Effective thermodynamics for a driven athermal system with dry and viscous friction

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STATISTICAL MECHANICS OF FRICTIONAL ATHERMAL SYSTEMS ?

Grains with friction → Dissipation → ~~$e^{-\beta\mathcal{H}}$~~

Edwards Hypothesis: All packings where grains occupy the same volume are equiprobable

Assumption: change of configuration due to ‘’extensive operations’’ guarantees ‘’ergodicity’’

$$\mathcal{C} = (\mathbf{x}_1, \phi_1, \ldots, \mathbf{x}_N, \phi_N)$$

$$S = \log \int \mathcal{DC}\ \delta(V - \mathcal{W}(\mathcal{C}))$$

S.F Edwards, C.C. Mounfield, Physica A (210), 1994

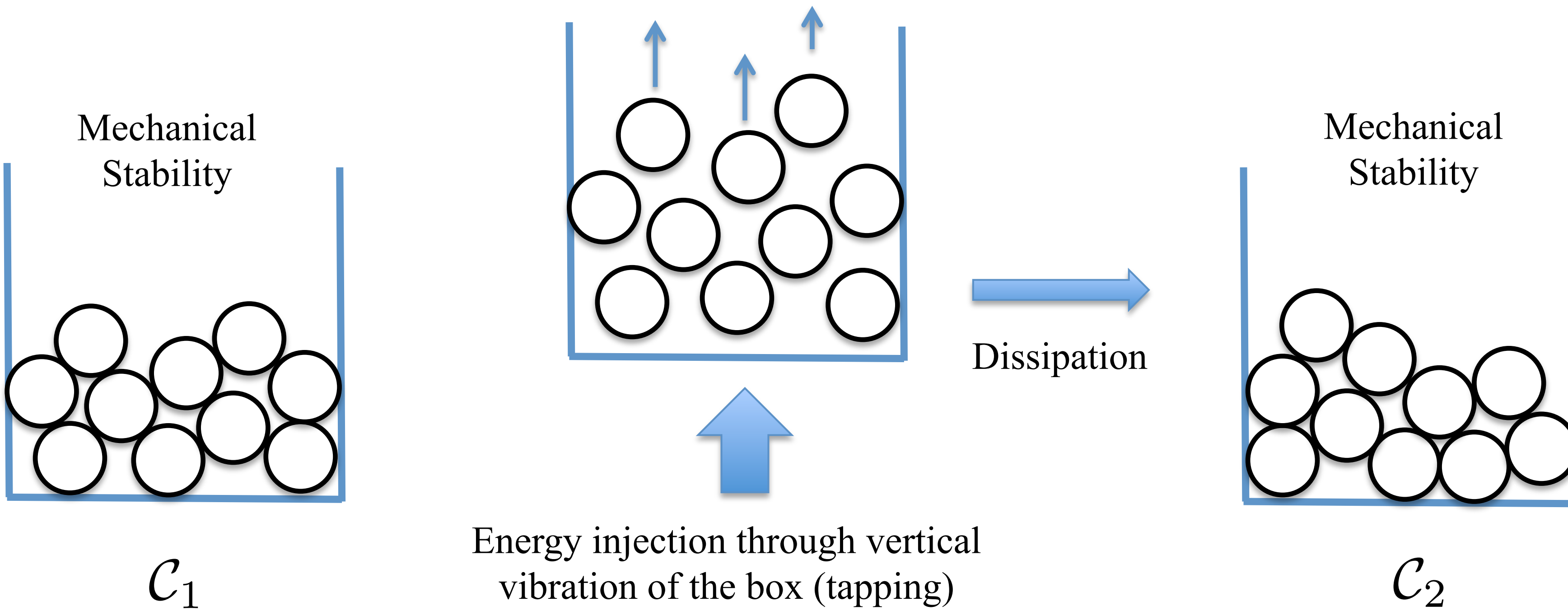
$$\frac{1}{T} = \frac{\partial S}{\partial E} \quad \Longrightarrow \quad \frac{1}{X} = \frac{\partial S}{\partial V}$$

Compactivity

TEST OF EDWARDS ASSUMPTION: TAPPING

DYNAMIC averages $\langle \mathcal{O} \rangle_V$

Mechanically stable configurations at fixed Volume



THERMODYNAMIC averages

$$X^{-1} = \frac{\partial S}{\partial V}$$

$$\langle \mathcal{O} \rangle_X = \mathcal{Z}^{-1} \sum_{\mathcal{C}} \mathcal{O}(\mathcal{C})\ e^{-W(\mathcal{C})/X}$$

HARMONIC CHAIN WITH DRY FRICTION

True Dynamics

- TAPPING DYNAMICS
- BLOCKED CONFIGURATIONS
- RELEVANT OBSERVABLES

Effective Thermodynamics

- DEFINITION OF THE EFFECTIVE THEORY
- PREDICTIONS EFFECTIVE THEORY
- COMPARISON BETWEEN EFFECTIVE THEORY AND DRIVEN ATHERMAL DYNAMICS

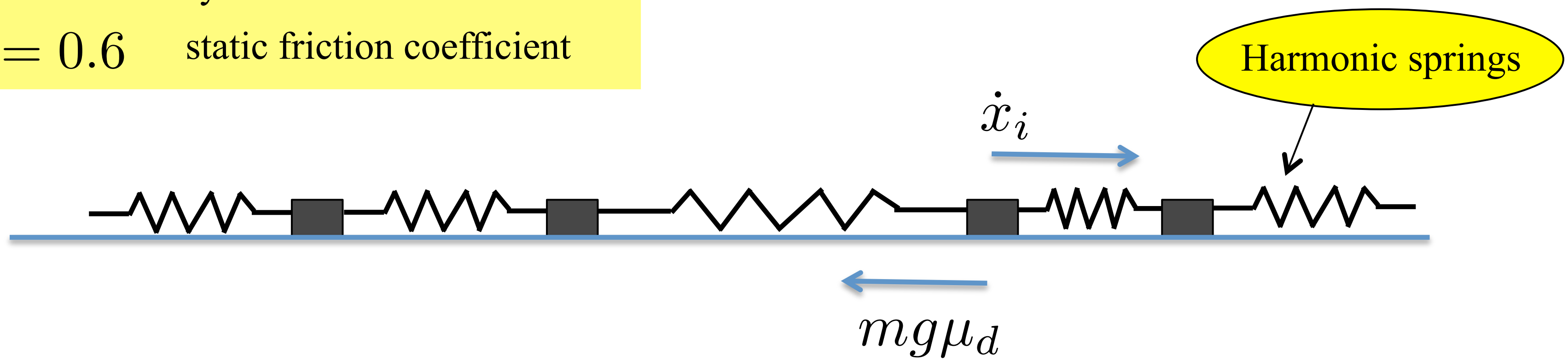
Viscous friction

- RESULTS OF SIMULATIONS
- FIELD THEORETIC APPROACH

HARMONIC CHAIN WITH DRY (Coulomb) FRICTION

$\mu_d = 0.5$ dynamic friction coefficient

$\mu = 0.6$ static friction coefficient



-Equations of motion (dynamic friction)

$$m\,\ddot{x}_i = \underbrace{-mg\mu_d\,\mathrm{sgn}(\dot{x}_i)}_{\text{Dynamic friction: energy dissipation}} + (x_{i+1} + x_{i-1} - 2x_i) + \underbrace{F_i(t)}_{\text{External Force: energy gain}}$$

Dynamic friction: **energy dissipation**

External Force: **energy gain**

-Condition to start moving (static friction)

$$\left.\begin{array}{c} \dot{x}_i = 0 \\ |(x_{i+1} + x_{i-1} - 2x_i + F(t))| > \mu\, mg \end{array}\right\} \longrightarrow \dot{x}_i > 0$$

TAPPING DYNAMICS

1) External force **switched on** for a fixed duration t: energy injection

$$m\,\ddot{x}_i = -mg\mu_d\,\mathrm{sgn}(\dot{x}_i) + (x_{i+1} + x_{i-1} - 2x_i) + F\,n_i$$

Not all the particles are pulled ! $\rho = \frac{1}{N}\sum_{i=1}^{N} n_i \qquad \rho < 1$

2) External force **switched off**: relaxation to a **MECHANICALLY STABLE** configuration with all particles are at rest (absorbing state)

$$m\,\ddot{x}_i = -mg\mu_d\,\mathrm{sgn}(\dot{x}_i) + (x_{i+1} + x_{i-1} - 2x_i)$$

Dynamics is arrested

$$\dot{x}_i = 0 \qquad \forall i$$

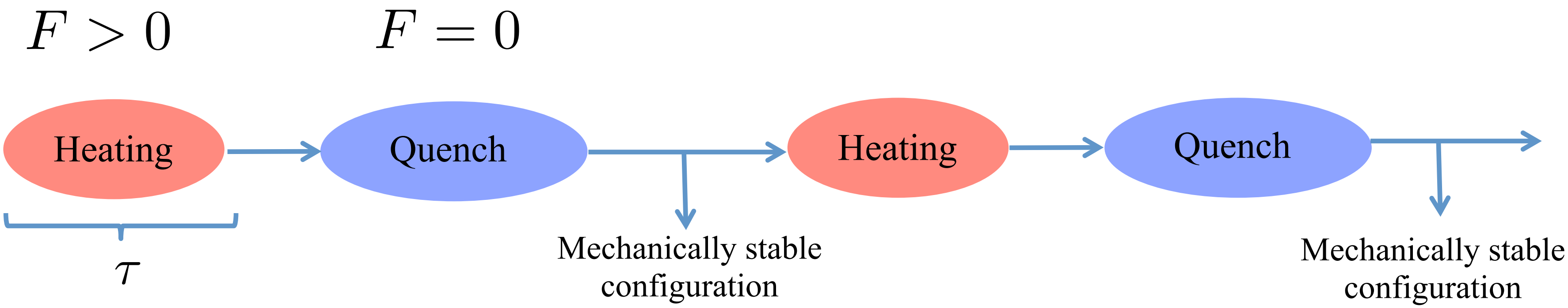
$$|x_{i+1} + x_{i-1} - 2x_i| < \mu m g$$

$$e = \frac{1}{N}\sum_{i=1}^{N}\frac{\xi_i^2}{2}$$

Energy of the mechanically stable configurations

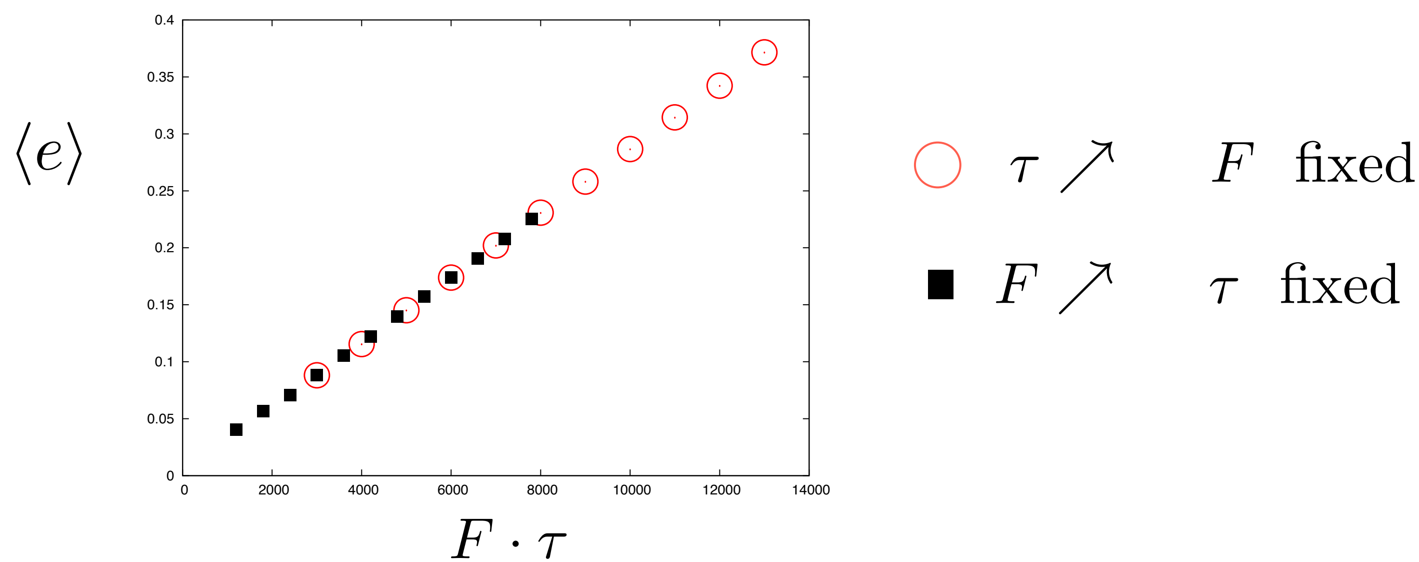
$\xi_i = x_i - x_{i-1} - \ell_0$ Spring elongation

HARMONIC CHAIN WITH DRY (Coulomb) FRICTION



After few cycles the energy of blocked configurations fluctuates around a stationary value

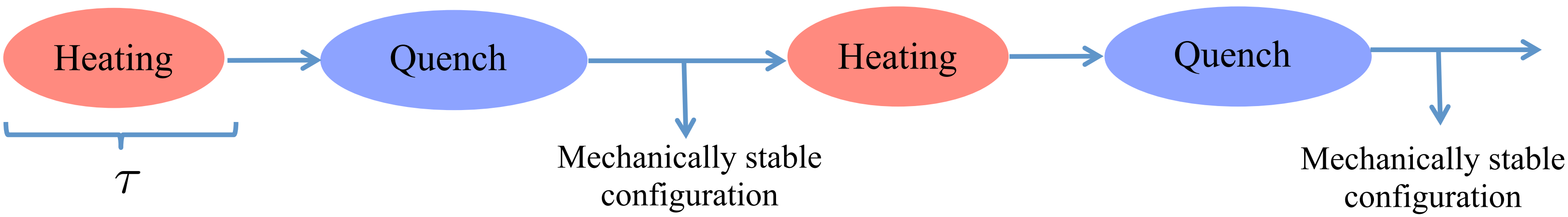
Energy of mechanically stable configurations (N=256)



