Extending a Generalized Parton Distribution from DGLAP to ERBL (Preliminary results!)

From an Overlap of Light-cone Wave-functions to a Double Distribution

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Outline

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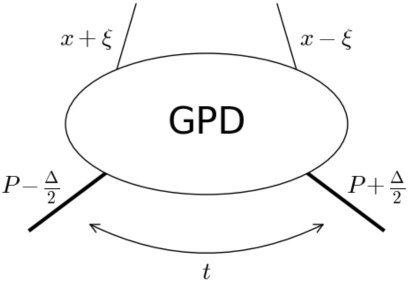
Definition of GPDs

- Quark GPD (twist-2, spin-0 hadron): **(Müller et al., 1994; Radyushkin, 1996; Ji, 1997)**

$$H^q(x,\xi,t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{i\,x\,P^+z^-} \left\langle P+\frac{\Delta}{2}\middle|\bar{q}(-z)\,\gamma^+ q(z)\middle| P-\frac{\Delta}{2}\right\rangle\Bigg|_{z^+=0,\; z_\perp=0} . \quad (1)$$



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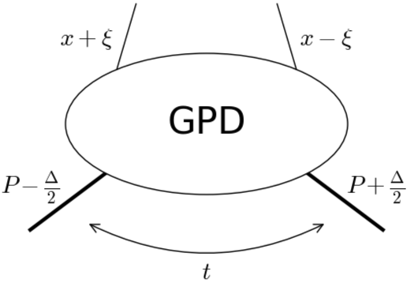

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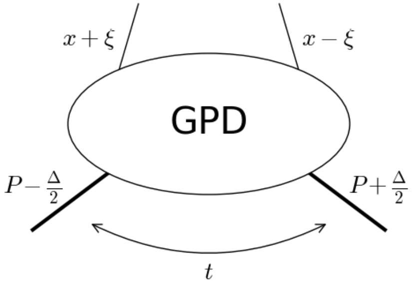

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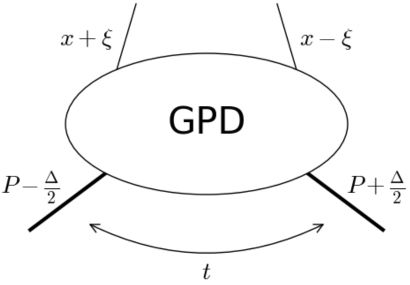

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- Impact parameter space GPD (at $\xi = 0$): **(Burkardt, 2000)**

$$q\left(x,\vec{b}_\perp\right) = \int \frac{\mathrm{d}^2\vec{\Delta}_\perp}{(2\pi)^2}\, e^{-i\,\vec{b}_\perp\cdot\vec{\Delta}_\perp}\, H^q\left(x,0,-\vec{\Delta}_\perp^{\,2}\right) . \quad (2)$$

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Main properties:

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- Link to PDFs and Form Factors:

$$\int \mathrm{d}x \, H^q\,(x, \xi, t) = F_1^q\,(t) \quad , \quad \int \mathrm{d}x \, E^q\,(x, \xi, t) = F_2^q\,(t) \,, \tag{5}$$

$$H^q\,(x, 0, 0) = \theta\,(x)\, q\,(x) - \theta\,(-x)\, \bar{q}\,(-x) \,. \tag{6}$$

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(Brodsky and Lepage, 1989)

- A given *hadronic state* is decomposed in a **Fock basis**:

$$|H; P, \lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x]_N \left[\mathrm{d}^2\mathbf{k}_\perp\right]_N \Psi^\lambda_{N,\beta}(x_1, \mathbf{k}_{\perp 1}, \ldots) \, |N, \beta; k_1, \ldots, k_N\rangle \ , \qquad (7)$$

