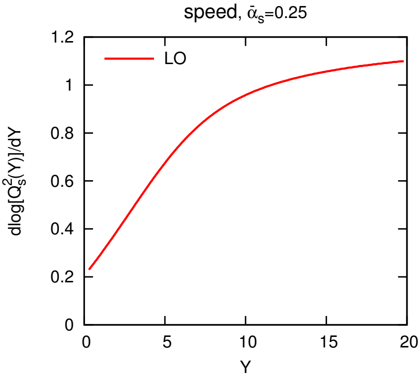
$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}} - T_{\boldsymbol{xz}} T_{\boldsymbol{zy}} \right]$$

  - $T_{\boldsymbol{xy}}(Y) = T(r, Y)$ with $r = |\boldsymbol{x} - \boldsymbol{y}|$ (dipole size)
  - weak scattering (dilute target): $T(r, Y) \ll 1 \Rightarrow$ BFKL equation
  - the BFKL solution increases exponentially with Y
  - non-linear term enforces unitarity bound: $T(r, Y) \leq 1$
  - saturation momentum $Q_s(Y)$: $T(r, Y) = 0.5$ when $r = 1/Q_s(Y)$
- The BK solution $T(r, Y)$:
  - a front which interpolates between weak scattering at $r \ll 1/Q_s(Y)$ and the 'black disk' limit at $r \gtrsim 1/Q_s(Y)$

The saturation front

- $T(r, Y)$ as a function of $\rho \equiv \ln(1/r^2)$ with increasing Y

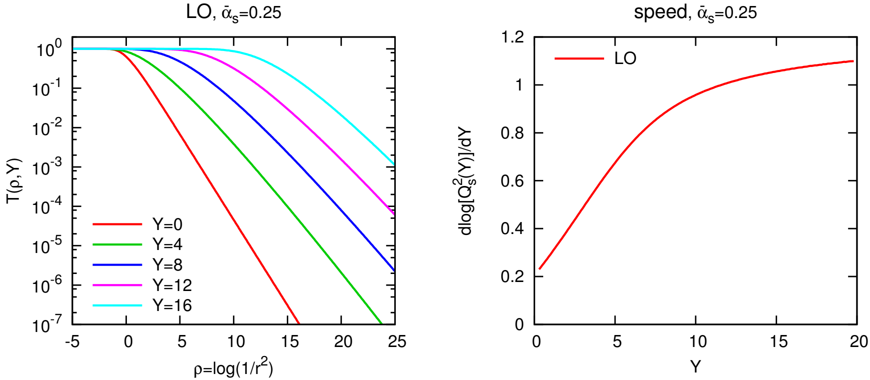




- color transparency at large ρ (small r) : $T \propto r^2 = \mathrm{e}^{-\rho}$
- unitarization at small ρ (large r) : $T = 1$ (black disk)
- saturation exponent: $\lambda_s \equiv \frac{\mathrm{d}\ln Q_s^2}{\mathrm{d}Y} \simeq 1$: way too large

The saturation front

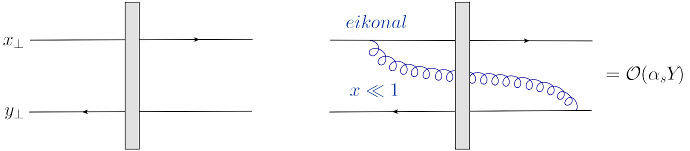
- $T(r, Y)$ as a function of $\rho \equiv \ln(1/r^2)$ with increasing Y



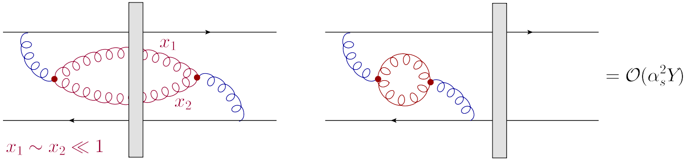
- color transparency at large ρ (small r) : $T \propto r^2 = \mathrm{e}^{-\rho}$
- unitarization at small ρ (large r) : $T = 1$ (black disk)
- saturation exponent: $\lambda_{\text{HERA}} = 0.2 \div 0.3$