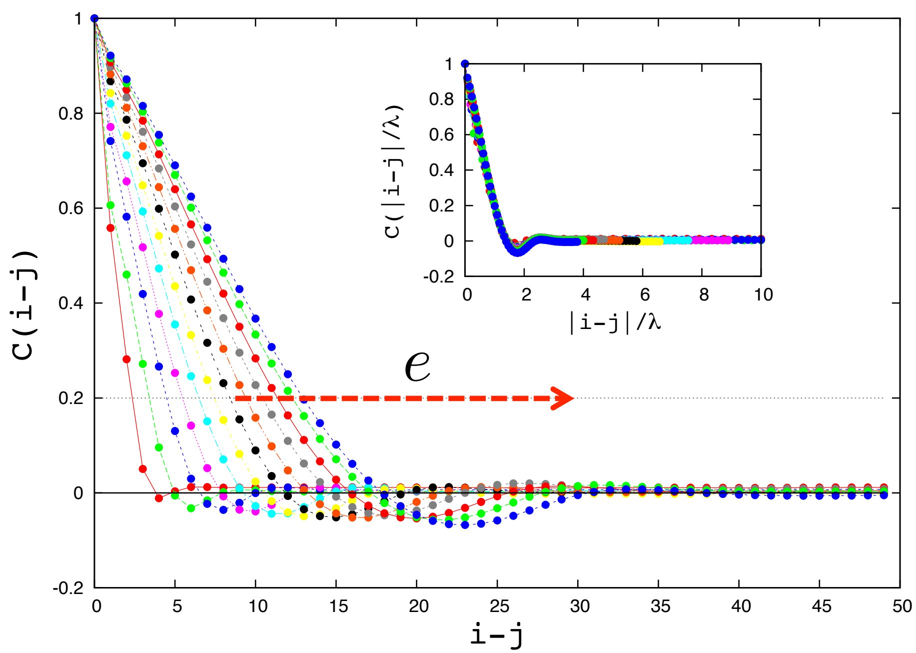
SPRING-SPRING CORRELATION

(IN MECHANICALLY STABLE CONFIGURATIONS)

$F > 0$ $F = 0$



$\langle \xi_i \xi_j \rangle$ Spring-spring correlation



$$\langle \xi_i \xi_j \rangle \sim C(|i-j|/\ell(e))$$

$$\ell(e) \sim e$$

Extent of correlation between springs grows as the energy stored by the springs

EFFECTIVE THERMODYNAMICS ''Á LA EDWARDS''

''Given a certain situation attained dynamically, physical observables are obtained by averaging over the ***usual equilibrium distribution*** at the corresponding volume, energy, etc. but restricting the sum to 'blocked' configurations.'' (Barrat, Kurchan, Loreto, Sellitto, PRL, 2000)

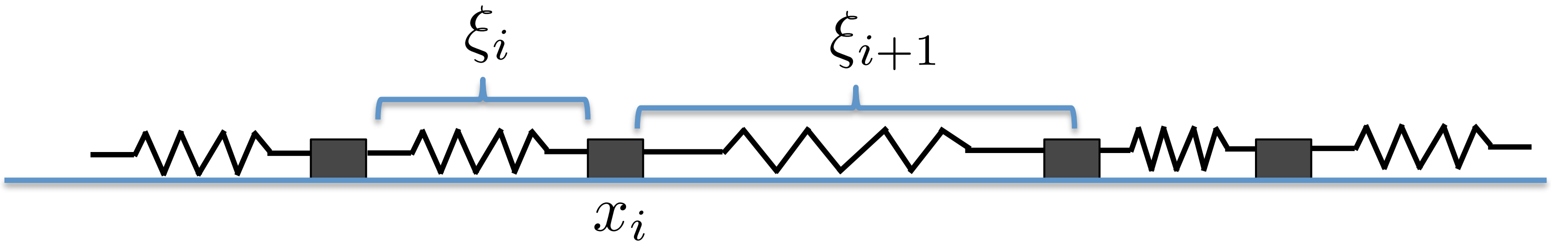
$$\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_N\}$$ Springs elongations

$$\int \mathcal{D}\boldsymbol{\xi}\ e^{-\beta_{\mathrm{Ed}} E[\boldsymbol{\xi}]}\ \delta(\mathcal{F}[\boldsymbol{\xi}])$$

Mechanical stability $\mathcal{F}(\boldsymbol{\xi}) = 0$

$$E[\boldsymbol{\xi}] = \sum_{i=1}^{N} \frac{\xi_i^2}{2} \qquad \beta_{\mathrm{Ed}} = \left[\frac{\partial S}{\partial E}\right]_{\mathcal{F}(\boldsymbol{\xi})=0}$$

EFFECTIVE THERMODYNAMICS "À LA EDWARDS"



$$\mathcal{Z} = \int d\xi_1 \dots d\xi_N \; e^{-\beta_{Ed} \sum_{i=1}^{N} \frac{\xi_i^2}{2}} \prod_{i=1}^{N} \Theta(\mu - |\xi_{i+1} - \xi_i|)$$

$$T(x,y) = e^{-\beta_{Ed} \frac{x^2}{4}} \Theta(\mu - |x-y|) e^{-\beta_{Ed} \frac{y^2}{4}}$$

$$\mathcal{Z} = \int d\xi_1 \dots d\xi_N \; T(\xi_1, \xi_2) \dots T(\xi_N, \xi_1) = \mathrm{Tr}(\mathcal{T}^N)$$

Transfer Operator Formalism $\quad \mathcal{T}[f](x) = \int_{-\infty}^{\infty} dy \; T(y,x) f(x)$

‘’THERMODYNAMIC’’ POTENTIALS

$$\mathcal{T}[f](x) = \int_{-\infty}^{\infty} dy \; T(y,x) f(x)$$

$$T(x,y) = e^{-\beta_{Ed}\frac{x^2}{4}} \Theta(\mu - |x-y|) e^{-\beta_{Ed}\frac{y^2}{4}}$$

Transfer operator is well behaved:
complete spectrum of eigenvalues and eigenvectors

$$f(\beta_{Ed}, \mu) = -\frac{1}{\beta_{Ed} N} \log(\mathcal{T}^N)$$

$$f(\beta_{Ed}, \mu) = -\frac{1}{\beta_{Ed}} \log[\lambda_{\max}(\beta_{Ed}, \mu)]$$

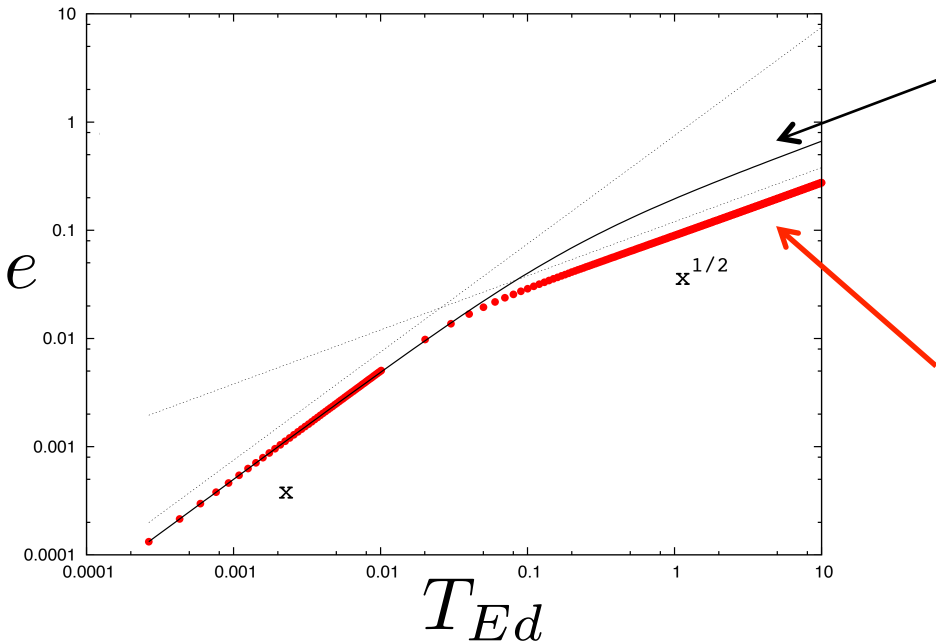
$$e = \partial_{\beta_{Ed}}(\beta_{Ed} f) \qquad e = -\lambda_{\max}^{-1} \langle \lambda_{\max} | \partial_{\beta_{Ed}} \mathcal{T} | \lambda_{\max} \rangle$$

SMOOTHENING OF THE DRY FRICTION CONSTRAINT

$$\Theta(\mu - |x-y|) \sim \frac{1}{\sqrt{\pi}} \exp\left(-\frac{|x-y|^2}{4\mu^2}\right)$$

$$\mathcal{Z} = \int d\xi_1 \ldots d\xi_N \quad \exp\left[-\boldsymbol{\xi} \cdot A\boldsymbol{\xi}\right]$$

Explicit expressions become available



$$e = \frac{1}{2} \frac{\mu\, T_{Ed}}{\sqrt{2T_{Ed} + \mu^2}}$$

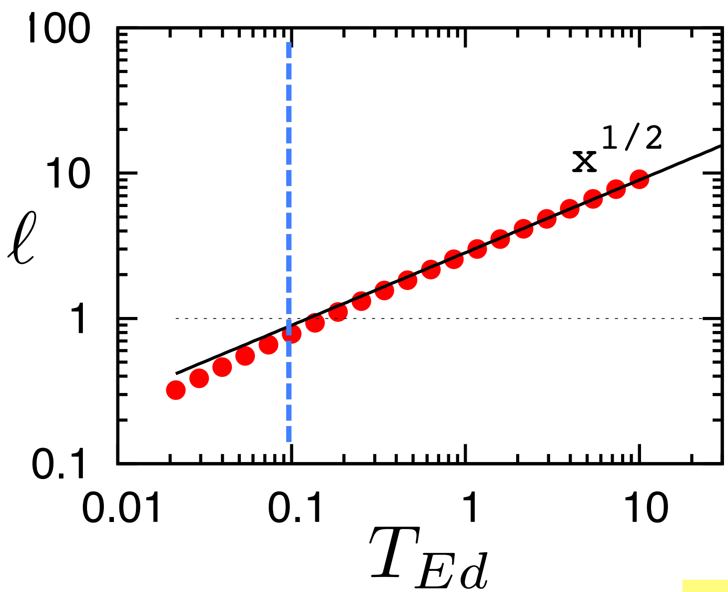
Transfer Operators exact result

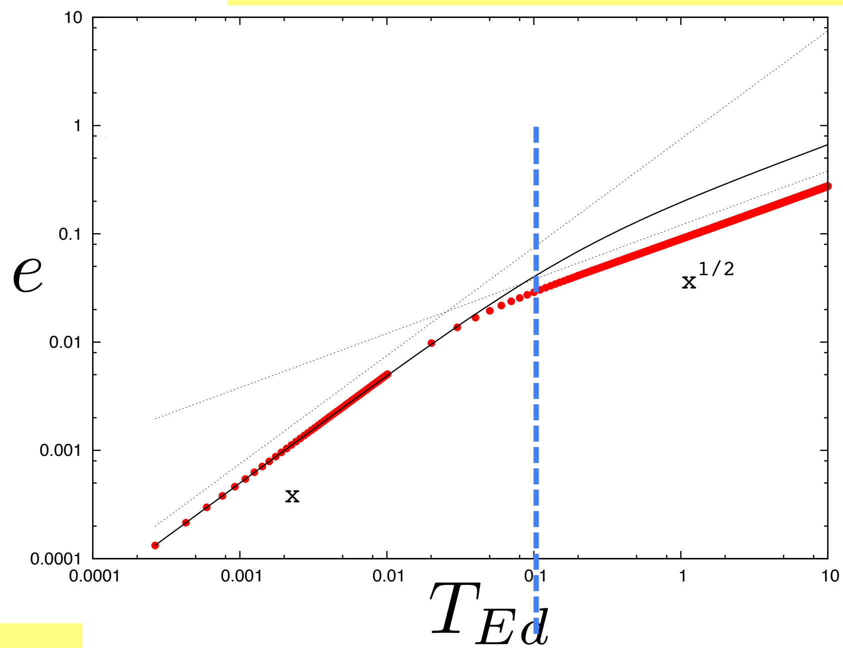
GAUSSIAN APPROXIMATION: CORRELATION FUNCTION

$$\langle \xi_i \xi_j \rangle = 2\, e(\mu_{\rm s}, T_{\rm Ed})\ \exp\left(-\frac{|i-j|}{\ell(\mu_{\rm s}, T_{\rm Ed})}\right)$$

$$\mu^2/T_{Ed} \ll 1 \quad \Longrightarrow \quad \ell \sim \frac{\sqrt{T_{Ed}}}{\mu}$$

Correlations appear in the ''out-of-equilibrium'' regime





$$\ell(e) \sim e$$

DRIVEN ATHERMAL DYNAMICS

$$m\,\ddot{x}_i = F_{\text{diss}} + F_{\text{el}} + F_{\text{ext}}$$

$$\downarrow$$

Blocked configurations

$$\langle \xi_i \xi_j \rangle \sim C\left(\frac{|i-j|}{\ell(e)}\right)$$

$$\ell(e) \sim e$$

EFFECTIVE THERMODYNAMICS

$$\mathcal{Z} = \int_{\boldsymbol{\xi} \in \text{blocked}} \mathcal{D}\boldsymbol{\xi}\; e^{-\beta_{Ed} E[\boldsymbol{\xi}]}$$

$$\langle \xi_i \xi_j \rangle \sim \exp\left(-\frac{|i-j|}{\ell(e)}\right)$$

$$\ell(e) \sim e$$

G. Gradenigo, E. Ferrero, E. Bertin, J.-L. Barrat, **Phys. Rev. Lett**. 115, 140601 (**2015**)

GAUSSIAN APPROXIMATION
Field-theoretic description

Edwards effective theory $$\mathcal{Z} = \int \mathcal{D}\boldsymbol{\xi}\ e^{-S(\boldsymbol{\xi})}$$

$$S(\boldsymbol{\xi}) = \frac{1}{2}\left[\underbrace{\beta_{\mathrm{Ed}}\sum_{i=1}^{N}\xi_i^2}_{\textbf{Energy}} + \underbrace{\frac{1}{2\mu_{\mathrm{s}}^2}\sum_{i=1}^{N}(\xi_{i+1}-\xi_i)^2}_{\textbf{Dry friction (smooth) constraint}}\right]$$

