Integrable dissipative exclusion process

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General context

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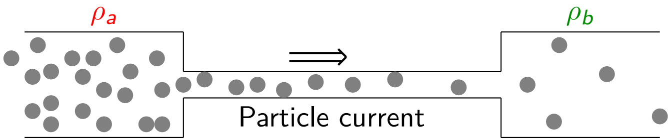

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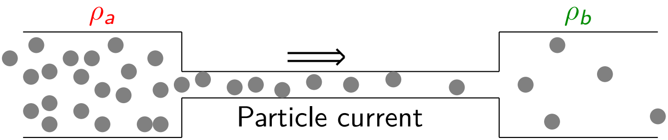

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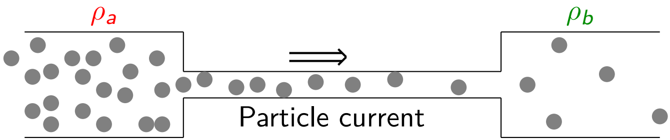

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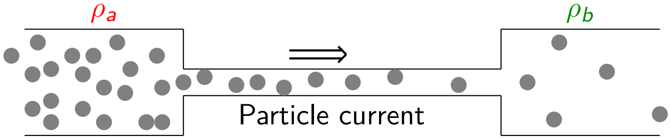
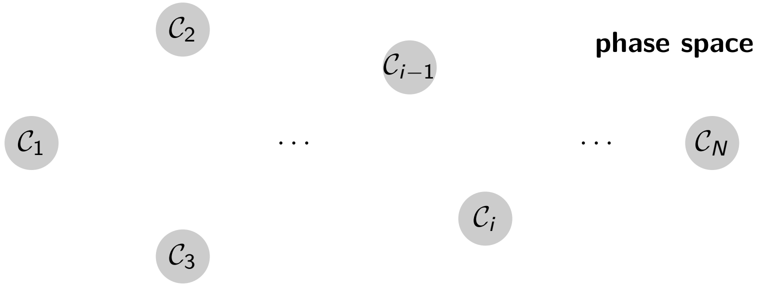


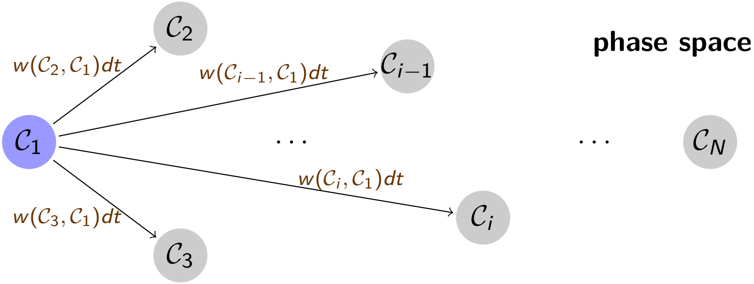
→ No general framework for such systems: $P_{stat}(\mathcal{C}) \sim$?

→ Exact computations have been done, and several techniques have been developed (matrix ansatz).

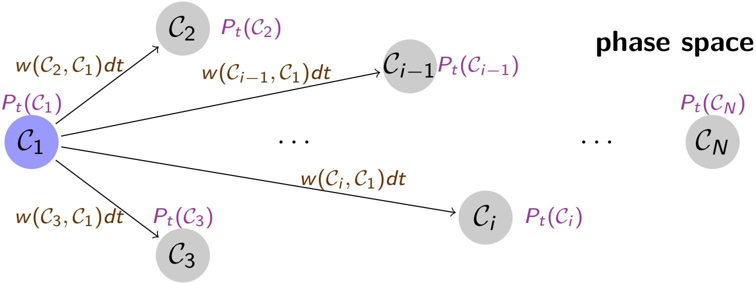
1. A simple out-of-equilibrium model.
    - Framework: Markov process, master equation.
    - Presentation of the model.
    - Configurations space, Markov matrix.
2. Stationary state and Matrix Ansatz.
    - Matrix Ansatz.
    - Commutation relations.
    - Computation of physical quantities.
3. Thermodynamical limit.
    - Scaling of the parameters.
    - Limit of the physical quantities.
    - Macroscopic fluctuation theory.