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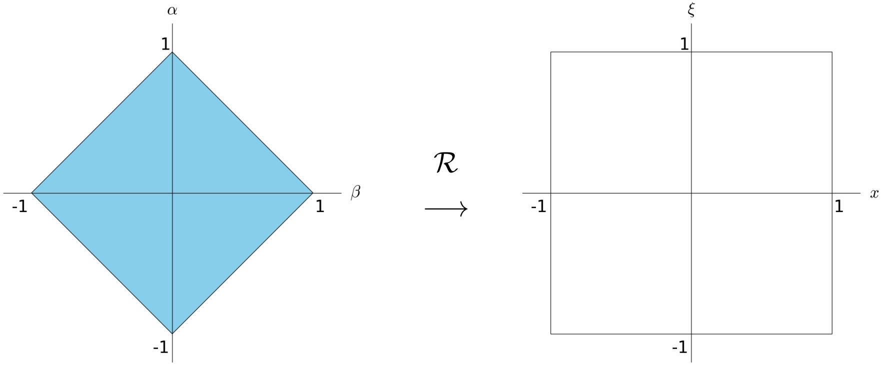
- Pobylitsa gauge (One Component DD): (Pobylitsa, 2003)

$$H(x,\xi,t) = (1-x)\int_\Omega \mathrm{d}\beta\,\mathrm{d}\alpha\,f_P(\beta,\alpha,t)\,\delta(x-\beta-\alpha\xi)\;, \quad (11)$$

with

$$\begin{cases} F(\beta,\alpha) & = & (1-\beta)\,f_P(\beta,\alpha) \\ G(\beta,\alpha) & = & -\alpha\,f_P(\beta,\alpha) \end{cases} \;. \quad (12)$$

Radon transform



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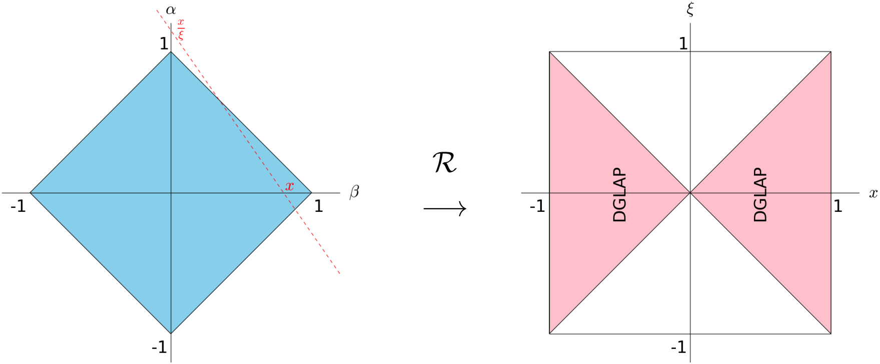

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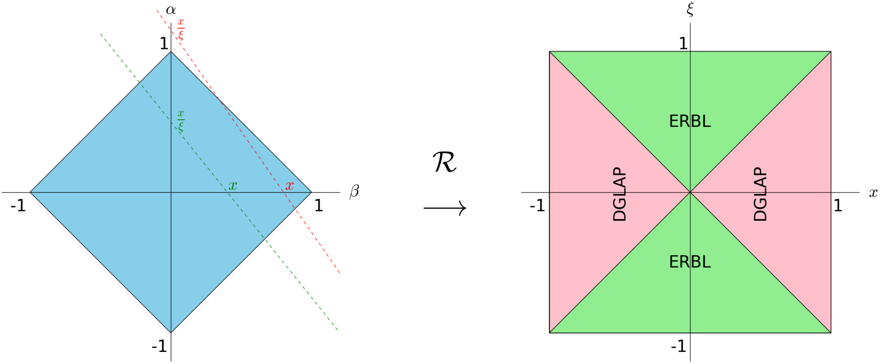

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1. Introduction to Generalized Parton Distributions
   - Definition and properties
2. Overlap and Double Distribution representations of GPDs
   - Overlap of Light-cone wave functions
   - Double Distributions
3. From an Overlap of LCWFs to a Double Distribution
   - Inversion of Incomplete Radon Transform
   - Results
4. Conclusion

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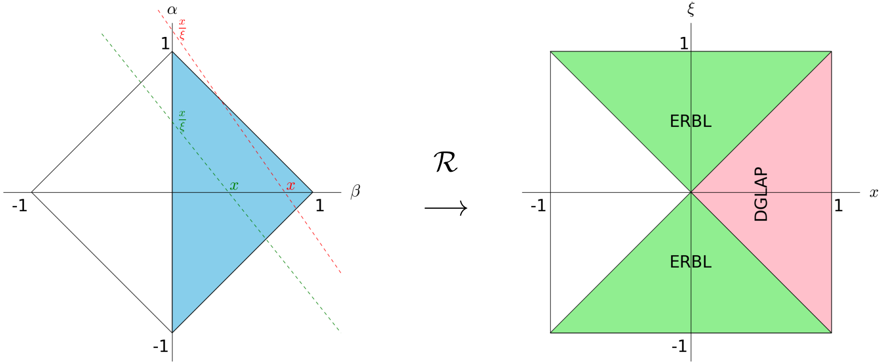
  - We can reconstruct the GPD everywhere.