Energy

Dry friction (smooth) constraint

Continuum limit

$$\xi_i \;\rightarrow\; \xi(x)$$

$$\xi_{i+1} - \xi_i \;\rightarrow\; \frac{\partial \xi(x)}{\partial x}$$

$$m^2 = 2\mu_{\mathrm{s}}^2 \beta_{\mathrm{Ed}}$$

$$S(\boldsymbol{\xi}) \sim \int dx \left[\frac{1}{2}\left(\frac{\partial \xi(x)}{\partial x}\right)^2 + \frac{1}{2}m^2\xi^2(x)\right]$$

GAUSSIAN APPROXIMATION
Field-theoretic description

Edwards effective theory $$\mathcal{Z} = \int \mathcal{D}\boldsymbol{\xi}\; e^{-S(\boldsymbol{\xi})}$$

$$\langle \xi(x)\xi(y) \rangle \sim e^{-m|x-y|}$$

$$\ell \sim \sqrt{T_{\mathrm{Ed}}}$$

Continuum limit

$$\xi_i \;\rightarrow\; \xi(x)$$

$$\xi_{i+1} - \xi_i \;\rightarrow\; \frac{\partial \xi(x)}{\partial x}$$

$$m^2 = 2\mu_{\mathrm{s}}^2 \beta_{\mathrm{Ed}}$$

$$S(\boldsymbol{\xi}) \sim \int \, dx \, \left[\frac{1}{2} \left(\frac{\partial \xi(x)}{\partial x} \right)^2 + \frac{1}{2} m^2 \xi^2(x) \right]$$

Mechanically stable configuration = Trajectories (of a fictitious stochastic process)

Partition sum $$\mathcal{Z} = \int \mathcal{D}\xi \exp\left(-\int dx \left[\frac{1}{2\mu^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\beta_{\mathrm{Ed}}}{2}\xi^2(x)\right]\right) \quad (1)$$

spring elongation $\xi \iff$ "space variable"

spring position $x \iff$ "time variable"

Typical configurations ''are generated" by a Langevin equation with white noise

$$\frac{d\xi}{dx} = -\mu\sqrt{\beta_{\mathrm{Ed}}}\,\xi(x) + \eta(x) \qquad \langle \eta(x)\eta(y)\rangle = \Gamma(x,y) = 2\mu^2\sqrt{2\pi}\,\delta(x-y)$$

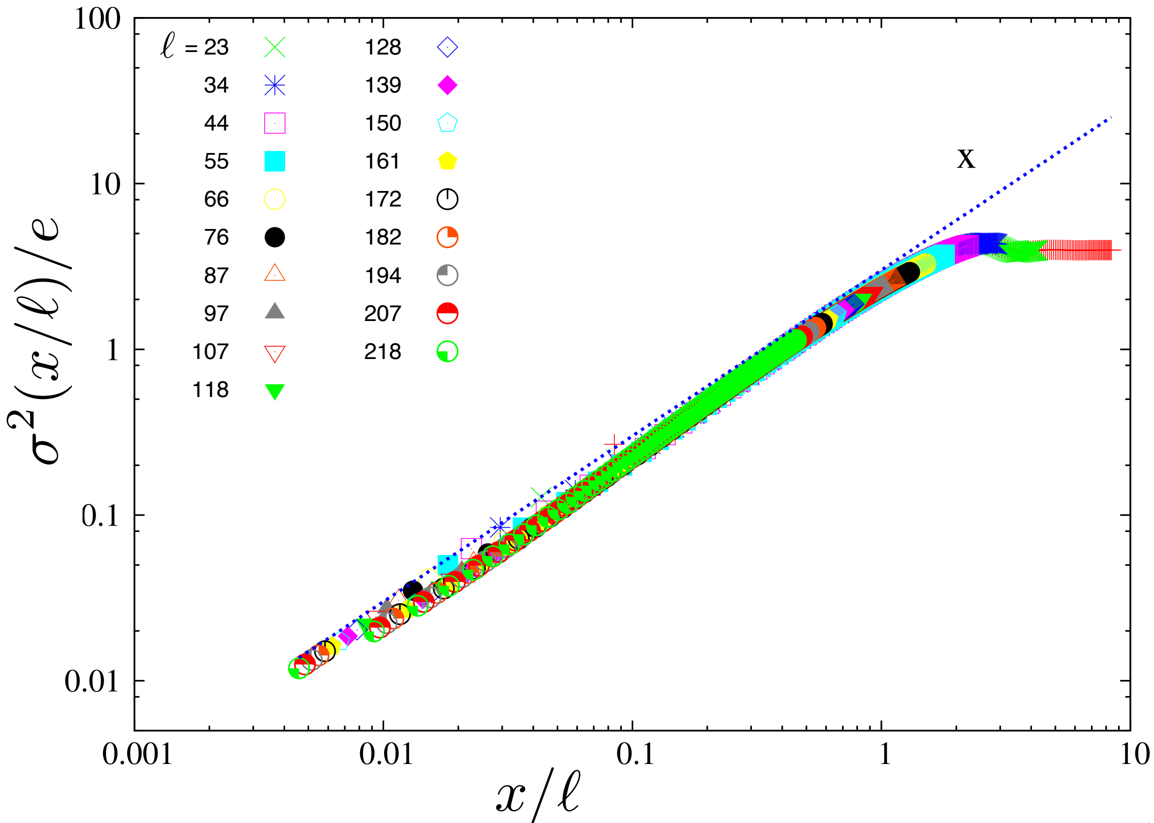
Sum over path probabilities (Langevin equation) (2)	$\iff$	Partition sum (Field-theory) (1)

$$\mathcal{Z} = \int \mathcal{D}\xi \exp\left(-\int dx\; dy\; L\xi(x)\;\Gamma^{-1}(x-y)\;L\xi(y)\right) \quad (2)$$

$$L = \frac{\partial}{\partial x} + \mu\sqrt{\beta_{\mathrm{Ed}}}$$

Mean square displacement (MSD) of spring elongations

- Numerical data from Mechanically Stable Configurations at different energies



$$\sigma^2(x) = \langle(\xi(x) - \xi(0))^2\rangle$$

Mean square displacement of spring elongation along the chain in a mechanically stable configuration

$$\sigma^2(x) \sim x$$

$$\ell(e) \iff \text{“characteristic time“}$$

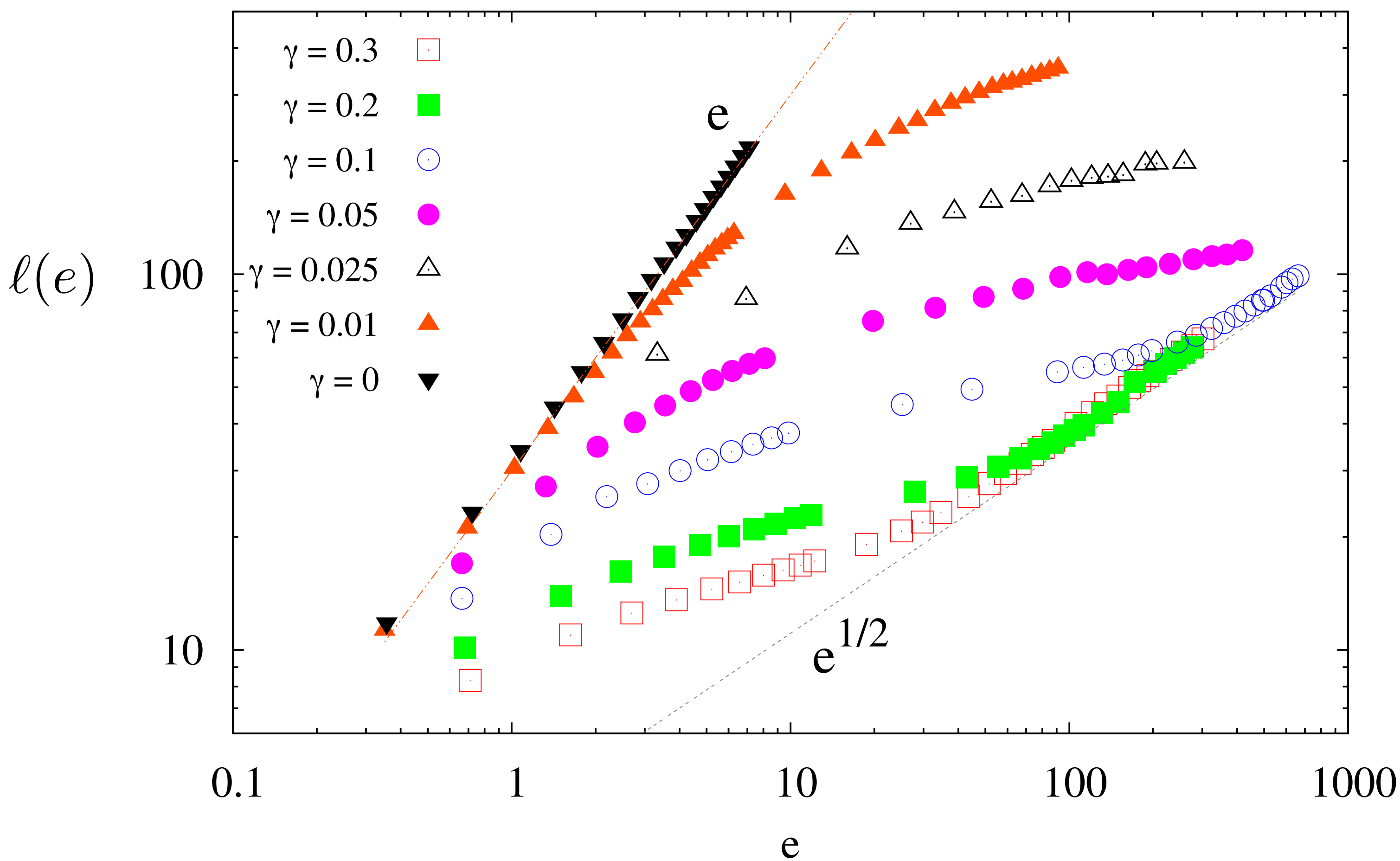
$$\frac{d\xi}{dx} = -\mu\sqrt{\beta_{\mathrm{Ed}}}\;\xi(x) + \eta(x)$$

$$\langle\eta(x)\eta(y)\rangle = \Gamma(x,y) = 2\mu^2\sqrt{2\pi}\;\delta(x-y)$$

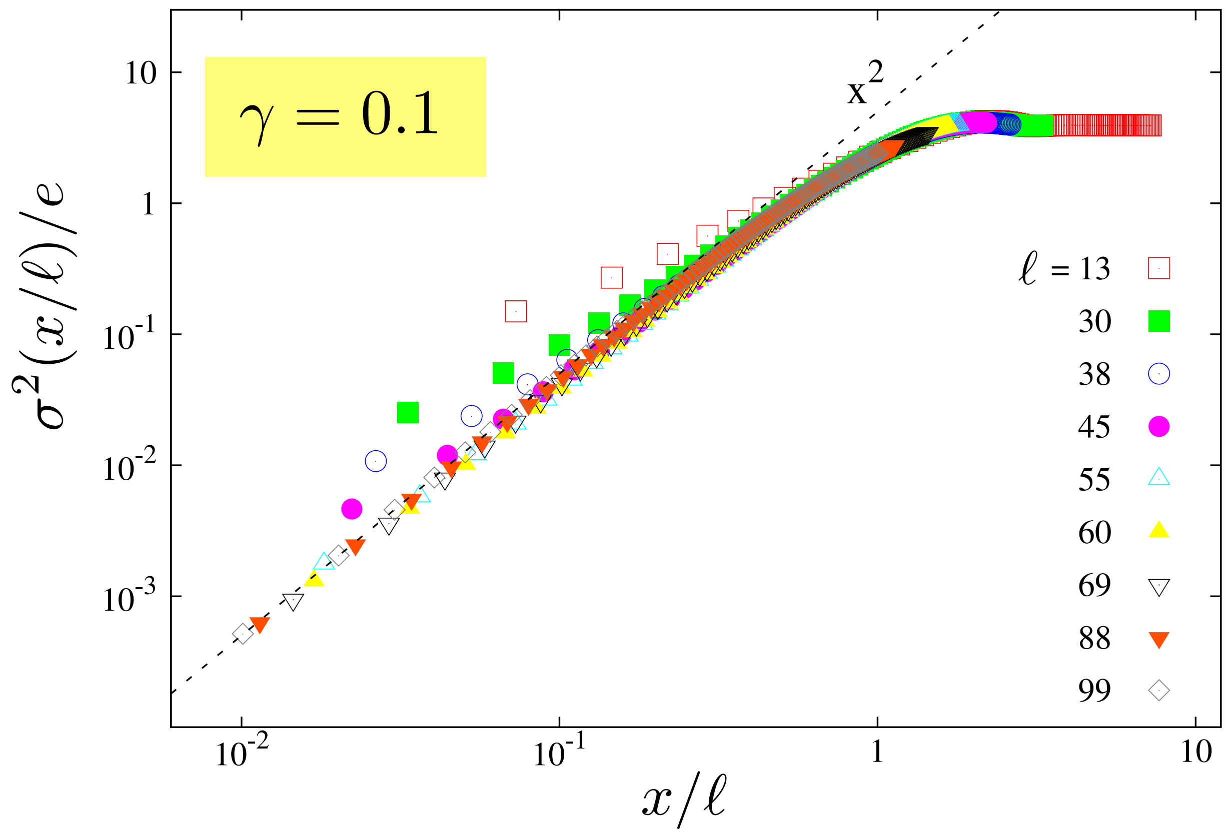
Viscous friction: Deviation from the Edwards theory !!