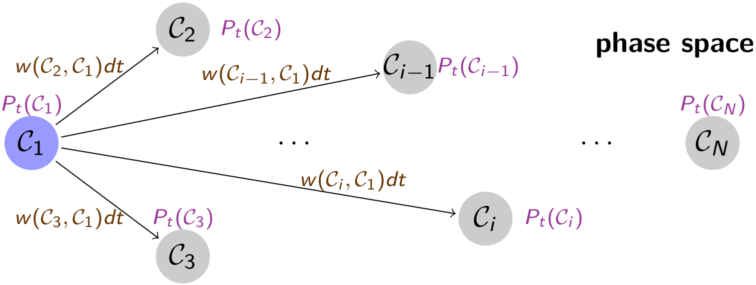
- The system can be in several different configurations.



- The system can be in several different configurations.
- During infinitesimal time dt, the system can jump from a configuration $\mathcal{C}$ to another configuration $\mathcal{C}'$ with probability $w(\mathcal{C}',\mathcal{C})dt$.

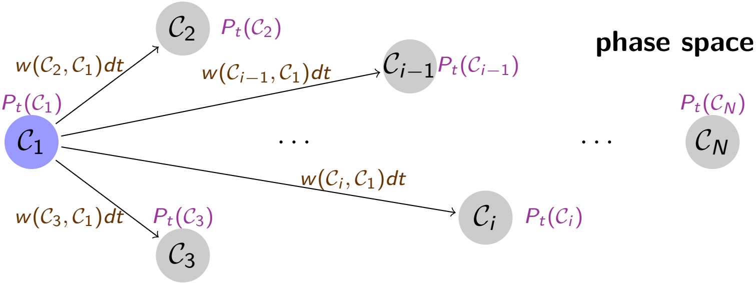


- Let $P_t(\mathcal{C})$ the probability for the system to be in configuration $\mathcal{C}$ at time t.



- Let $P_t(\mathcal{C})$ the probability for the system to be in configuration $\mathcal{C}$ at time t.
- The time evolution is governed by the master equation

$$P_{t+dt}(\mathcal{C}) = \sum_{\mathcal{C}'\neq\mathcal{C}} w(\mathcal{C},\mathcal{C}')dt P_t(\mathcal{C}') + \left(1 - \sum_{\mathcal{C}'\neq\mathcal{C}} w(\mathcal{C}',\mathcal{C})dt\right) P_t(\mathcal{C}) \ .$$



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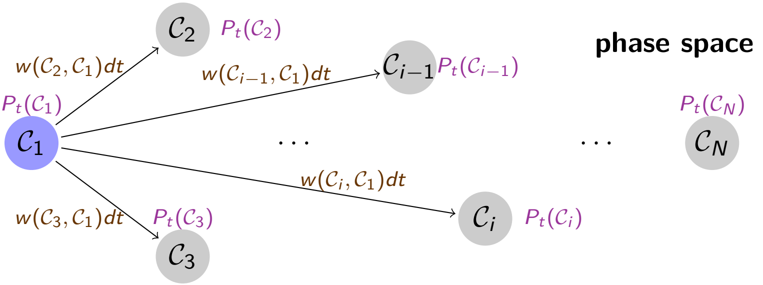
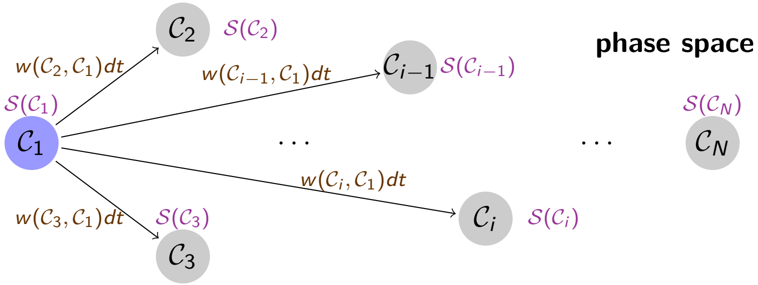
$$\frac{dP_t(\mathcal{C})}{dt} = \sum_{\mathcal{C}'\neq\mathcal{C}} w(\mathcal{C},\mathcal{C}')P_t(\mathcal{C}') - \sum_{\mathcal{C}'\neq\mathcal{C}} w(\mathcal{C}',\mathcal{C})P_t(\mathcal{C})\,.$$