Beyond leading order

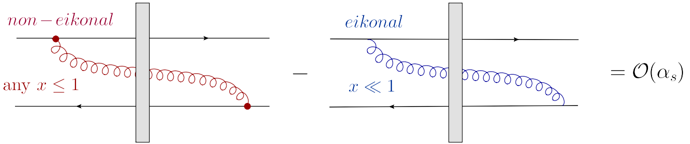
- LO approximation: any number $n \geq 0$ of soft emissions $\Longrightarrow (\alpha_s Y)^n$



- NLO corrections to the evolution: 2 soft gluons, with similar values of x



- NLO correction to impact factor: the first gluon is hard



BK equation at NLO *Balitsky, Chirilli (arXiv:0710.4330)*

- "Reasonably simple" (= it fits into one slide)
- Note however: $N_f = 0$, large N_c, tiny fonts

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\, (S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}) \Bigg\{ 1 +$$

$$+ \bar{\alpha}_s \Bigg[\bar{b}\, \ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\, \frac{(\boldsymbol{x}-\boldsymbol{z})^2 - (\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}$$

$$+ \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \Bigg] \Bigg\}$$

$$+ \frac{\bar{\alpha}_s^2}{8\pi^2} \int \frac{\mathrm{d}^2\boldsymbol{u}\, \mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\, (S_{\boldsymbol{xu}} S_{\boldsymbol{uz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}} S_{\boldsymbol{uy}})$$

$$\Bigg\{ -2 + \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 + (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2 - 4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \ln \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

$$+ \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2} \Bigg[1 + \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \Bigg] \ln \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \Bigg\}$$

Deconstructing NLO BK

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\, (S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}) \Bigg\{ 1 +$$

$$+ \bar{\alpha}_s \left[\bar{b}\, \ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\, \frac{(\boldsymbol{x}-\boldsymbol{z})^2 - (\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2} \right.$$

$$\left. + \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \right] \Bigg\}$$

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- green : leading-order (LO) terms
- violet : one-loop running coupling corrections
- red : NLO term enhanced by a double collinear logarithm
- blue : NLO term enhanced by a single logarithm

Deconstructing NLO BK

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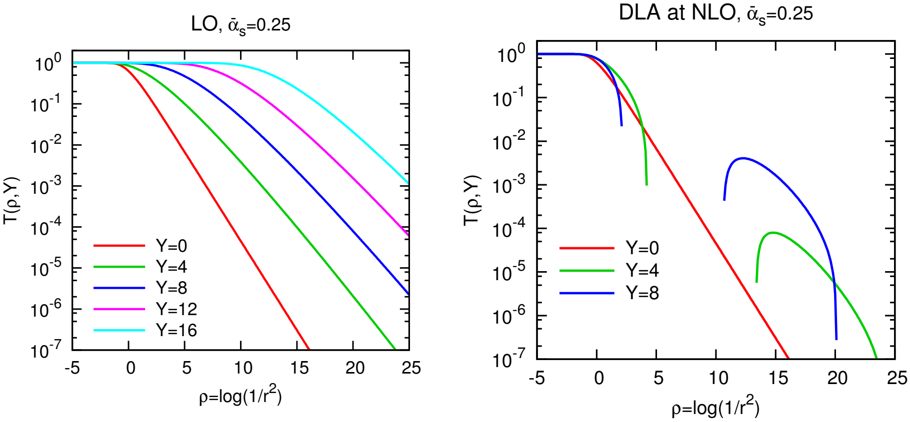
- green : leading-order (LO) terms
- violet : one-loop running coupling corrections
- double logs : kinematical origin — time ordering
- single logs : dynamical origin — DGLAP evolution

Unstable numerical solution

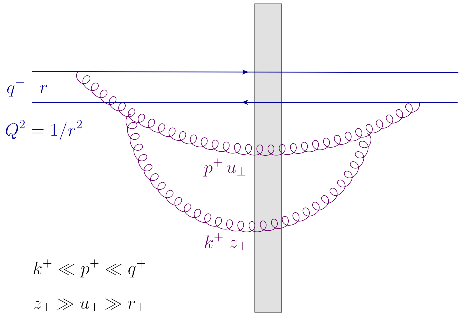
- Keep the double logarithm alone: very large daughter dipoles

$$-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\simeq-\frac{1}{2}\ln^2\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2}\quad \text{if}\quad |\boldsymbol{z}-\boldsymbol{x}|\simeq|\boldsymbol{z}-\boldsymbol{y}|\gg r$$