Support properties



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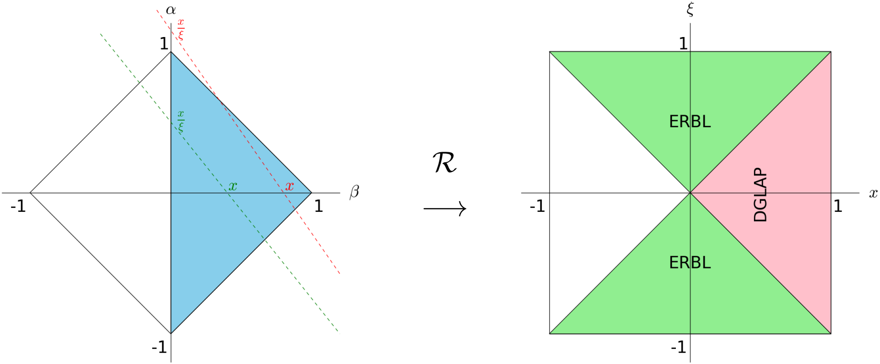

- Quark GPD: $H(x, \xi) = 0$ for $-1 < x < -|\xi| \implies f(\beta, \alpha) = 0$ for $\beta < 0$.

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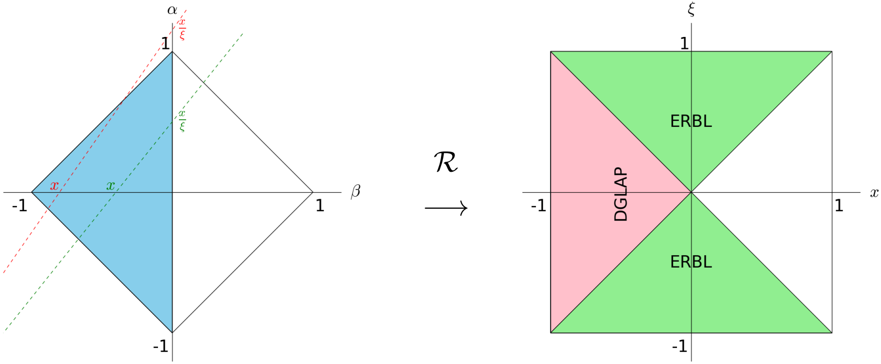

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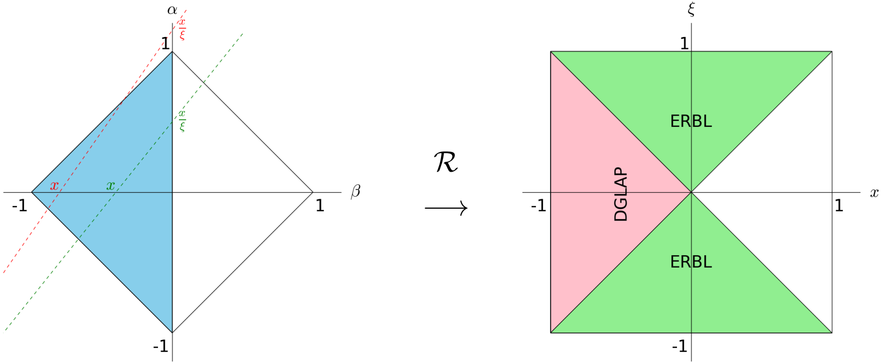