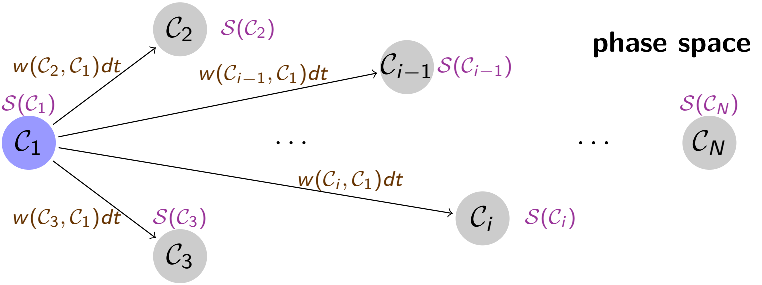


- Let $P_t(\mathcal{C})$ the probability for the system to be in configuration $\mathcal{C}$ at time t.
- The time evolution is governed by the master equation

$$0 = \frac{dP_t(\mathcal{C})}{dt} = \sum_{\mathcal{C}' \neq \mathcal{C}} w(\mathcal{C}, \mathcal{C}')P_t(\mathcal{C}') - \sum_{\mathcal{C}' \neq \mathcal{C}} w(\mathcal{C}', \mathcal{C})P_t(\mathcal{C}) \, .$$



- Let $\mathcal{S}(\mathcal{C})$ the probability for the system to be in configuration $\mathcal{C}$ in the stationary state.



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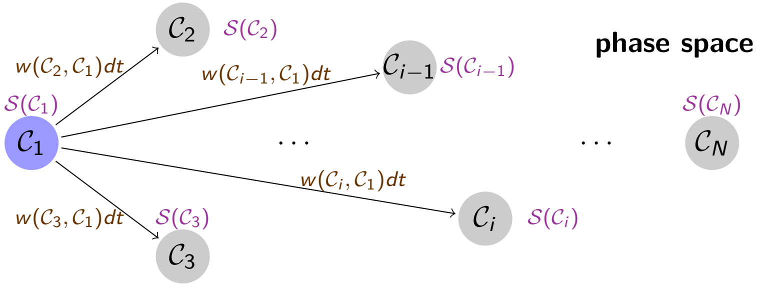
$$0 = \sum_{\mathcal{C}'\neq\mathcal{C}} w(\mathcal{C},\mathcal{C}')\mathcal{S}(\mathcal{C}') - \sum_{\mathcal{C}'\neq\mathcal{C}} w(\mathcal{C}',\mathcal{C})\mathcal{S}(\mathcal{C})\ .$$



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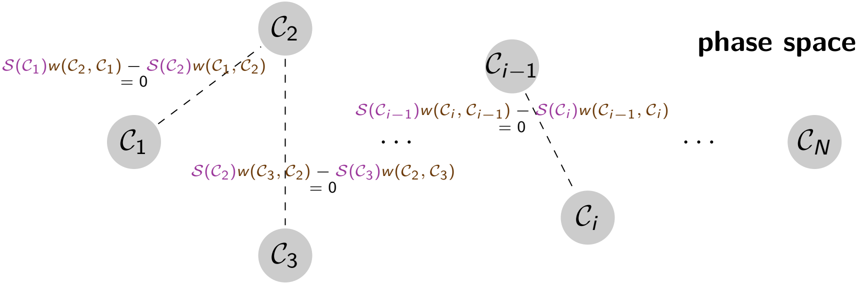
$$0 = \sum_{\mathcal{C}' \neq \mathcal{C}} \left(w(\mathcal{C}, \mathcal{C}')\mathcal{S}(\mathcal{C}') - w(\mathcal{C}', \mathcal{C})\mathcal{S}(\mathcal{C}) \right) \ .$$