$$m\ \ddot{x}_i = -\gamma \dot{x}_i - m g \mu_d\ \mathrm{sgn}(\dot{x}_i) + (\xi_{i+1} - \xi_i) + F_i^{\mathrm{ext}}(t)$$

Edwards → Iso-energetic mechanically stable configurations have identical probability → $\ell(e) \sim e$



MSD of spring elongations with viscous friction



$$\sigma^2(x) = \langle (\xi(x) - \xi(0))^2 \rangle$$

Mean square displacement of spring elongation along the chain in a mechanically stable configuration

Ballistic increase

$$\sigma^2(x) \sim x^2$$

Langevin equation with COLOURED noise

$$\frac{d\xi}{dx} = -\mu\sqrt{\beta_{\mathrm{Ed}}}\;\xi(x) + \eta(x)$$

$$\langle \eta(x)\eta(y) \rangle = \Gamma(x,y) = 2\mu^2\sqrt{2\pi}\;\delta(x-y) + \frac{D_{\mathrm{col}}}{\tau} e^{-|x-y|/\tau}$$

Langevin equation with COLOURED noise

$$\frac{d\xi}{dx} = -\mu\sqrt{\beta_{\mathrm{Ed}}}\;\xi(x) + \eta(x)$$

$$\langle \eta(x)\eta(y)\rangle = \Gamma(x,y) = 2\mu^2\sqrt{2\pi}\;\delta(x-y) + \frac{D_{\mathrm{col}}}{\tau}e^{-|x-y|/\tau}$$

Sum over path probabilities

$$\mathcal{Z} = \int \mathcal{D}\xi \exp\left(-\int dx\; dy\; L\xi(x)\;\Gamma^{-1}(x-y)\;L\xi(y)\right)$$

$$L = \frac{\partial}{\partial x} + \mu\sqrt{\beta_{\mathrm{Ed}}}$$

Sum over path probabilities (Langevin equation) ⟺ **Partition sum** (Field-theory)

⇓

Partition sum: NON LOCAL GAUSSIAN FIELD THEORY

$$\mathcal{Z} = \int \mathcal{D}\xi \exp\left(-\int dx\; dy\; \mathcal{L}\xi(x)\; g(x-y)\; \mathcal{L}\xi(y)\right)$$

Non-local field theory for the mechanically stable configurations sampled with viscous (+dry) friction

$$\mathcal{Z} = \int \mathcal{D}\xi \exp\left(- \int dx\ dy\ \mathcal{L}\xi(x)\ g(x-y)\ \mathcal{L}\xi(y) \right)$$

$$g(x-y) = \frac{1}{2\sqrt{2}\mu\ M\ \tau} \exp\left(-M \frac{|x-y|}{\tau} \right) \qquad M = \sqrt{1 + \frac{D_{\text{col}}}{\mu\sqrt{2\pi}}}$$

$$\mathcal{L} = \tau \frac{\partial^2}{\partial x^2} + (1 + 2\sqrt{\beta_{\text{Ed}}}\mu\ \tau + \tau^2) \frac{\partial}{\partial x} + \mu\sqrt{\beta_{\text{Ed}}}$$

Dry friction only $\quad D_{\text{col}} \Longrightarrow 0 \qquad \tau \Longrightarrow 0 \quad$ Local theory is recovered

$$\mathcal{Z} = \int \mathcal{D}\xi \exp\left(- \int dx \left[\frac{1}{2\mu^2} \left(\frac{\partial \xi}{\partial x} \right)^2 + \frac{\beta_{\text{Ed}}}{2} \xi^2(x) \right] \right)$$

CONCLUSIONS

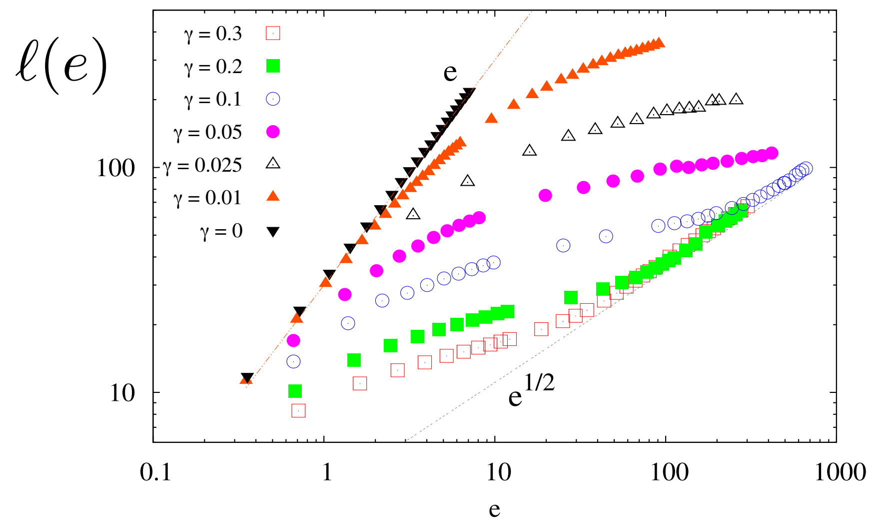
- We presented a 1D model with **dry friction** where the effective thermodynamics *à la* Edwards works very well: transfer operator techniques, gaussian approximation of the constraint

- Uniform Edwards theory works well for dry friction: iso-energetic mechanically stable configurations are sample with identical probability

- What about viscous friction? Non-uniform Edwards theory!

- What about viscous friction? Non-local field theory, coloured noise

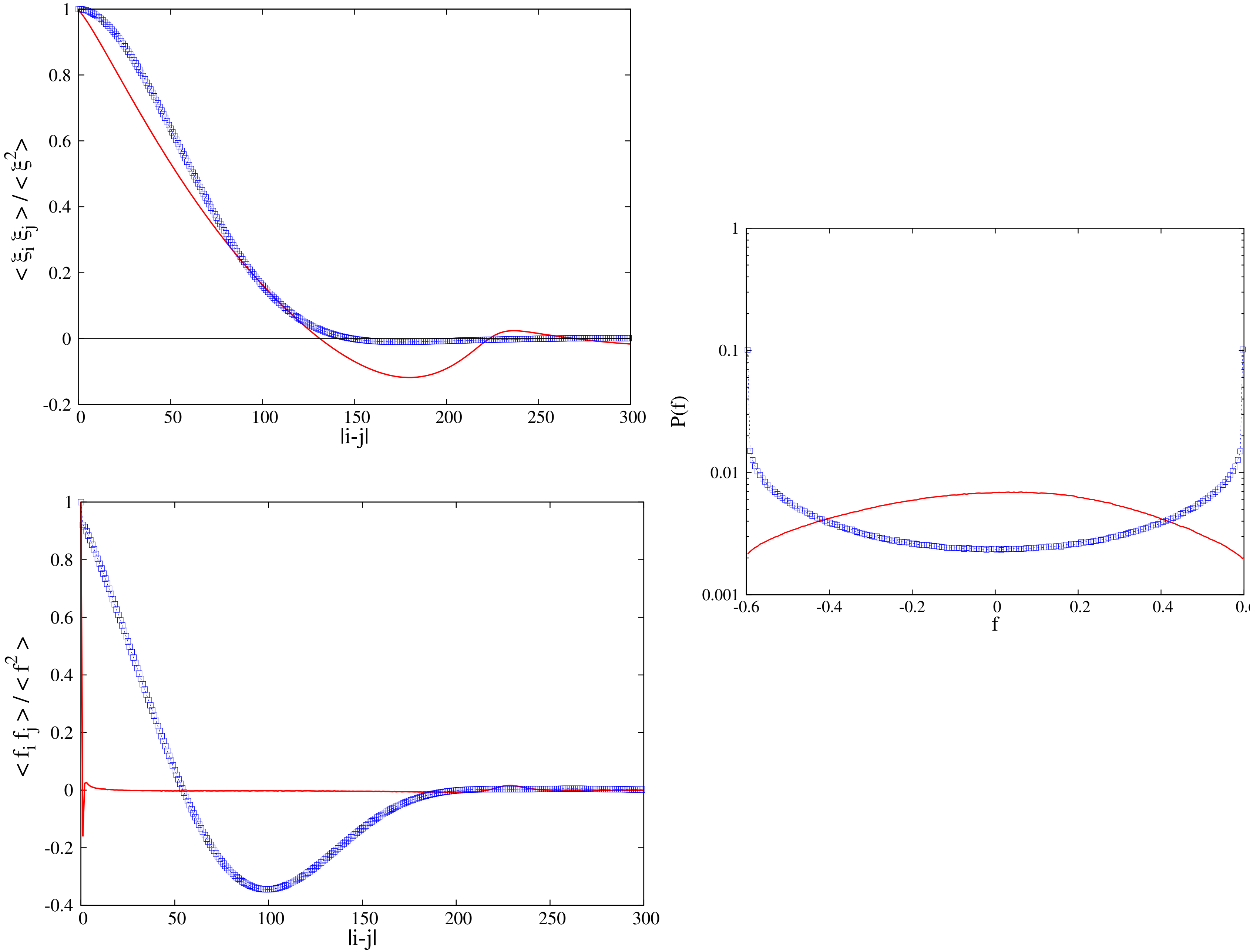
$$D_{\mathrm{col}}(T_{\mathrm{Ed}},\gamma) \;=\; ?$$
$$\tau(T_{\mathrm{Ed}},\gamma) \;=\; ?$$



Thermodynamics = ?

$$f = (N\beta_{\mathrm{Ed}})^{-1}\log(\mathcal{Z}) \qquad = \quad ?$$
$$e = \partial_{\beta_{Ed}}(\beta_{Ed} f)$$

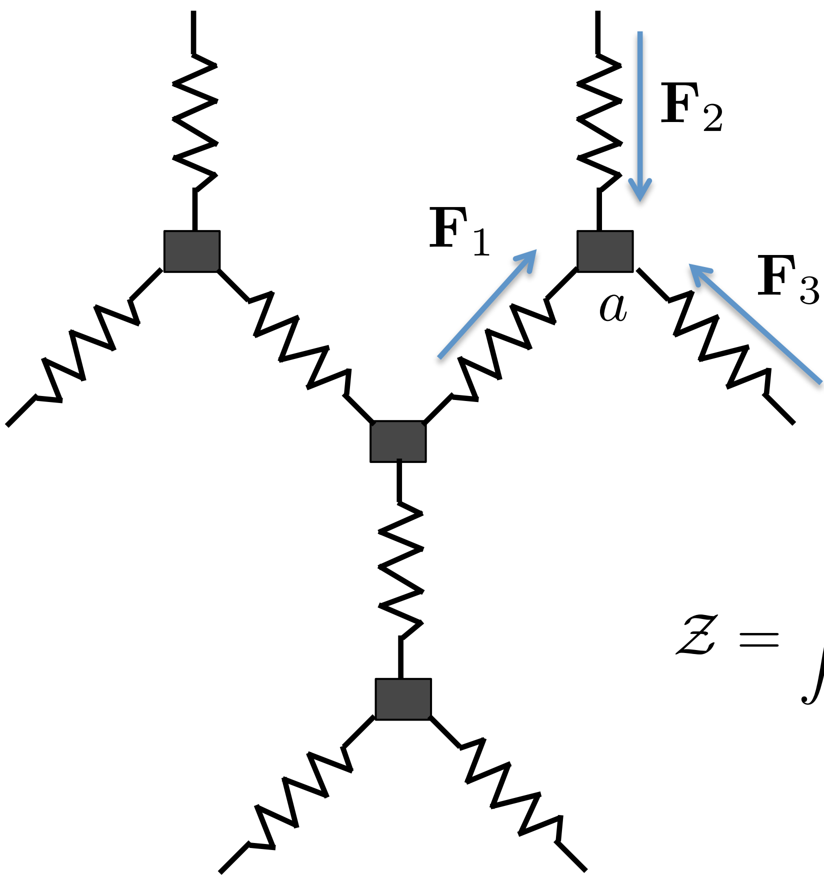
THANK FOR YOUR ATTENTION



$< \xi_i \xi_j > / < \xi^2 >$
|i-j|
$< f_i f_j > / < f^2 >$
|i-j|
P(f)
f

PERSPECTIVES

- Transfer operator approach on the tree-like random graph (cavity equations)



$$\Theta\left(\mu - \left|\sum_{i\in\partial a} \mathbf{F}_i\right|\right)$$

$$\mathbf{F} = (F_i^x, F_i^y)$$

$$\mathcal{Z} = \int \mathcal{D}\boldsymbol{\xi}\; e^{-\beta_{\mathrm{Ed}} \sum_i \frac{\xi_i^2}{2}} \prod_a \Theta\left(\mu - \left|\sum_{i\in\partial a} \mathbf{F}_i\right|\right)$$

PERSPECTIVES

- Study of non-linear springs (1D)

Tapping dynamics

$$m\ \ddot{x}_i = -mg\mu_d\ \mathrm{sgn}(\dot{x}_i) + (\xi_{i+1} - \xi_i) + \alpha(\xi_{i+1}^3 - \xi_i^3) + F_i^{\mathrm{ext}}(t)$$

Effective theory

$$\mathcal{Z} = \int \mathcal{D}\boldsymbol{\xi}\ e^{-\beta_{\mathrm{Ed}} \sum_{i=1}^{N} [\frac{1}{2}\xi_i^2 + \frac{\alpha}{4}\xi_i^4]}\ \delta[\mathcal{F}(\boldsymbol{\xi}) - 1]$$

Gaussian approximation: very well known field theory

$$S(\boldsymbol{\xi}) \sim \int\ dx\ \left[\frac{1}{2}\left(\frac{\partial\xi(x)}{\partial x}\right)^2 + \frac{1}{2}m^2\xi^2(x) + \frac{\alpha m^2}{4}\xi^4(x)\right]$$