- Quark GPD: $H(x,\xi) = 0$ for $-1 < x < -|\xi| \Longrightarrow f(\beta,\alpha) = 0$ for $\beta < 0$.
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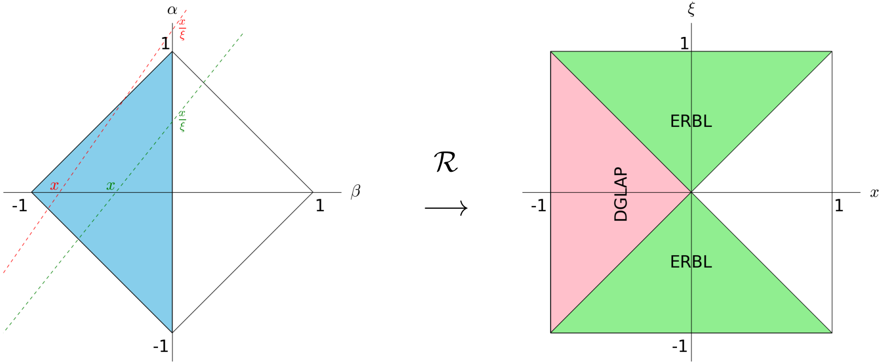
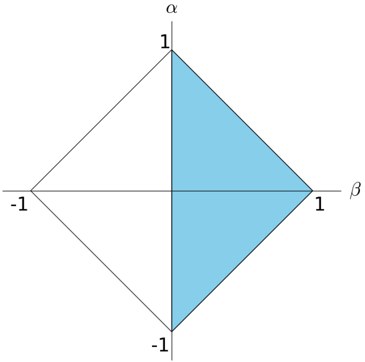


- Quark GPD: $H(x,\xi) = 0$ for $-1 < x < -|\xi| \implies f(\beta,\alpha) = 0$ for $\beta < 0$.
- Domains $\beta < 0$ and $\beta > 0$ are uncorrelated in the DGLAP region.
- Divide and conquer:
  - Better numerical stability.
  - Lesser complexity: $O(N^p + N^p) \ll O((N+N)^p)$.

Domain for the inversion



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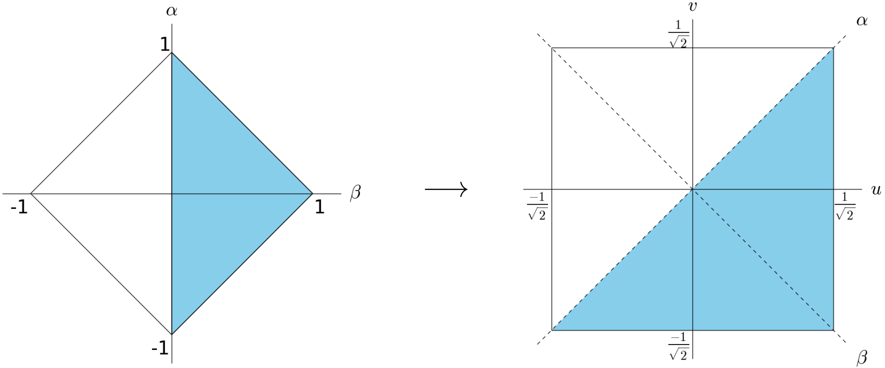

- Rotated square $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \times \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$:

$$\begin{cases} u = \dfrac{\beta + \alpha}{\sqrt{2}} & , \\ v = \dfrac{\alpha - \beta}{\sqrt{2}} & . \end{cases} \qquad (14)$$

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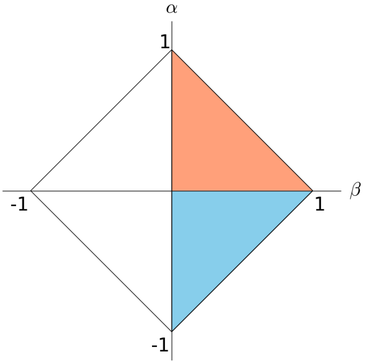
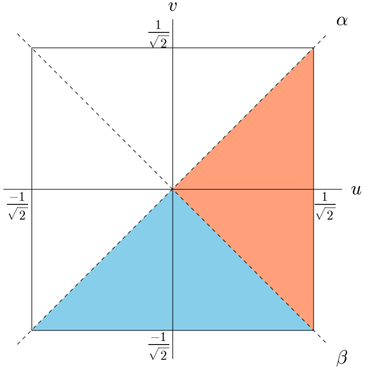


$\longrightarrow$



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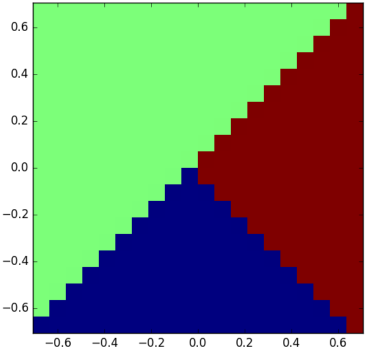
- α-parity of the DD:

$$f(\beta, -\alpha) = f(\beta, \alpha)\ . \tag{15}$$

Discretization

- Discretization of the DD (piece-wise constant):

$$\tilde{f}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \tilde{f}_{ij}\, \mathbf{1}_{[u_i,u_{i+1}]}(u)\, \mathbf{1}_{[v_j,v_{j+1}]}(v)\ , \tag{16}$$