- LO approximation implicitly assumes that lifetimes are strongly ordered

$$\Delta t_p \sim p^+ u_\perp^2 \gg \Delta t_k \sim k^+ z_\perp^2$$



- lifetime of a gluon fluctuation

$$\Delta t_p \simeq \frac{2p^+}{p_\perp^2} \sim p^+ u_\perp^2$$

- satisfied by the LO kinematics

$$p^+ \gg k^+ \text{ and } u_\perp^2 \sim z_\perp^2$$

- violated when $z_\perp$ is large enough

- Integrate out the harder gluon $(p^+, u_\perp)$ to double-log accuracy:

$$\text{LO :} \quad \bar{\alpha}_s \int_{k^+}^{q^+} \frac{\mathrm{d}p^+}{p^+} \int_{r^2}^{z^2} \frac{\mathrm{d}u^2}{u^2} = \bar{\alpha}_s Y \ln \frac{z^2}{r^2}$$

Time ordering

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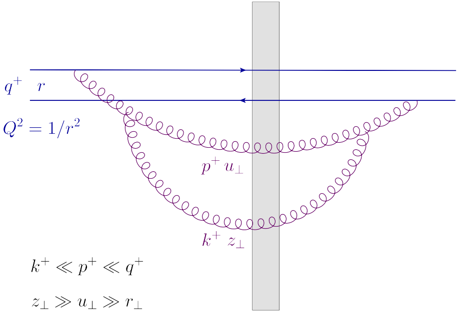

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$$\text{NLO}: \quad \bar{\alpha}_s \int_{k^+}^{q^+} \frac{\mathrm{d}p^+}{p^+} \int_{r^2}^{z^2} \frac{\mathrm{d}u^2}{u^2}\, \Theta(p^+ u^2 - k^+ z^2) \;=\; \bar{\alpha}_s Y \ln\frac{z^2}{r^2} - \frac{\bar{\alpha}_s}{2} \ln^2 \frac{z^2}{r^2}$$

Resummation of double logs in BK

- The double collinear logs can be resummed to all orders:
  - by enforcing time-ordering within the BK equation (non-local in Y)
    Ciafaloni (88), Andersson et al (96), ... G. Beuf (14)
  - by modifying the BK kernel to all orders (local in Y)
    E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos (15)

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \bar{\alpha}_s \int \frac{\mathrm{d}^2\boldsymbol{z}}{2\pi} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \mathcal{K}_{\mathrm{DLA}}\big(\rho(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\big)\left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)$$

- $\mathcal{K}_{\mathrm{DLA}}(\rho)$ resums powers of $\bar{\alpha}_s\rho^2$ to all orders:

$$\mathcal{K}_{\mathrm{DLA}}(\rho) \equiv \frac{\mathrm{J}_1\big(2\sqrt{\bar{\alpha}_s\rho^2}\big)}{\sqrt{\bar{\alpha}_s\rho^2}} = 1 - \frac{\bar{\alpha}_s\rho^2}{2} + \frac{(\bar{\alpha}_s\rho^2)^2}{12} + \cdots$$

$$\rho^2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}$$

Extending to single-logs & running coupling

- The NLO equation with all the transverse logs

$$\frac{\mathrm{d}T(r)}{\mathrm{d}Y} = \bar{\alpha}_s \int \mathrm{d}z^2 \, \frac{r^2}{z^4} \left\{ 1 - \bar{\alpha}_s \left(\frac{1}{2} \ln^2 \frac{z^2}{r^2} + \frac{11}{12} \ln \frac{z^2}{r^2} - \bar{b} \ln r^2 \mu^2 \right) \right\} T(z)$$