Career

Ph. D. : **Supercooled liquids** ; Trento (Italy), 2007-2009

Supervisor : P. Verrocchio.
Collaborations: G. Parisi, A. Cavagna. I Giardina, T. Grigera, C. Cammarota

Post-Doc : **Non-equilibrium Statistical Mechanis;** Rome, 2010-2012

Advisor: A. Puglisi, A. Vulpiani

Collaborations: H. Touchette, R. Burioni, U. Marconi, A. Cavagna, T. Grigera, P. Verrocchio, A. Sarracino, D. Villamaina

Post-Doc : **Glass transition;** Paris, 2013-2014

Advisor: G. Biroli, S. Franz

Post-Doc : **Non-equilibrium Statistical Mechanics;** Grenoble, 2015-present

Advisor: E. Bertin, J.-L. Barrat

Collaborations: E. Ferrero

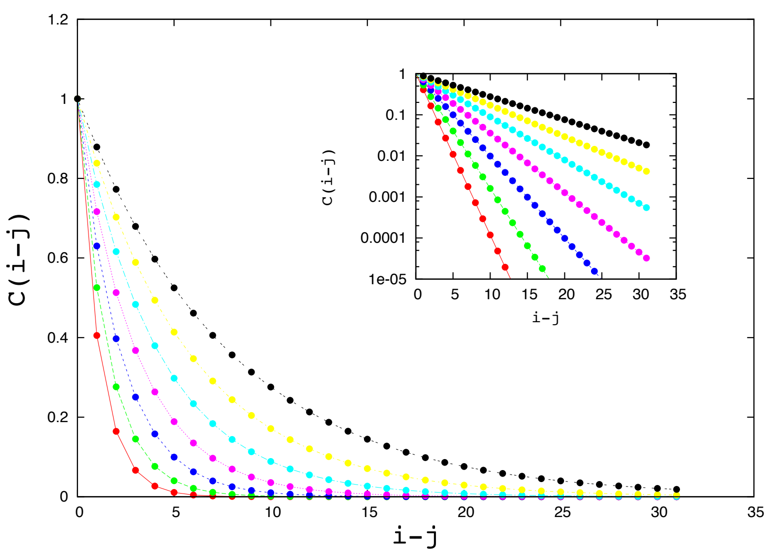
CORRELATION FUNCTION

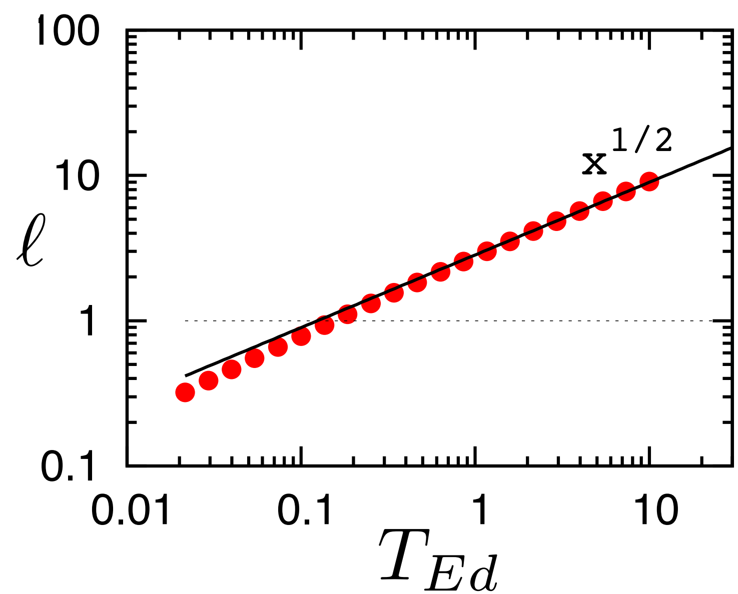
$$C(|n-m|) \sim \exp(-|n-m|/\ell(e))$$

The operator (real, symmetric kernel) has an orthonormal basis

$$\mathcal{T}[f_b](x) = \lambda_b f_b(x) \qquad \int_{-\infty}^{\infty} f_b(x) f_a(x) = \delta_{a,b}$$

$$\lim_{N\to\infty} \langle \xi_m \xi_n \rangle_{Ed} = \sum_{b\in Sp(\mathcal{T})} \left(\frac{\lambda_b}{\lambda_{\max}}\right)^{n-m} \left| \int_{-\infty}^{\infty} dx\ x f_b(x) f_{\lambda_{\max}}(x) \right|^2$$





ENTROPY

Non-interacting springs approximation

Marginal distribution $$p(\xi) = \mathcal{Z}^{-1} \int d\xi_1 \ldots d\xi_{N-1} P(\boldsymbol{\xi}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\xi^2/(2\sigma^2)}$$

$$\sigma \sim \sqrt{T_{\mathrm{Ed}}}$$

$$s = -\int d\xi \; p(\xi) \; \log(p(\xi)) \sim \log(T_{\mathrm{Ed}})$$

Exact result

$$Q(\boldsymbol{\xi}) = P(\boldsymbol{\xi})/\mathcal{Z} \quad s = -\int \mathcal{D}\boldsymbol{\xi} \; Q(\boldsymbol{\xi}) \; \log(Q(\boldsymbol{\xi})) = \beta_{\mathrm{Ed}} e - \beta_{\mathrm{Ed}} f$$

$$\lim_{T_{\mathrm{Ed}} \to \infty} = \frac{1}{2} \log 2 + \log \mu$$

SCALINGS

Fluctuations of chain length

$$\mathcal{L} = L_0 + \sum_{i=1}^{N} \xi_i$$

$$\langle \mathcal{L}^2 \rangle - \langle \mathcal{L} \rangle^2 = \sum_{ij} \langle \xi_i \xi_j \rangle$$

$$\langle \mathcal{L}^2 \rangle - \langle \mathcal{L} \rangle^2 \sim \xi(\beta_{\mathrm{Ed}}, \mu_{\mathrm{s}}) e(\beta_{\mathrm{Ed}}, \mu_{\mathrm{s}}) L_0$$

$$\langle \mathcal{L}^2 \rangle - \langle \mathcal{L} \rangle^2 \sim T_{\mathrm{Ed}}$$

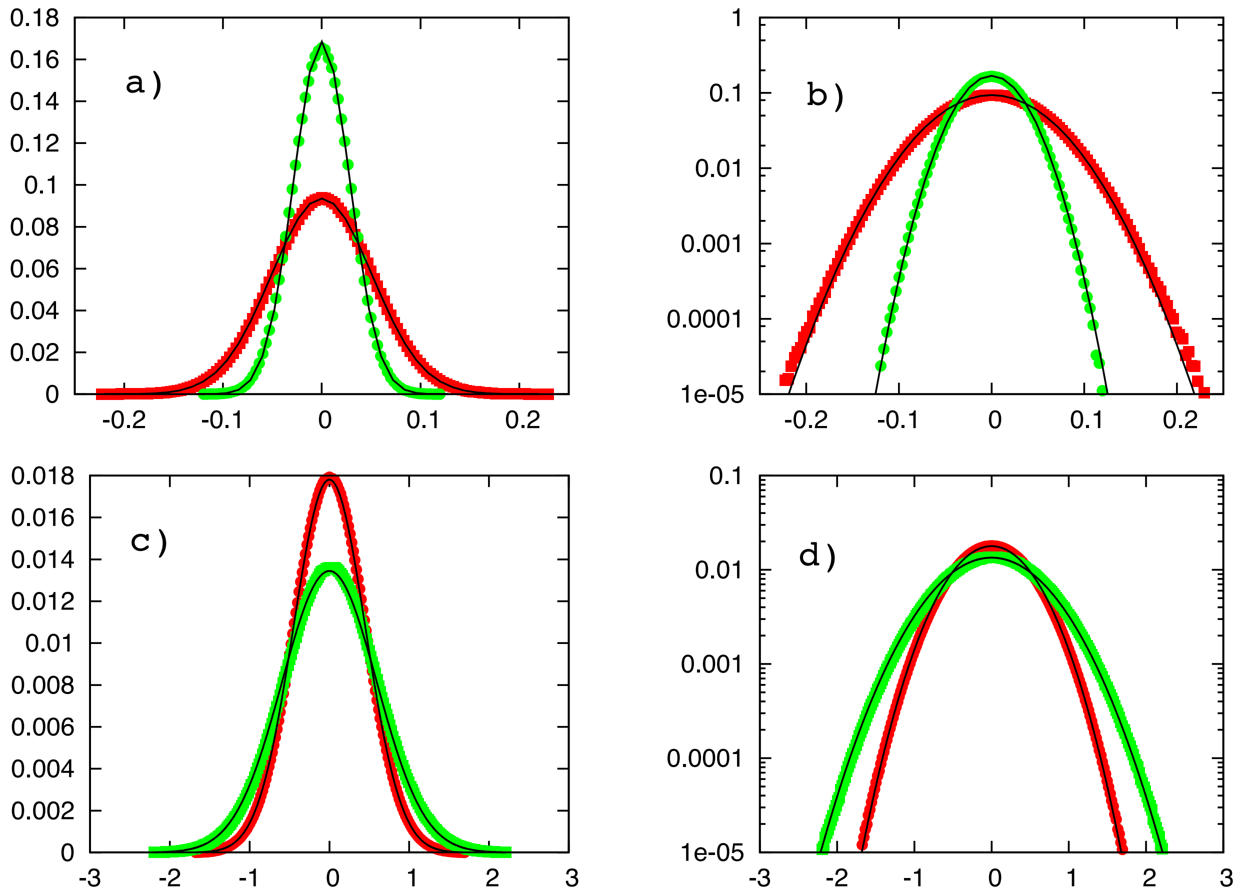
SPRING-LENGHT PROBABILITY DISTRIBUTIONS $P(\xi)$