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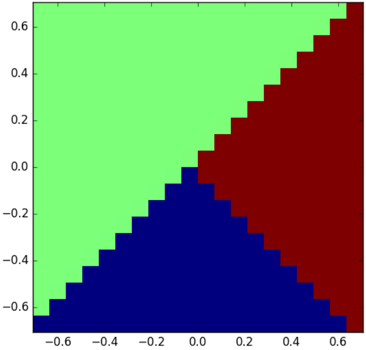
- Mesh:



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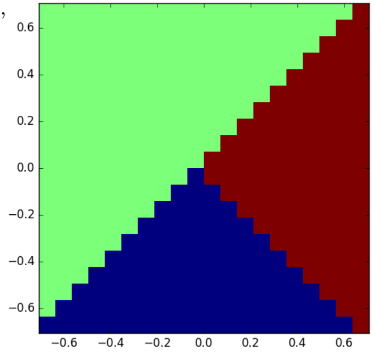

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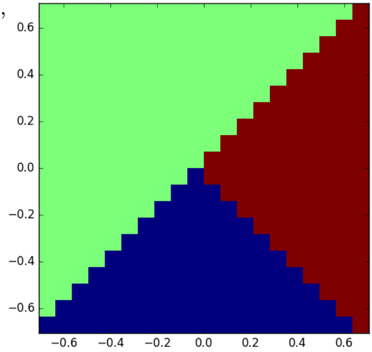
- Sampling:



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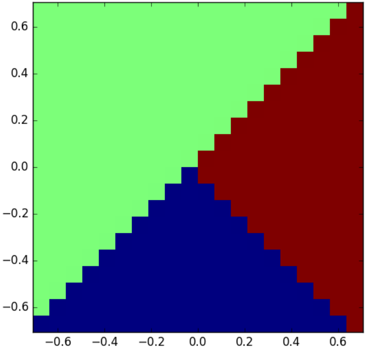

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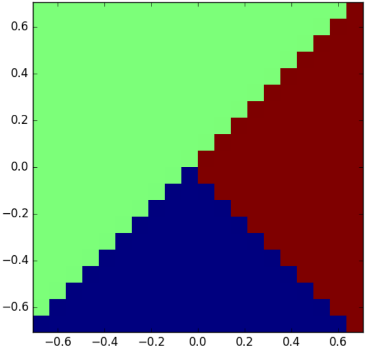

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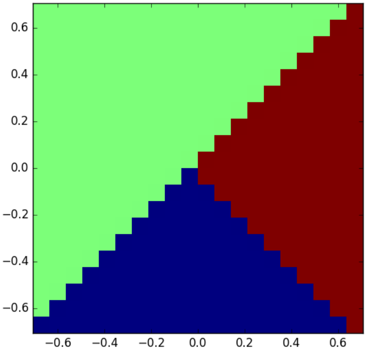
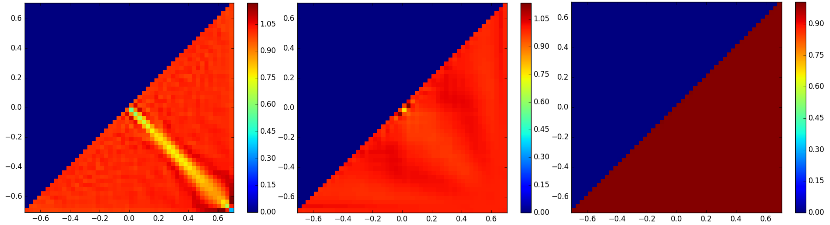
  - A full-rank: more information but also more noise.