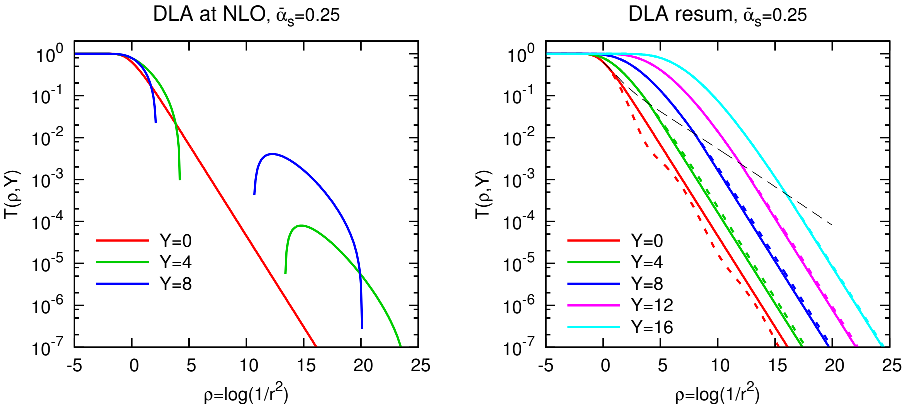
  - the double-logarithm is already included within $\mathcal{K}_{\mathrm{DLA}}(\rho)$ ✓
  - the collinear single-log is part of the DGLAP anomalous dimension ✓
  - the running coupling log is resummed by replacing $\bar{\alpha}_s \to \bar{\alpha}_s(r_{\min})$ ✓

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \int \frac{\mathrm{d}^2 \boldsymbol{z}}{2\pi} \, \bar{\alpha}_s(r_{\min}) \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left[\frac{r^2}{z_<^2} \right]^{\pm A_1 \bar{\alpha}_s} \mathcal{K}_{\mathrm{DLA}}\big(\rho(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\big) \times \big(S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}} \big)$$

$$A_1 \equiv \frac{11}{12}\,, \qquad z_<^2 \equiv \min\{(\boldsymbol{x}-\boldsymbol{z})^2, (\boldsymbol{y}-\boldsymbol{z})^2\}$$

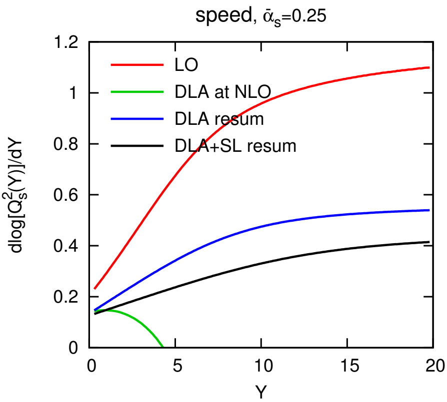
Numerical solutions: saturation front

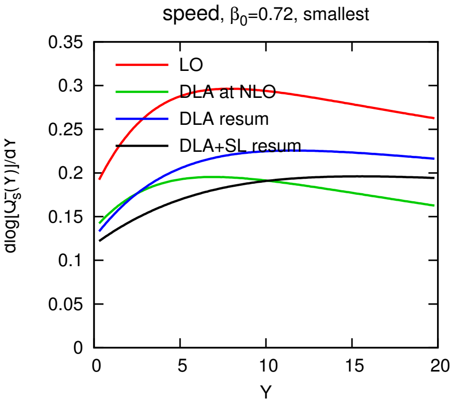
- The resummation stabilizes and slows down the evolution



- left: pure NLO
- right: double collinear logs resummed to all orders

Saturation exponent $\lambda_s \equiv \mathrm{d}\ln Q_s^2/\mathrm{d}Y$

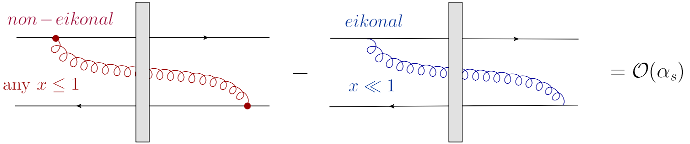




- Fixed coupling
  - LO: $\lambda_s \simeq 4.88\bar{\alpha}_s \simeq 1$
  - resummed DL: $\lambda_s \simeq 0.5$
  - DL + SL: $\lambda_s \simeq 0.4$
- Running coupling
  - LO: $\lambda_s = 0.25 \div 0.30$
  - DL + SL: $\lambda_s \simeq 0.2$
  - better convergence