Lines: effective theory

Points: simulations

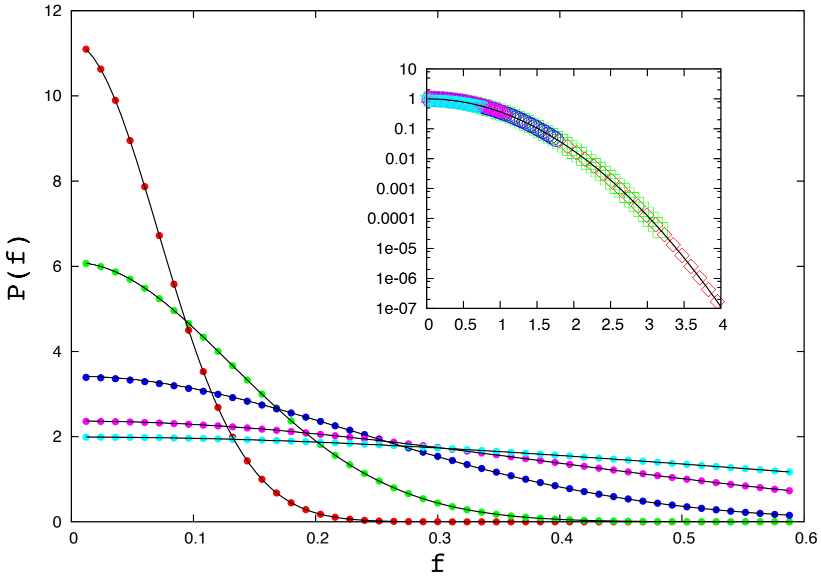
$$e \sim T_{Ed}$$

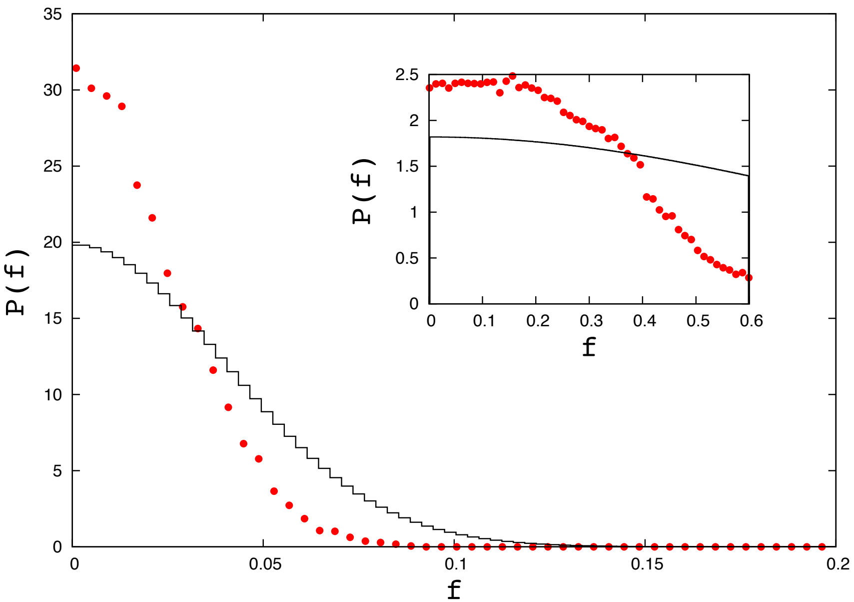
$$e \sim \sqrt{T_{Ed}}$$



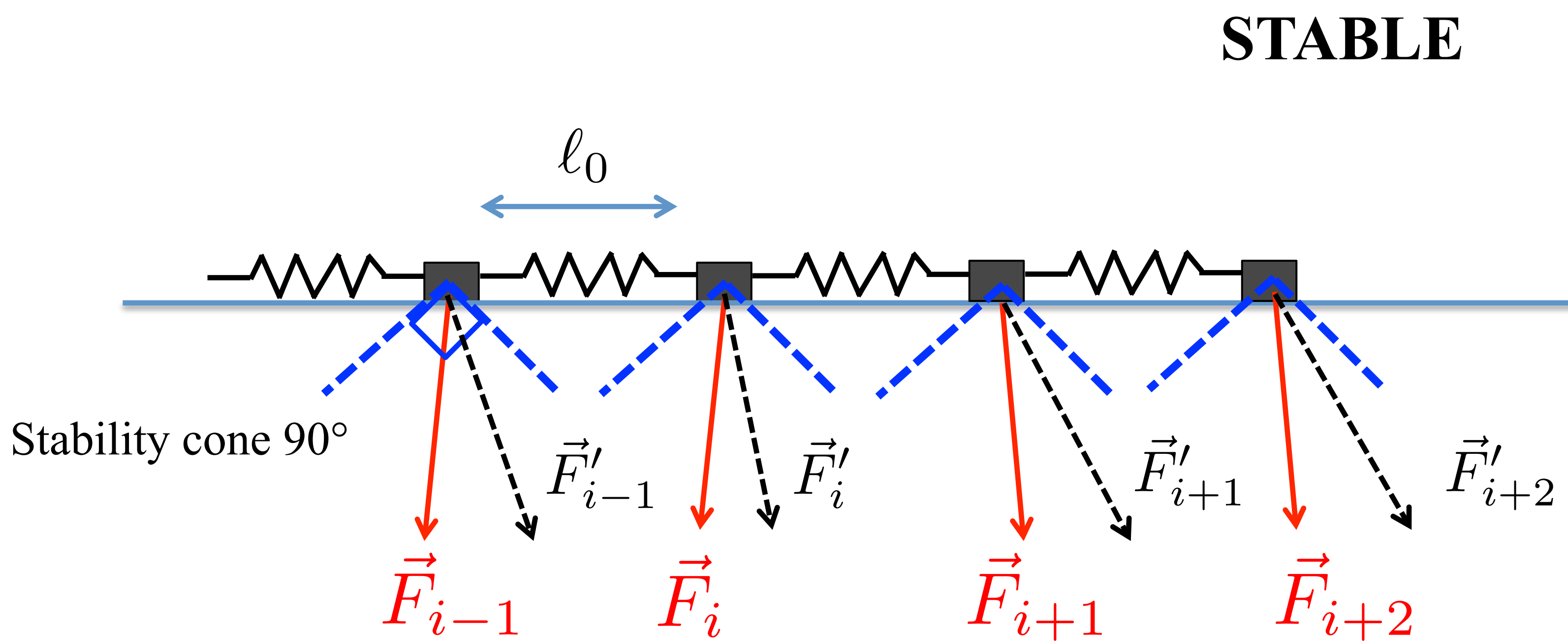
PROBABILITY DISTRIBUTION OF FORCE

Effective theory

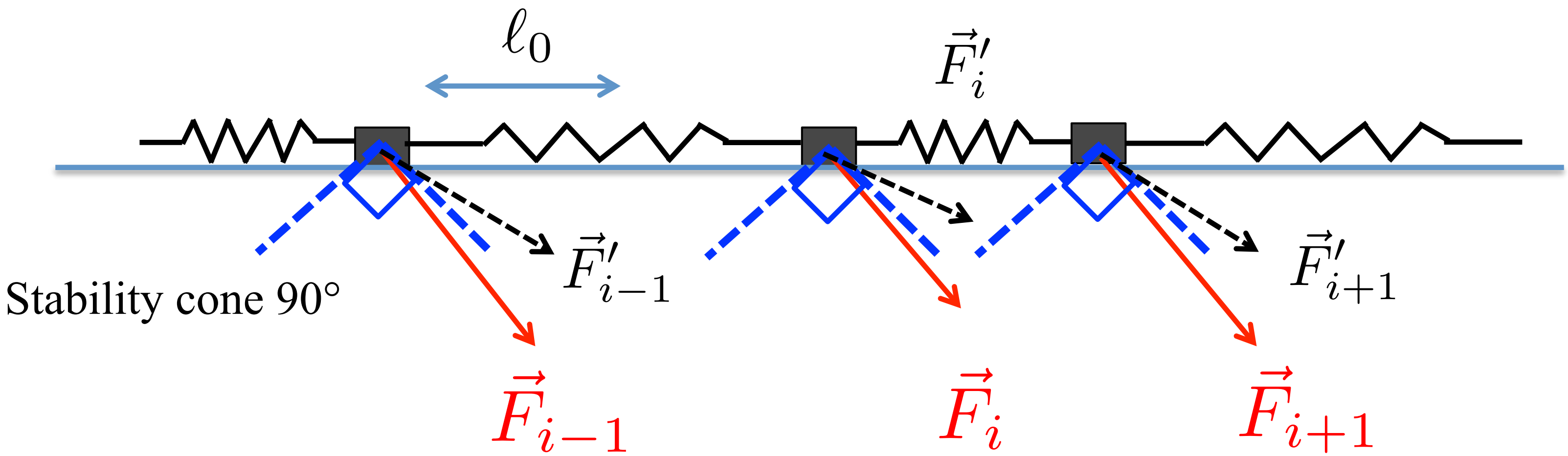




Effective theory (lines) VS simulations (points)


STABLE
ℓ_0
Stability cone 90°
$\vec{F}'_{i-1}$
$\vec{F}'_{i}$
$\vec{F}'_{i+1}$
$\vec{F}'_{i+2}$
$\vec{F}_{i-1}$
$\vec{F}_{i}$
$\vec{F}_{i+1}$
$\vec{F}_{i+2}$

UNSTABLE



ℓ_0
$\vec{F}_i'$
$\vec{F}_{i-1}'$
$\vec{F}_{i+1}'$
Stability cone 90°
$\vec{F}_{i-1}$
$\vec{F}_i$
$\vec{F}_{i+1}$

AMORPHOUS PACKINGS & GLASSES

Number of blocked structures in frictional granular assemblies at given Volume

$$\log[\mathcal{N}_{\text{blocked}}(V)] \sim N$$

$$\frac{1}{X} = \frac{\partial S_{\text{blocked}}}{\partial V}$$

Number of energy minima in models of glasses at given Energy

$$\log[\mathcal{N}_{\text{minima}}(E)] \sim N$$

$$\frac{1}{T_{\text{eff}}} = \frac{\partial S_{\text{states}}}{\partial E}$$

ANGORICITY

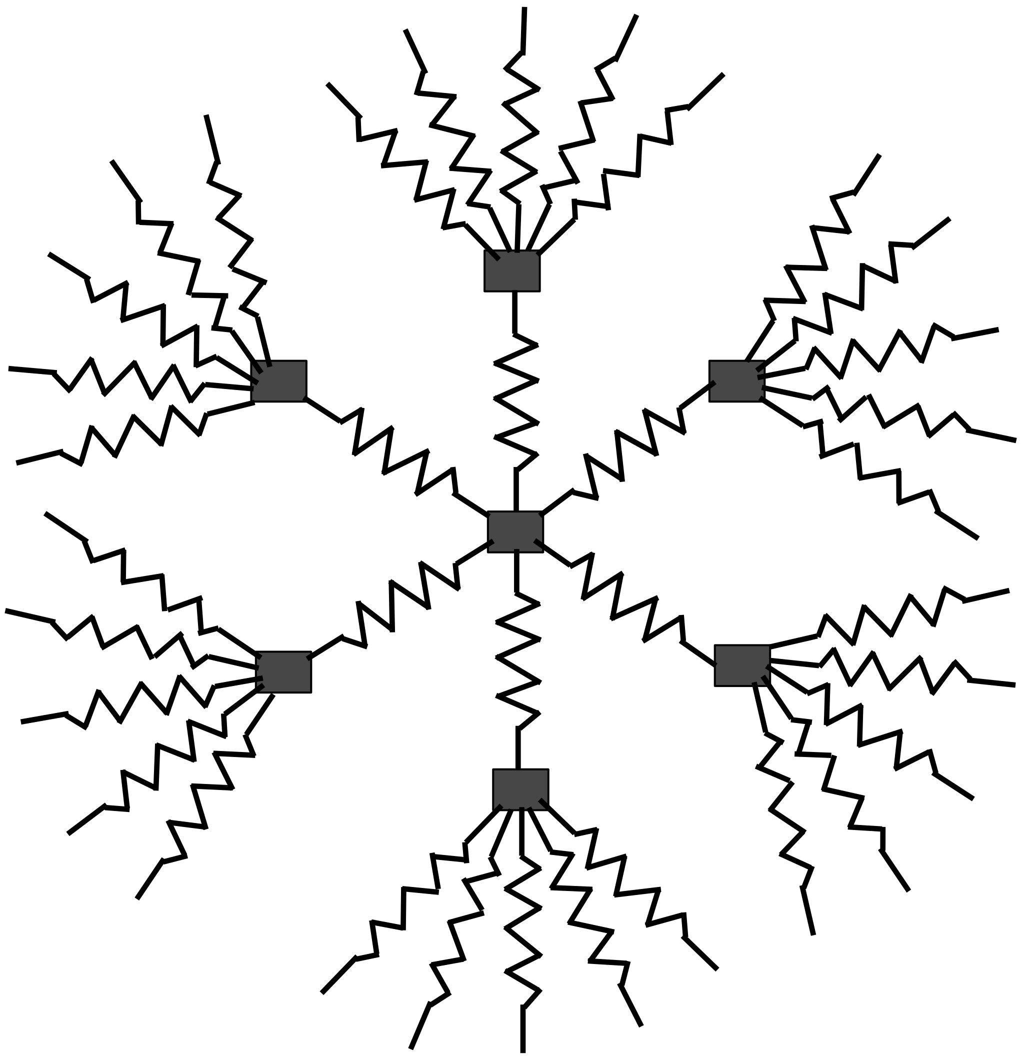
$$\Sigma_{\mu\nu} = \sum_{ij} [\vec{d}_{ij} \otimes \vec{f}_{ij}]_{\mu\nu}$$

Force-momentum tensor is conserved

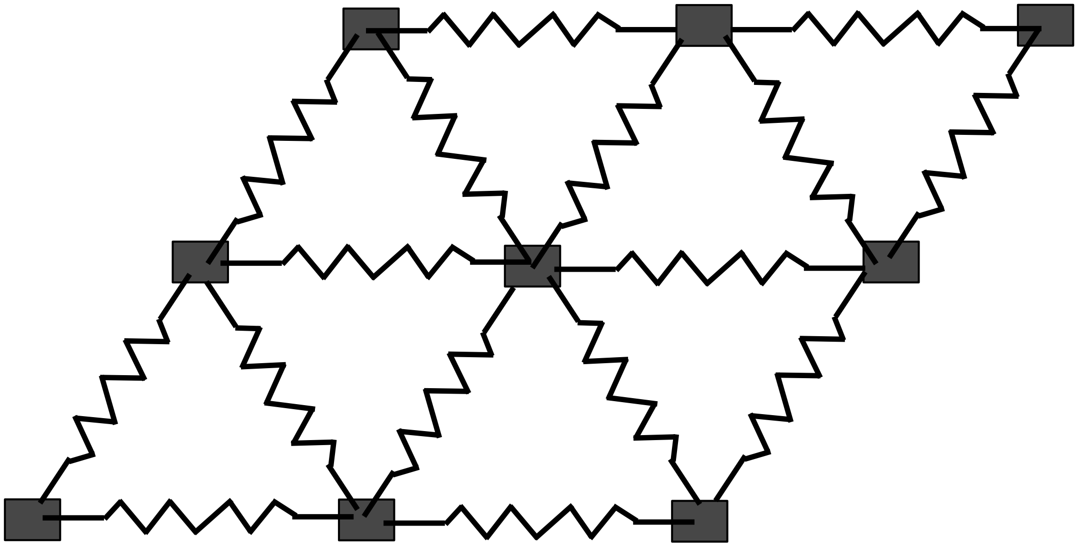
$$\alpha_{\mu\nu} = \frac{\partial S}{\partial \Sigma_{\mu\nu}}$$

''The Statistical Physics of athermal materials'',
D. Bi, **S. Henkes**, K. E. Daniels, B. Chakraborty
arXiv:1404.1854

BETHE LATTICE



2D LATTICE



AMORPHOUS PACKINGS & GLASSES

Number of blocked structures in frictional granular assemblies at given Volume

$$\log[\mathcal{N}_{\rm blocked}(V)] \sim N$$

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Number of energy minima in models of glasses at given Energy

$$\log[\mathcal{N}_{\rm minima}(E)] \sim N$$

$$\frac{1}{T_{\rm eff}} = \frac{\partial S_{\rm states}}{\partial E}$$

VOLUME 85, NUMBER 24 PHYSICAL REVIEW LETTERS 11 DECEMBER 2000

Edwards' Measures for Powders and Glasses

Alain Barrat,[1] Jorge Kurchan,[2] Vittorio Loreto,[3] and Mauro Sellitto[4]

VOLUME 90, NUMBER 19 PHYSICAL REVIEW LETTERS week ending 16 MAY 2003

Possible Test of the Thermodynamic Approach to Granular Media

David S. Dean and Alexandre Lefèvre

TEST OF EDWARDS IN ISING MODEL

A. Lefèvre & D. Dean, *J. Phys. A: Math. Gen.* **34** (2001)

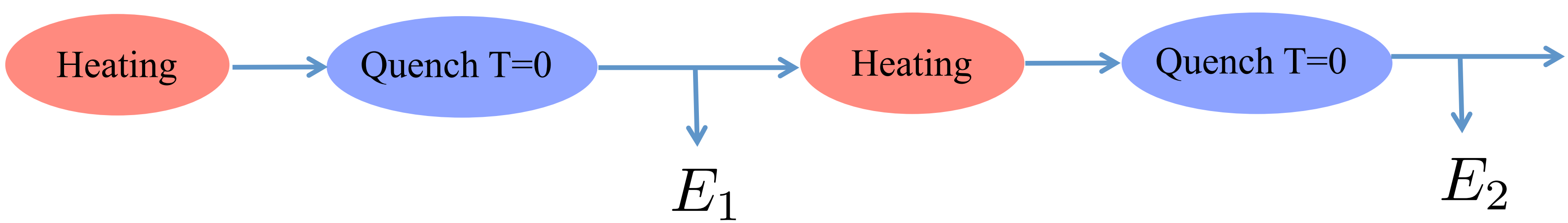
TAPPING DYNAMICS

1) Heating: all spins are flipped with probability p

$$p \in [0, 1/2[$$

2) Quench at T=0: only spin flips which lower the energy are allowed

''BLOCKED CONFIGURATIONS'' Energy cannot be lowered with a single spin flip



Average over states collected via tapping dynamics

$$\langle \mathcal{O} \rangle_{E(p)} = \mathcal{N}^{-1} \sum_{i=1}^{N} \mathcal{O}(\mathcal{C}_i)\ \delta\left(E - E_i\right)$$

$$\beta(p) = \left[\frac{\partial S}{\partial E}\right]_{E(p)} \qquad \langle \mathcal{O} \rangle_{\beta(p)} = \mathcal{Z}^{-1} \sum_{\boldsymbol{\sigma} | \boldsymbol{\sigma} \in \text{blocked}} \mathcal{O}(\boldsymbol{\sigma})\ e^{-\beta(p) E(\boldsymbol{\sigma})}$$