Test (constant DD)



- Test with Constant DD.

$$f(\beta,\alpha) = \begin{cases} 1 & \beta > 0 \\ 0 & \beta < 0 \end{cases}$$

$$\downarrow$$

$$H(x,\xi)|_{\mathrm{DGLAP}} = \begin{cases} \frac{2x(1-x)}{1-\xi^2} & |\xi| < x < 1 \\ 0 & -1 < x < -|\xi| \end{cases}$$

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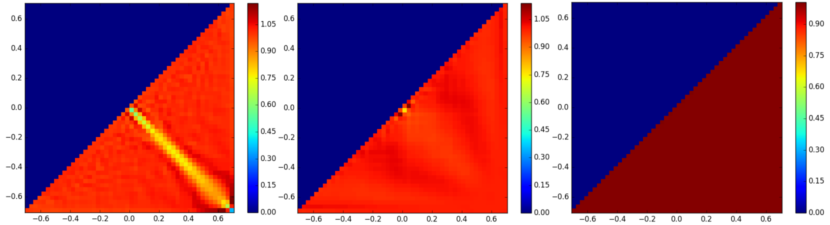

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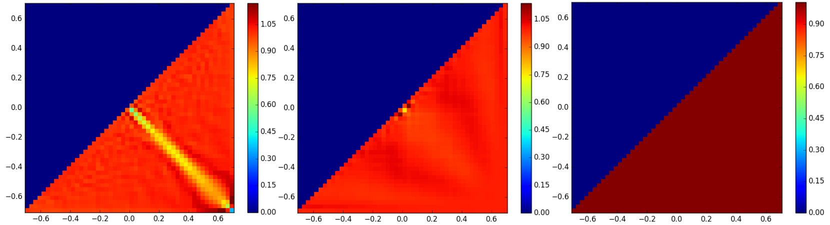

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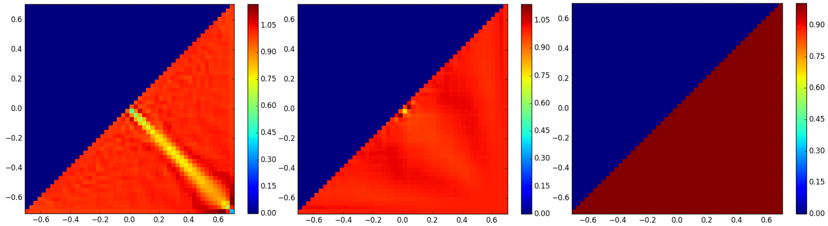

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- Consistent problem (discretized DD = theoretical DD):
  - Objective DD retrieved at arbitrary precision: residual decreases to 0 (machine precision).

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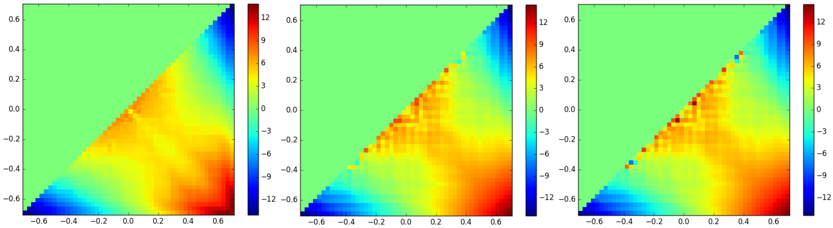
First result



- Real application to a DSE toy model.

$$f(\beta,\alpha) = \begin{cases} ? & \beta > 0 \\ 0 & \beta < 0 \end{cases}$$

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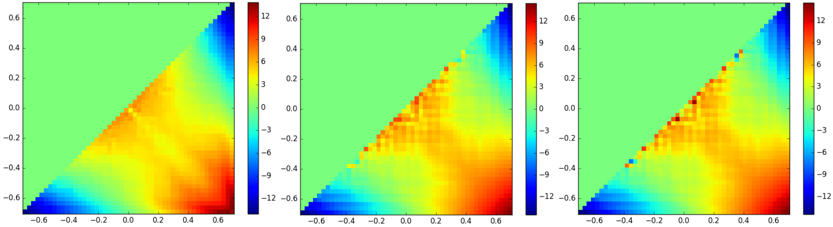
$$H(x,\xi)|_{x>|\xi|} = 30\frac{(1-x)^2\left(x^2-\xi^2\right)}{\left(1-\xi^2\right)^2}$$

First result



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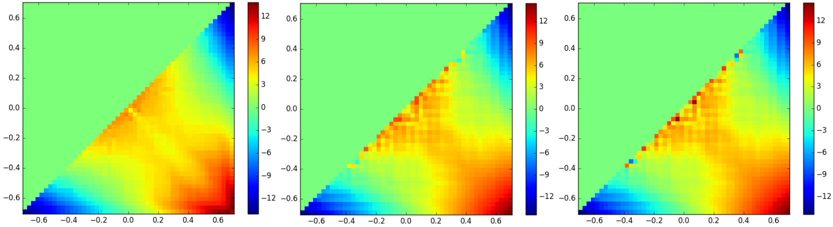