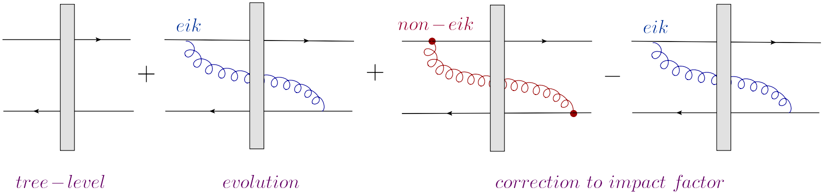
Particle production in pA collisions

- Recall: the NLO correction to impact factor



- The first gluon emission must be computed with exact kinematics (beyond the eikonal approximation)

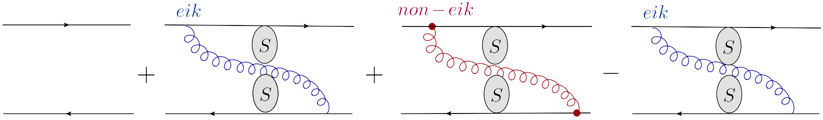
- The effect of one gluon emission: LO + NLO



Deconstructing the NLO approximation

(E.I., A. Mueller and D. Triantafyllopoulos, arXiv:1608.05293)

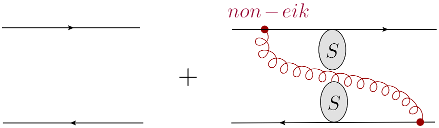
- Completing the evolution:



$\mathcal{S}$ (solution to $\mathcal{BK}$ equation) $\qquad$ $\mathcal{NLO}$ correction to impact factor

$$\mathcal{N} \;=\; \mathcal{S}(\boldsymbol{k}, X_g) + \bar{\alpha}_s \int_{X_g}^{1} \frac{\mathrm{d}x}{x} \left[\mathcal{K}(x) - \mathcal{K}(0)\right] \mathcal{S}\big(\boldsymbol{k}, X(x)\big)$$

- This is in principle the same as



$$\mathcal{N} = \mathcal{S}_0(\boldsymbol{k}) + \bar{\alpha}_s \int_{X_g}^{1} \frac{\mathrm{d}x}{x} \mathcal{K}(x) \mathcal{S}\big(\boldsymbol{k}, X(x)\big)$$

tree − level $\qquad$ *impact factor + evolution*

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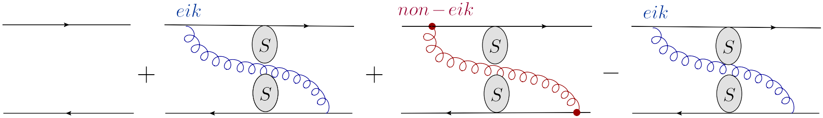

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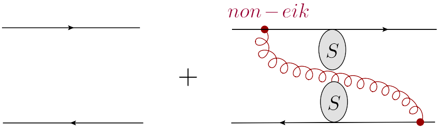
- This is in principle the same as ... but only in principle

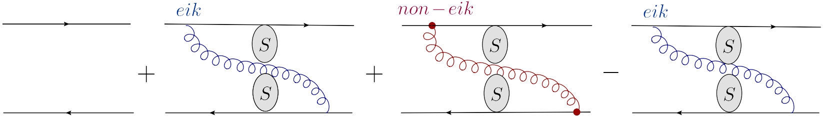


$$\mathcal{N} = \mathcal{S}_0(\boldsymbol{k}) + \bar{\alpha}_s \int_{X_g}^{1} \frac{\mathrm{d}x}{x} \mathcal{K}(x)\mathcal{S}\big(\boldsymbol{k}, X(x)\big)$$

tree − level $\qquad$ *impact factor + evolution*

The fine-tuning problem

- One adds and subtracts the LO evolution (the dominant contribution !)