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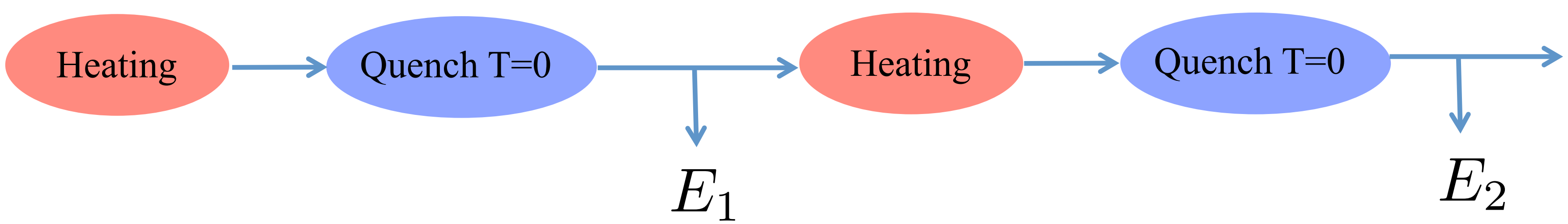


$$\mathcal{Z} = \sum_{\boldsymbol{\sigma}} e^{\beta_{Ed} \sum_i \sigma_i \sigma_{i+1}} \prod_i \Theta(\sigma_{i-1}\sigma_i + \sigma_i \sigma_{i+1})$$

$$x \geq 0 \;\rightarrow\; \Theta(x) = 1$$
$$x < 0 \;\rightarrow\; \Theta(x) = 0$$

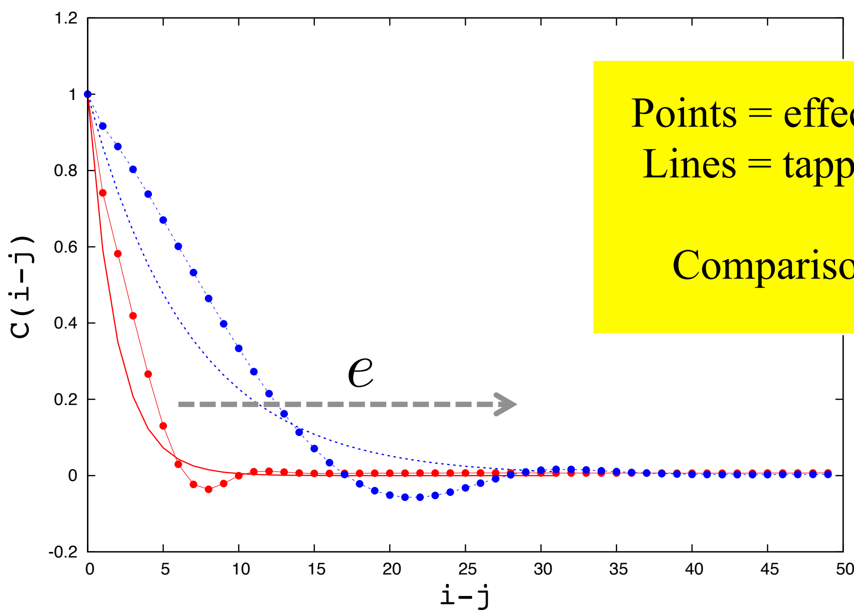
$$\beta(p) = \left[\frac{\partial S}{\partial E}\right]_{E(p)} \qquad \langle \mathcal{O} \rangle_{\beta(p)} = \mathcal{Z}^{-1} \sum_{\boldsymbol{\sigma} | \boldsymbol{\sigma} \in \mathrm{blocked}} \mathcal{O}(\boldsymbol{\sigma})\; e^{-\beta(p) E(\boldsymbol{\sigma})}$$

SPRING-SPRING CORRELATION FUNCTION

Make use of eigenvalues and eigenvectors

$$\mathcal{T}[f_b](x) = \lambda_b f_b(x) \qquad \int_{-\infty}^{\infty} f_b(x) f_a(x) = \delta_{a,b}$$

$$\lim_{N\to\infty} \langle \xi_m \xi_n \rangle_{Ed} = \sum_{b \in Sp(\mathcal{T})} \left(\frac{\lambda_b}{\lambda_{\max}} \right)^{n-m} \left| \int_{-\infty}^{\infty} dx \; x f_b(x) f_{\lambda_{\max}}(x) \right|^2$$



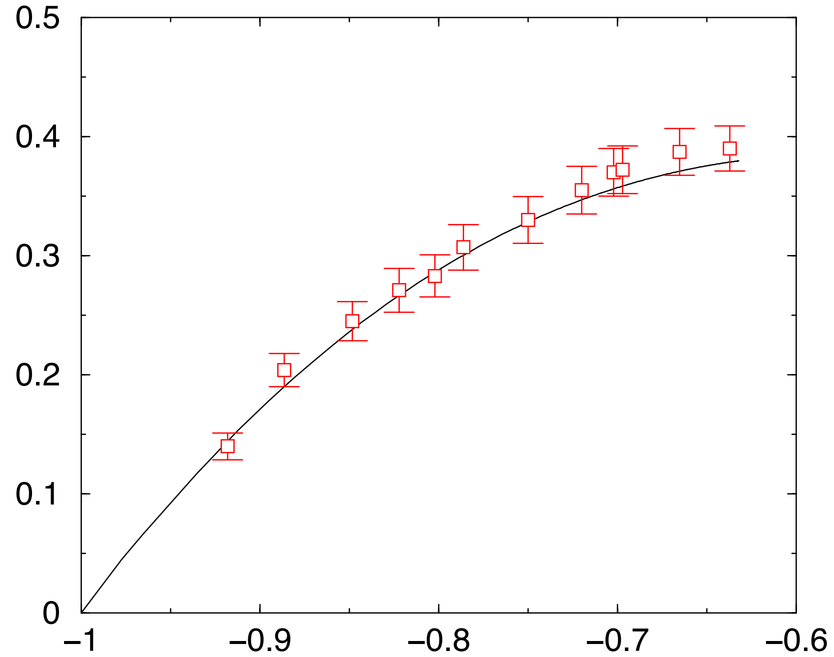
Points = effective thermodynamics
Lines = tapping dynamics (r=0.3)

Comparison is at fixed energy

TEST OF EDWARDS IN ISING MODEL

A. Lefèvre & D. Dean, *J. Phys. A: Math. Gen.* **34** (2001)

$$\sigma_E^2 = -\frac{\partial E}{\partial \beta}$$



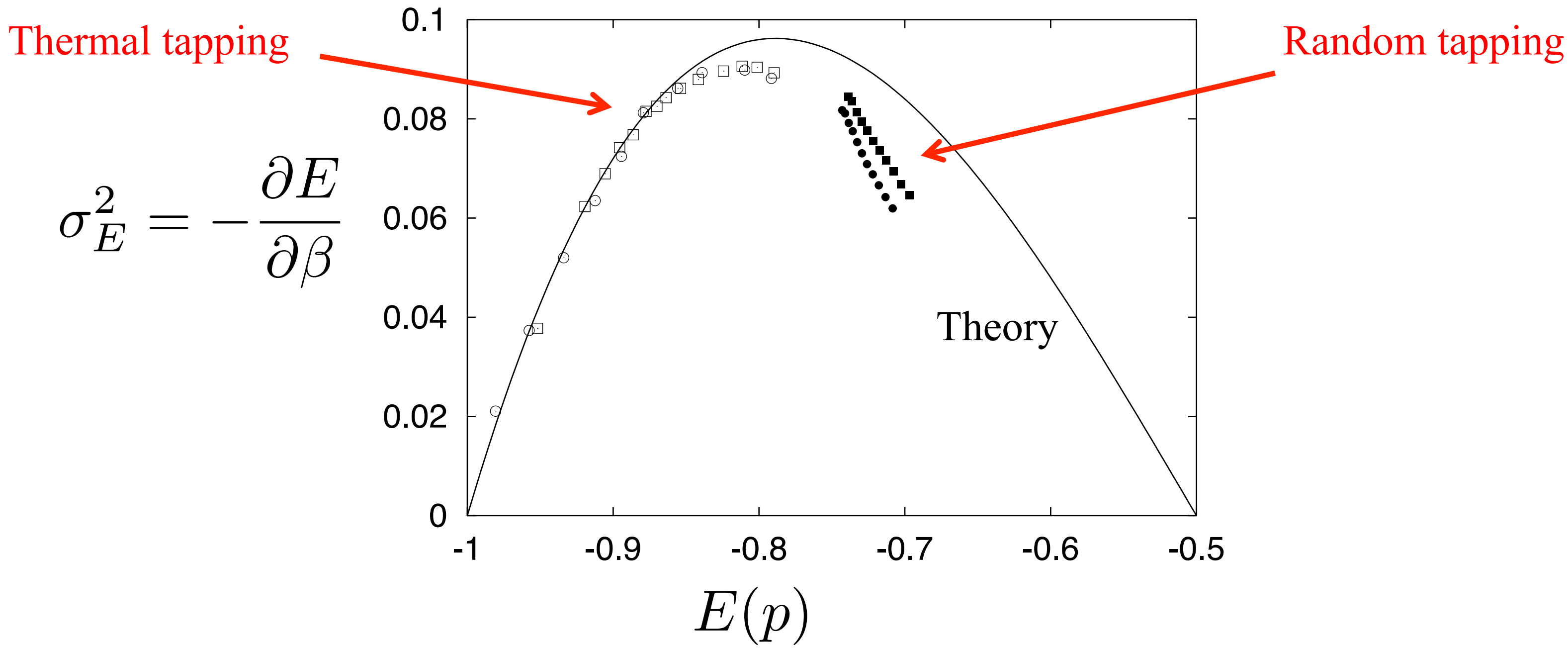
$$E(p)$$

$$\mathcal{Z} = \sum_{\boldsymbol{\sigma}} e^{\beta_{Ed} \sum_i \sigma_i \sigma_{i+1}} \prod_i \Theta(\sigma_{i-1}\sigma_i + \sigma_i \sigma_{i+1}) \qquad \begin{array}{l} x \geq 0 \;\rightarrow\; \Theta(x) = 1 \\ x < 0 \;\rightarrow\; \Theta(x) = 0 \end{array}$$

$$\beta(p) = \left[\frac{\partial S}{\partial E}\right]_{E(p)} \qquad \langle \mathcal{O} \rangle_{\beta(p)} = \mathcal{Z}^{-1} \sum_{\boldsymbol{\sigma}|\boldsymbol{\sigma} \in \text{blocked}} \mathcal{O}(\boldsymbol{\sigma})\; e^{-\beta(p) E(\boldsymbol{\sigma})}$$

TEST OF EDWARDS IN ISING MODEL + KINETIC CONSTRAINTS

$$\sigma_E^2 = -\frac{\partial E}{\partial \beta}$$



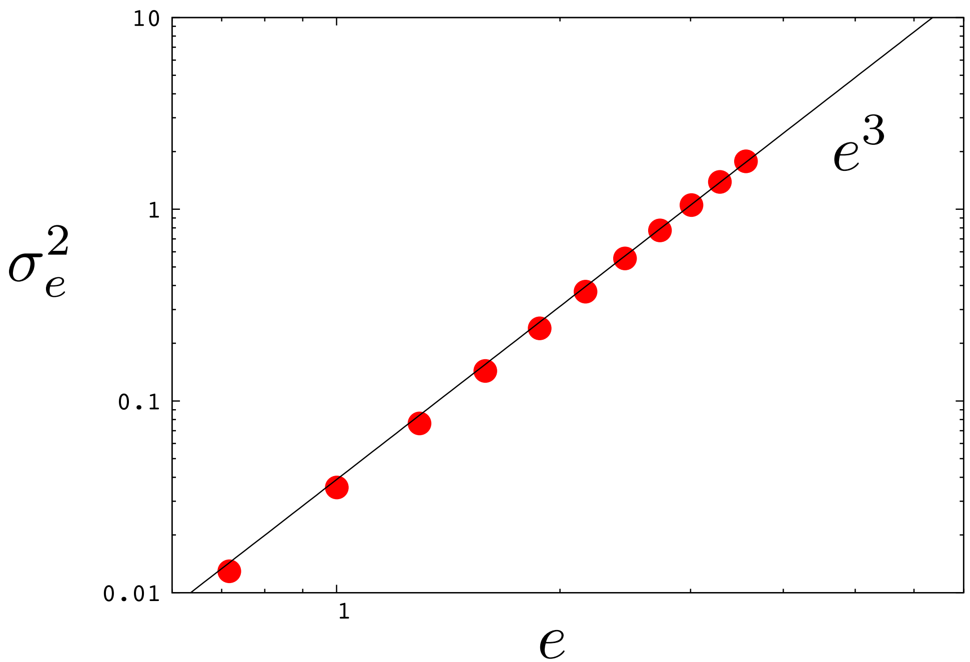
J. Berg, S. Franz, M. Sellitto, *EPJ B* (2002)

- Same test on one-dimensional Kinetically Constrained Models

- Disagreement between dynamical averages and Edwards effective theory

De Smedt, Godrèche, Luck, *EPJ B* (2002)