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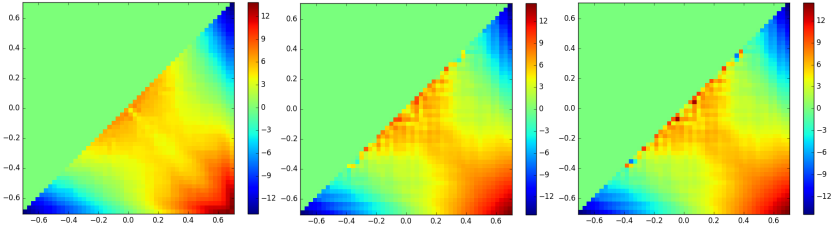

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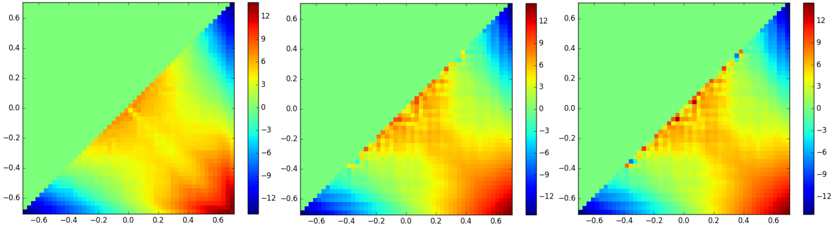

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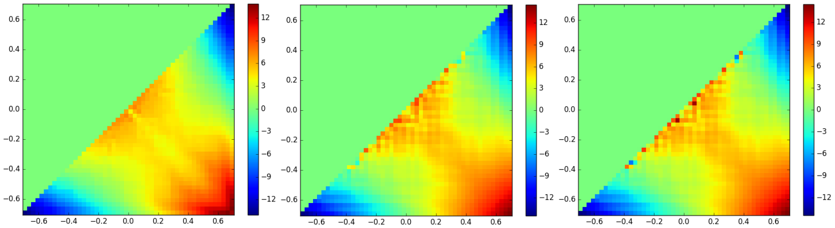

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$$f(\beta,\alpha) = \begin{cases} \frac{30}{4}\left(1 - 3\alpha^2 - 2\beta + 3\beta^2\right) & \beta > 0 \\ 0 & \beta < 0 \end{cases}$$

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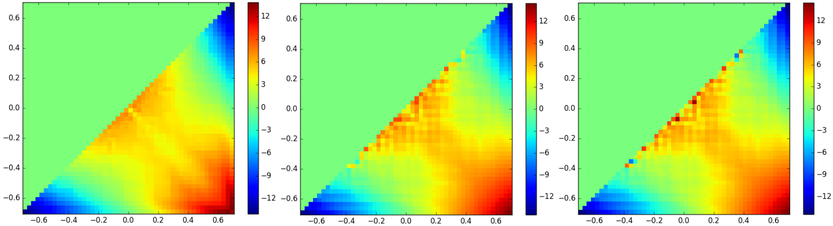

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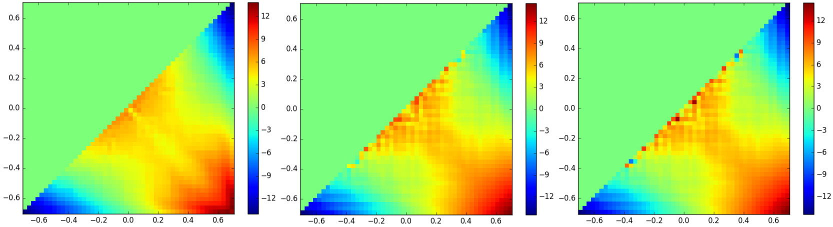

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Quantitative comparison of DDs

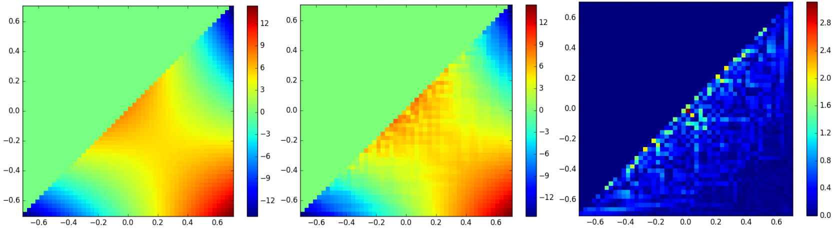

Figure: Quantitative comparison for the Overlap results. Left: Theoretical discretized DD. Middle: Numerical solution at tolerance 10^{-6}. Right: Absolute difference.