$\mathcal{S}$ (solution to BK equation)   NLO correction to impact factor

$$\mathcal{N} = \mathcal{S}(\boldsymbol{k}, X_g) + \bar{\alpha}_s \int_{X_g}^{1} \frac{\mathrm{d}x}{x} \left[\mathcal{K}(x) - \mathcal{K}(0)\right] \mathcal{S}\big(\boldsymbol{k}, X(x)\big)$$

- The 'added' and 'subtracted' pieces are treated differently
  - the 'added' piece is used to reconstruct the solution to BK

$$\mathcal{S}(\boldsymbol{k}, X_g) = \mathcal{S}_0(\boldsymbol{k}) + \bar{\alpha}_s \int_{X_g}^{1} \frac{\mathrm{d}x}{x} \mathcal{K}(0)\, \mathcal{S}\big(\boldsymbol{k}, X(x)\big)$$

  - the 'subtracted' piece is used to isolate the NLO impact factor

Why is this a problem ?

- Any approximation/numerical error in the solution to the BK equation may lead to a mismatch between the 'added' piece and the 'subtracted' one
- A widely used 'approximation' (toy model): the GBW saturation model

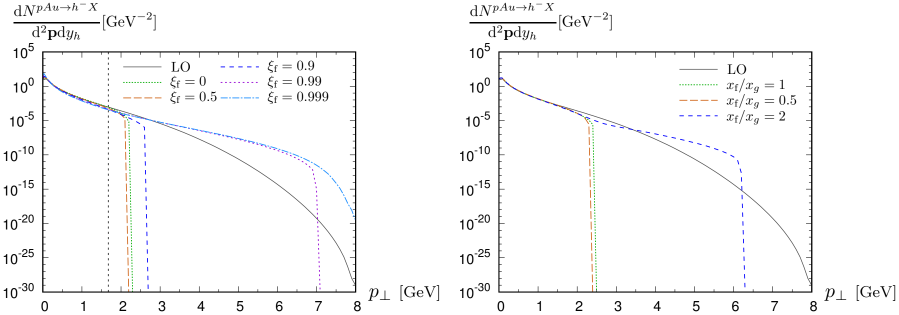
$$\mathcal{S}_{\rm GBW}(\boldsymbol{k}, X) \propto \mathrm{e}^{-\frac{k_\perp^2}{Q_s^2}}$$

  - the 'added' piece is exponentially suppressed at $k_\perp \gg Q_s$
  - the 'subtracted' piece develops a power-law tail $\propto 1/k_\perp^4$

$$\mathcal{N} = \mathcal{S}_{\rm GBW}(\boldsymbol{k}, X_g) + \bar{\alpha}_s \int_{X_g}^1 \frac{\mathrm{d}x}{x} \left[\mathcal{K}(x) - \mathcal{K}(0)\right] \mathcal{S}_{\rm GBW}(\boldsymbol{k}, X(x))$$

  - one gluon emission in pQCD always has a power-like tail !
  - the overall result becomes negative at sufficiently large $k_\perp$

CXY factorization + GBW model for $\mathcal{S}$



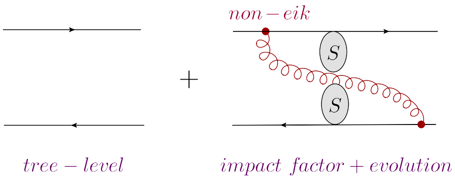
(Ducloué, Lappi, and Zhu, arXiv:1604.00225)

- This behavior is indeed visible in the numerical results
- Rapidity factorization scale $x_0 \equiv 1 - \xi_f$
- Decreasing x_0 pushes the problem to higher $k_\perp$
  - strongly dependent upon the precise implementation of x_0

A new factorization scheme

(E.I., A. Mueller and D. Triantafyllopoulos, arXiv:1608.05293)

- Back to basics: undo the 'rapidity' subtraction



$$\mathcal{N} = \mathcal{S}_0(\boldsymbol{k}) + \bar{\alpha}_s \int_{X_g}^{1} \frac{\mathrm{d}x}{x} \mathcal{K}(x) \mathcal{S}\big(\boldsymbol{k}, X(x)\big)$$

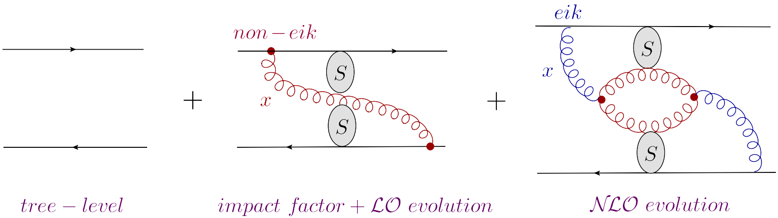
- LO evolution mixed up with NLO corrections to impact factor
  - not a strict perturbative expansion: it goes beyond NLO
  - respects the skeleton structure of the perturbative expansion