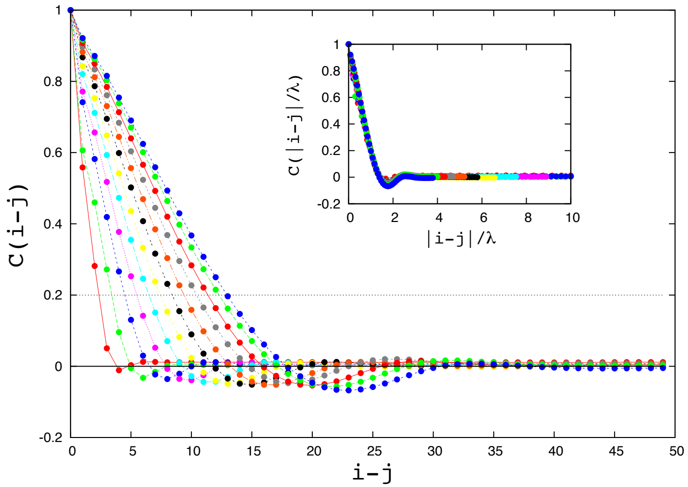
FLUCTUATIONS OF ENERGY



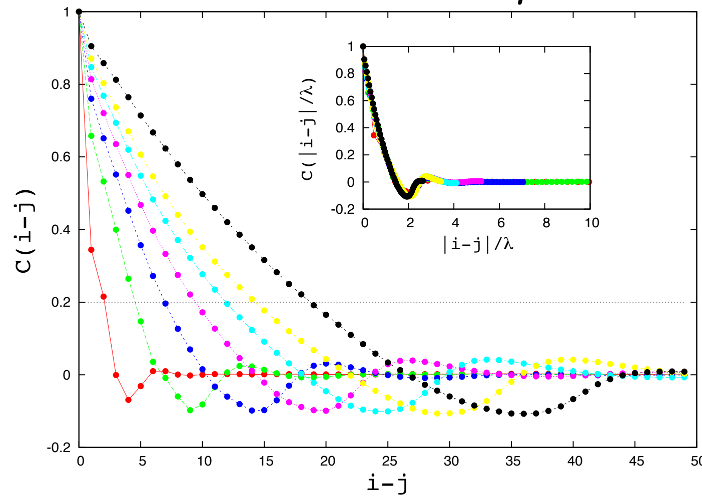
Effective theory

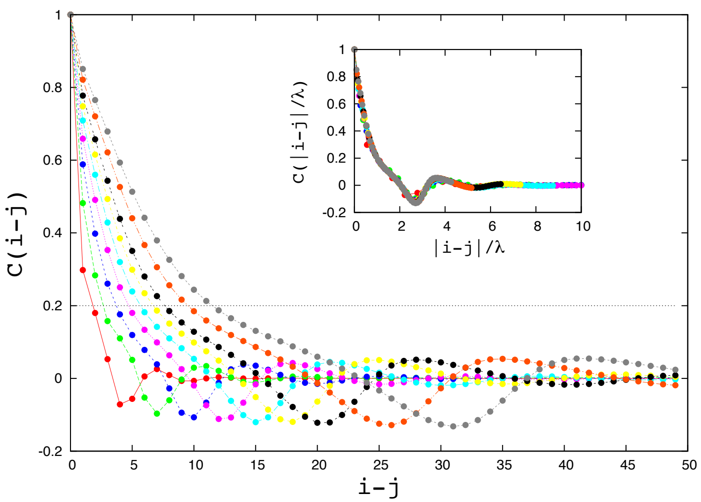
$$\left.\begin{aligned} e &\sim \sqrt{T_{Ed}} \\ \sigma_e^2 &= -\frac{\partial e}{\partial \beta_{Ed}} \sim T_{Ed}^{3/2} \end{aligned}\right\} \quad \sigma_e^2 \sim e^3$$

$f_i = F$ $\rho = 0.3$

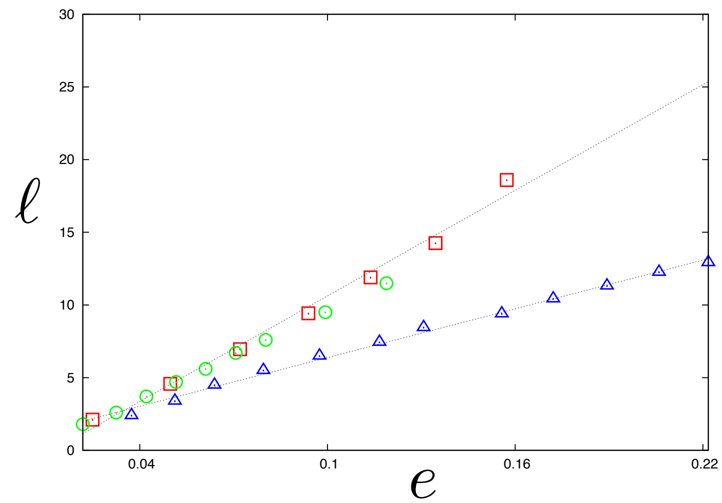


$f_i = F$ $\rho = 0.8$





$f_i \sim e^{-\frac{(f-F)^2}{\sigma}}$ $\rho = 1.0$



$\ell(e) \sim e$

Linear increase of correlation length for all the ''tapping'' dynamics we used

GAUSSIAN APPROXIMATION OF THE CONSTRAINT

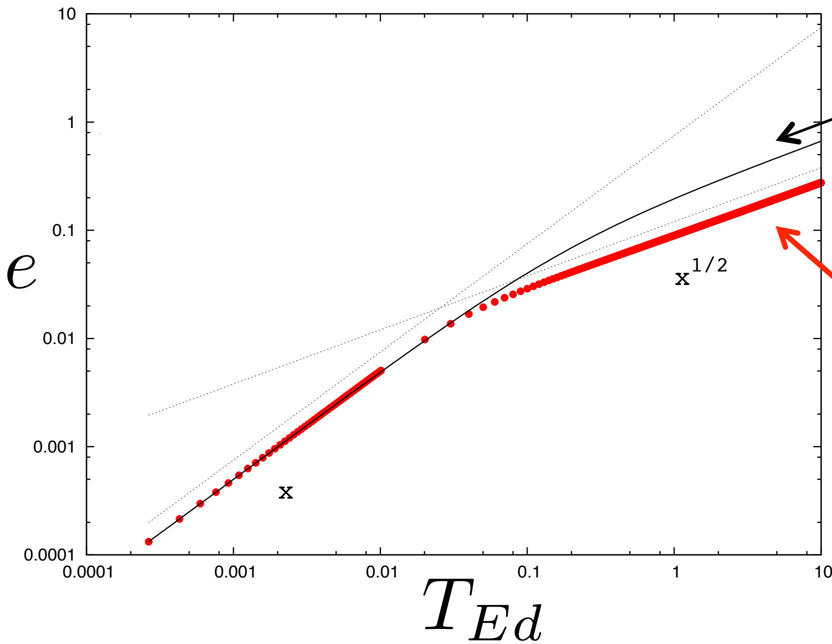
$$\mathcal{Z} = \mathrm{Tr}[\mathcal{T}^N]$$

$$T(x,y) = e^{-\beta_{Ed}\frac{x^2}{4}}\Theta(\mu - |x-y|)e^{-\beta_{Ed}\frac{y^2}{4}}$$

Smooth approximation of the constraint

$$\Theta(\mu - |x-y|) \sim \frac{1}{\sqrt{\pi}} \exp\left(-\frac{|x-y|^2}{4\mu^2}\right)$$

$$\mathcal{Z} = \int d\xi_1 \dots d\xi_N \quad \exp\left[-\boldsymbol{\xi}\cdot A\boldsymbol{\xi}\right]$$



$$e = \frac{1}{2}\frac{\mu\, T_{Ed}}{\sqrt{2T_{Ed} + \mu^2}}$$

$$e = -\lambda_{\max}^{-1}\langle\lambda_{\max}|\partial_{\beta_{Ed}}\mathcal{T}|\lambda_{\max}\rangle$$

DRIVEN ATHERMAL DYNAMICS

$$m\ \ddot{x}_i = F_{\rm diss} + F_{\rm el} + F_{\rm ext}$$

$\downarrow$

Blocked configurations

$$\langle \xi_i \xi_j \rangle \sim C\left(\frac{|i-j|}{\ell(e)}\right)$$

$$\ell(e) \sim e$$

$$e \sim \sqrt{?}$$

EFFECTIVE THERMODYNAMICS

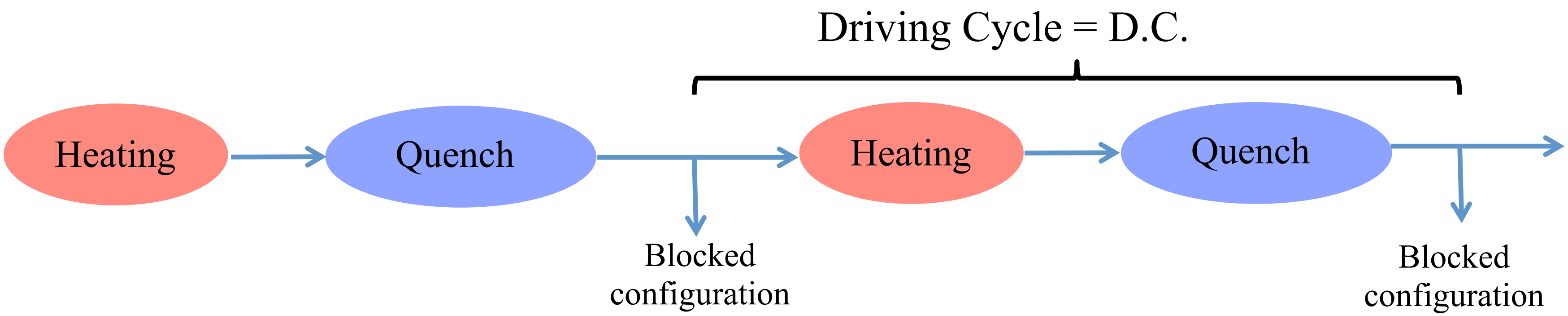
$$\mathcal{Z} = \int_{\boldsymbol{\xi} \in \text{blocked}} \mathcal{D}\boldsymbol{\xi}\ e^{-\beta_{Ed} E[\boldsymbol{\xi}]}$$

$$\langle \xi_i \xi_j \rangle \sim \exp\left(-\frac{|i-j|}{\ell(e)}\right)$$

$$\ell(e) \sim e$$

$$e \sim \sqrt{T_{Ed}}$$

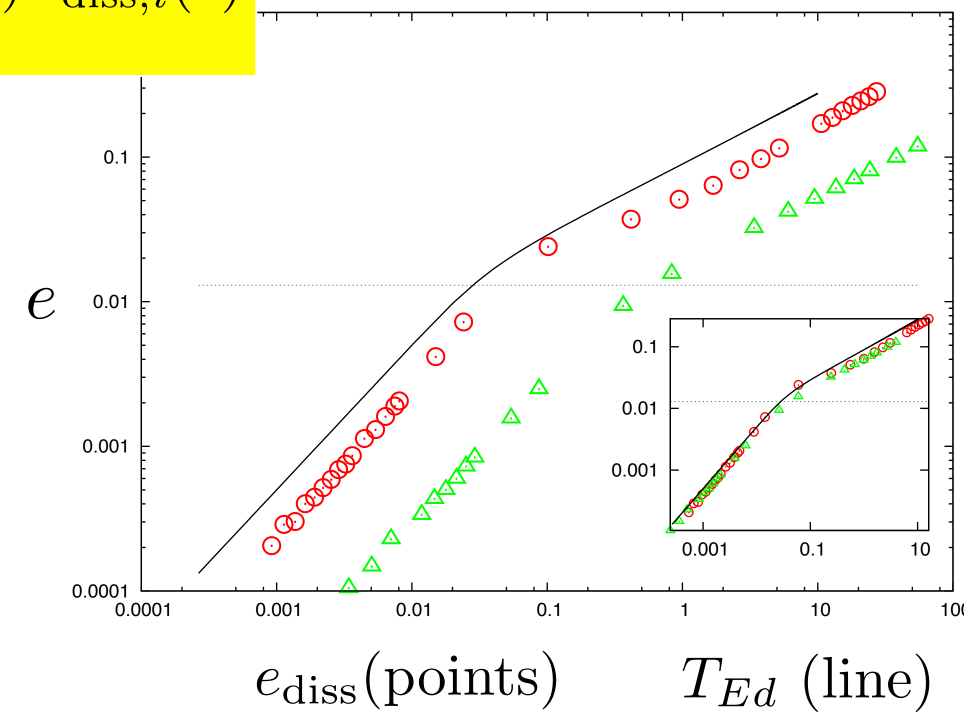
EDWARDS PARAMETER = DISSIPATED ENERGY



$$e_{\text{diss}} = -\frac{1}{N}\sum_{i=1}^{N}\int_{D.C.} ds\; v_i(s) F_{\text{diss},i}(s)$$

$e_{\text{diss}} \ll 1 \qquad e \sim e_{\text{diss}}$

$e_{\text{diss}} \gg 1 \qquad e \sim \sqrt{e_{\text{diss}}}$



DRIVEN ATHERMAL DYNAMICS

$$m\,\ddot{x}_i = F_{\text{diss}} + F_{\text{el}} + F_{\text{ext}}$$

↓

Blocked configurations

$$\langle \xi_i \xi_j \rangle \sim C\left(\frac{|i-j|}{\ell(e)}\right)$$

$$\ell(e) \sim e$$

$$e \sim \sqrt{e_{\text{diss}}}$$

EFFECTIVE THERMODYNAMICS

$$\mathcal{Z} = \int_{\boldsymbol{\xi}\in\text{blocked}} \mathcal{D}\boldsymbol{\xi}\; e^{-\beta_{Ed} E[\boldsymbol{\xi}]}$$

$$\langle \xi_i \xi_j \rangle \sim \exp\left(-\frac{|i-j|}{\ell(e)}\right)$$

$$\ell(e) \sim e$$

$$e \sim \sqrt{T_{Ed}}$$

$$e_{\text{diss}} = T_{Ed}$$

G. Gradenigo, E. Ferrero, E. Bertin, J.-L. Barrat, **Phys. Rev. Lett**. 115, 140601 (**2015**)