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Quantitative comparison of GPDs

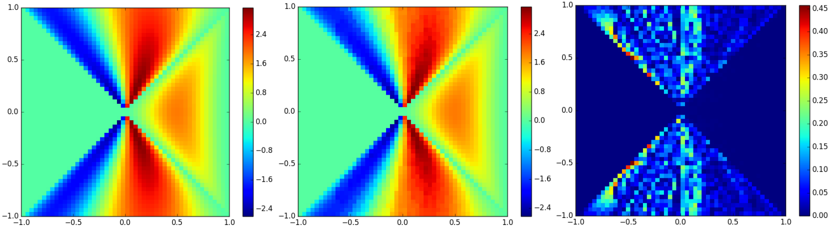

Figure: Quantitative comparison for the Overlap GPD obtained from the numerical DD solution. Left: Theoretical GPD. Middle: Numerical solution at tolerance 10^{-6}. Right: Absolute difference.

$$H(x,\xi) = \begin{cases} 30\dfrac{(1-x)^2\left(x^2-\xi^2\right)}{(1-\xi^2)^2} & x > |\xi| \\[2ex] \dfrac{15\,(x-1)\left(x^2-\xi^2\right)\left(\xi^2+2\,|\xi|\,x+x\right)}{2\,|\xi|^3\left(|\xi|+1\right)^2} & |x| < |\xi| \end{cases}$$

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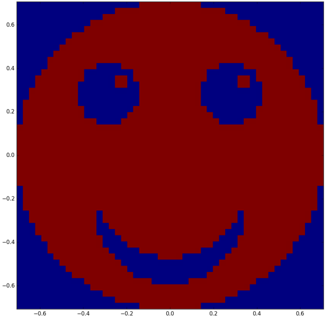
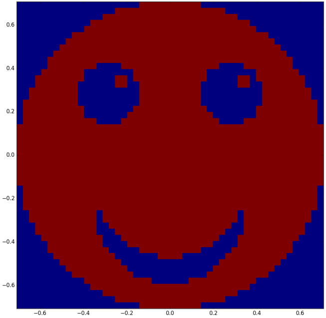
  - Handling of errors.
- Thank you!



Summary

- Extension of the DSE overlap toy model.
- Systematic procedure for GPD modeling from first principles:
  - LCWFs $\underset{\text{Overlap}}{\longrightarrow}$ GPD in DGLAP $\underset{\text{Inverse Radon Transform}}{\longrightarrow}$ DD $\underset{\text{RT}}{\longrightarrow}$ GPD.
  - Both polynomiality and positivity!
- Important points:
  - Compromise with respect to noise and convergence.
  - Pobylitsa gauge is promising.
- In the future:
  - Different methods: Basis functions, Bayesian methods, etc.
  - Handling of errors.
- Thank you!
  - Any questions?



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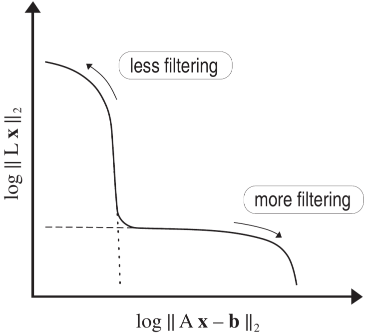
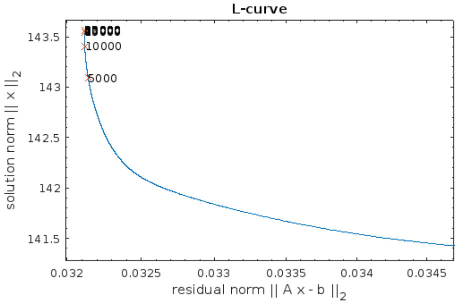
Discrete ill-posed problem



Theoretical "L-curve": curve parameterized by the regularization factor.

(fig. taken from Ref. **(Hansen, 2007)**)



L-curve with the iteration number as regularization factor.