- Second term guaranteed to be positive definite
  - with $\mathcal{K}(x) \to \mathcal{K}(0)$: the r.h.s. of the LO BK equation

Restoring full NLO evolution

(E.I., A. Mueller and D. Triantafyllopoulos, arXiv:1608.05293)

- Recall: the NLO BK evolution also involves 2-loop graphs



$$\mathcal{N} = \mathcal{S}_0 + \bar{\alpha}_s \int_{X_g}^{1} \frac{\mathrm{d}x}{x} \mathcal{K}(x)\, \mathcal{S}\big(X(x)\big) + \bar{\alpha}_s^2 \int_{X_g}^{1} \frac{\mathrm{d}x}{x} \mathcal{K}_2(0)\, \mathcal{S}\big(X(x)\big)$$

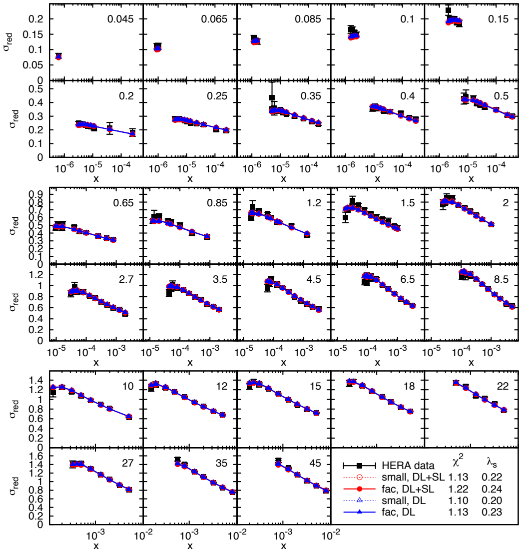
- $\mathcal{K}_2(0)$: NLO correction to the BK kernel with collinear improvement
- Complicated in practice ... but one can start with $\mathcal{S} \approx \mathcal{S}_{\mathrm{rcBK}}$ and $\mathcal{K}_2 = 0$

Fitting the HERA data

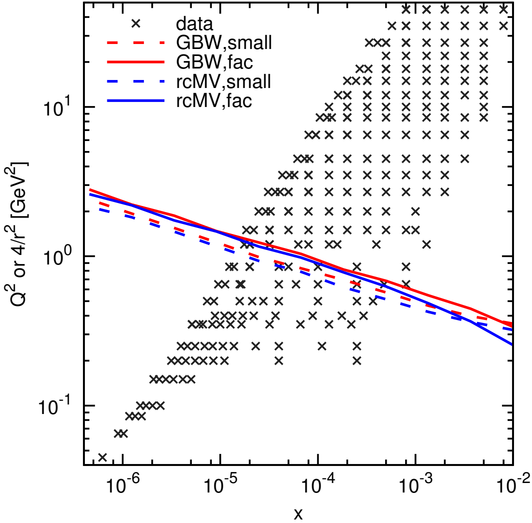
(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1507.03651)

- Numerical solutions to the collinearly-improved BK equation using initial conditions (at $x_0 = 0.01$) which involve 4 free parameters
  - a similar strategy as for the DGLAP fits
- Combined analysis by ZEUS and H1 (2009): small error bars
  - Bjorken' $x \leq 0.01$
  - $Q^2 < Q^2_{\max}$ with $Q^2_{\max} = 50 \div 400$ GeV^2
- 3 light quarks + charm quark, all treated on the same footing
  - good quality fits for $m_{u,d,s} = 0 \div 140$ MeV and $m_c = 1.3$ or 1.4 GeV
- χ^2 per point around 1.1-1.2
- Very discriminatory: the fits favor
  - initial condition: MV model with running coupling
  - smallest-dipole prescription for the running

The HERA fit: rcMV initial condition



The HERA fit: rcMV initial condition



- Saturation line $Q_s^2(x)$ superposed over the data points
  - saturation exponent: $\lambda_s = 0.20 \div 0.24$