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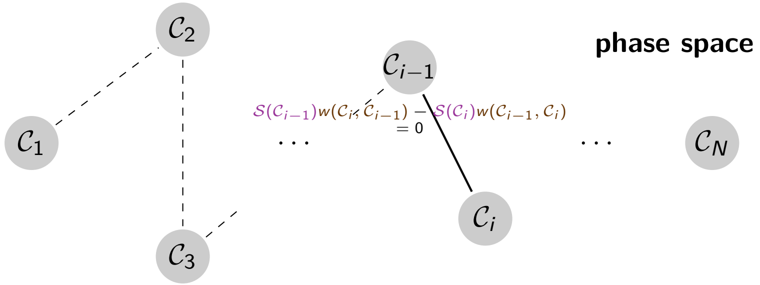

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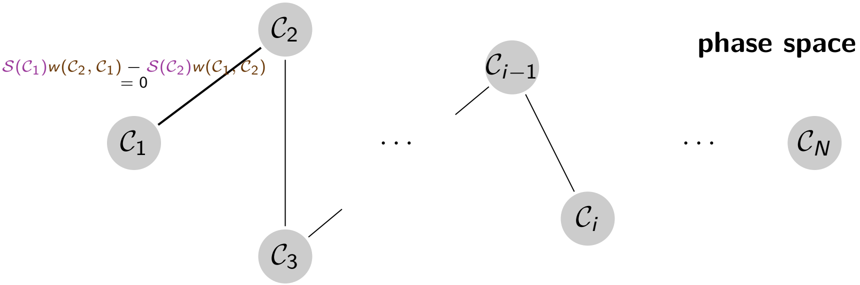

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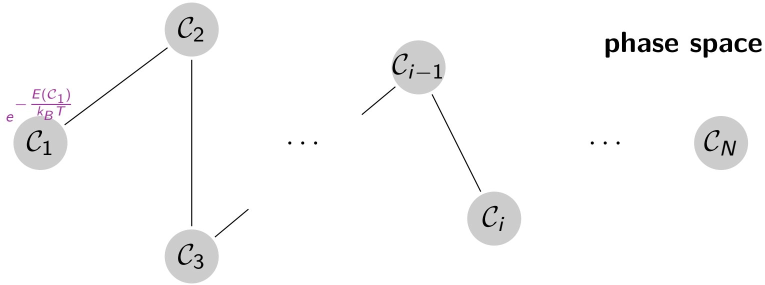

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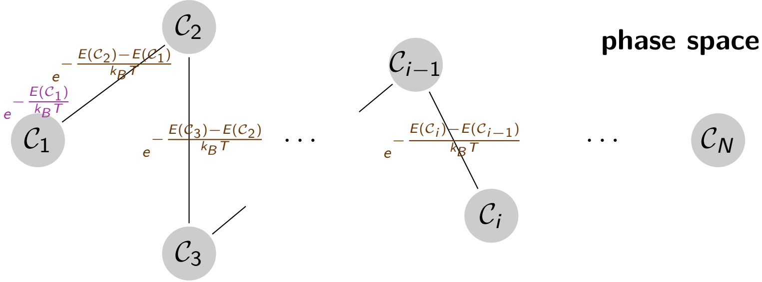

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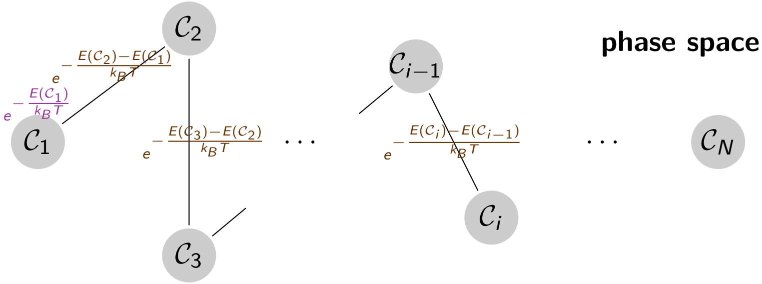

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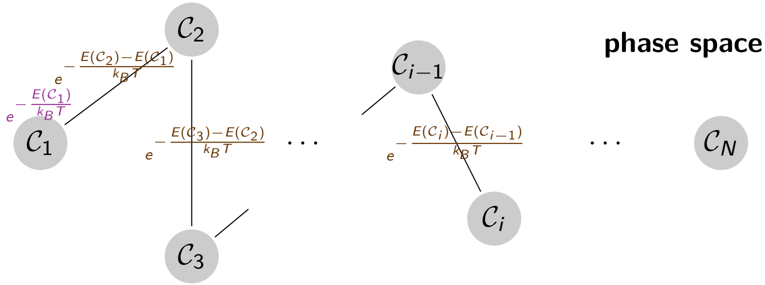

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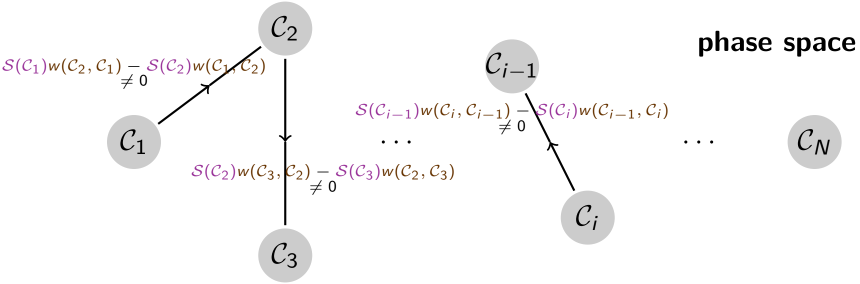
Ok with Boltzmann statistics!



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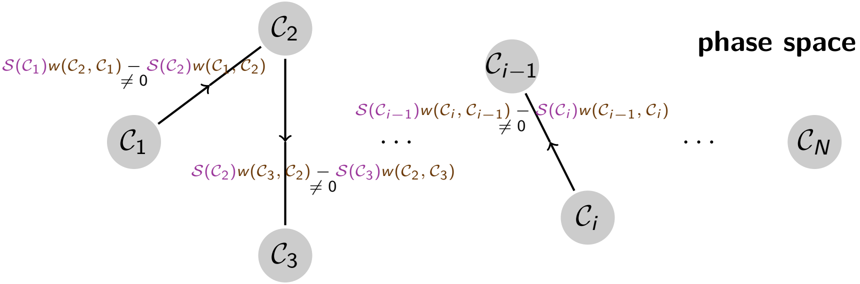

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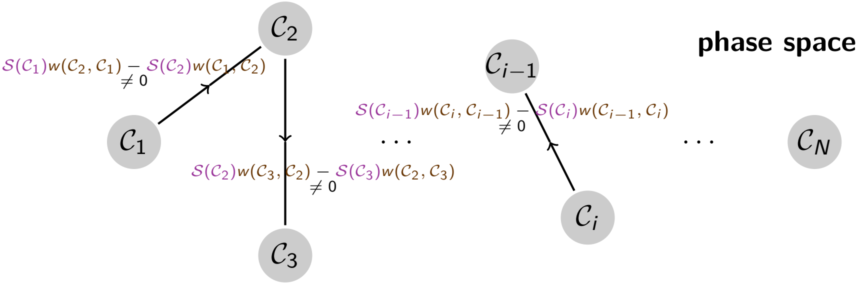

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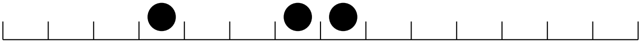
The system does not obey a Boltzmann statistic!

Dissipative symmetric simple exclusion process (DiSSEP)



Stochastic process on a one dimensional lattice with boundaries

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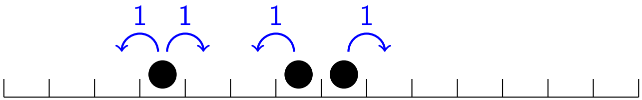

Stochastic process on a one dimensional lattice with boundaries

- in the bulk, particles can jump to the left or to the right with rate 1

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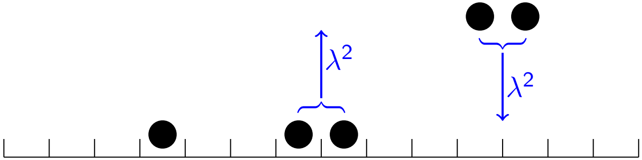

Stochastic process on a one dimensional lattice with boundaries

- in the bulk, particles can jump to the left or to the right with rate 1
- in the bulk, particle pairs can attach or detach with rate λ^2

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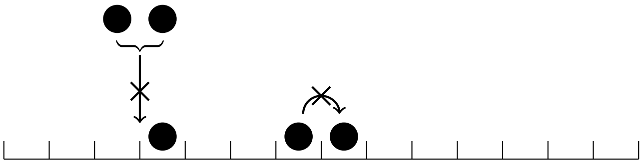

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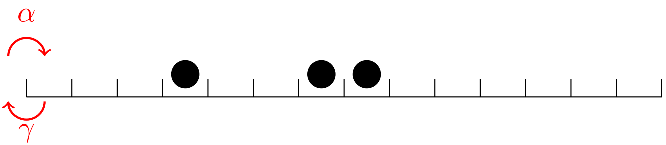

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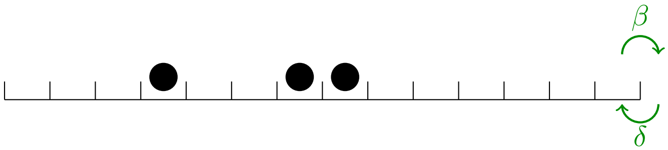

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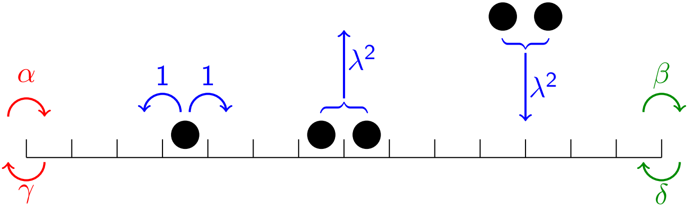

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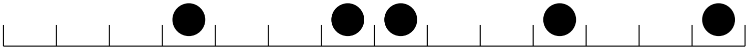
The system is driven out-of-equilibrium by the reservoirs: there are **particle currents** in the stationary state.

What is the configurations space?



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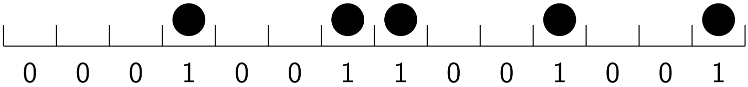

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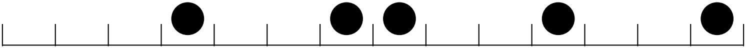

$$|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle$$

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- Attach to each site a two dimensional vector space $\mathbb{C}^2$ with basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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$$|P_t\rangle = \begin{pmatrix} P_t(\ (0,\dots,0,0,0)\ ) \\ P_t(\ (0,\dots,0,0,1)\ ) \\ P_t(\ (0,\dots,0,1,0)\ ) \\ \vdots \\ P_t(\ (1,\dots,1,1,1)\ ) \end{pmatrix} = \sum_{\tau_1,\dots,\tau_L\in\{0,1\}} P_t(\ (\tau_1,\dots,\tau_L)\ )\ |\tau_1\rangle\otimes\cdots\otimes|\tau_L\rangle$$

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The master equation rewrite in a vector form:

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$$\frac{d|P_t\rangle}{dt} = M|P_t\rangle,$$

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where M is the Markov matrix whose entries are $M_{\mathcal{C},\mathcal{C}'} = w(\mathcal{C},\mathcal{C}')$ and $M_{\mathcal{C},\mathcal{C}} = -\sum_{\mathcal{C}'\neq\mathcal{C}} w(\mathcal{C}',\mathcal{C})$.

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where

$$B = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix} \; ; \; w = \begin{pmatrix} -\lambda^2 & 0 & 0 & \lambda^2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \lambda^2 & 0 & 0 & -\lambda^2 \end{pmatrix} \; ; \; \overline{B} = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}$$

$End(\mathbb{C}^2)$ $\quad$ $End(\mathbb{C}^2 \otimes \mathbb{C}^2)$ $\quad$ $End(\mathbb{C}^2)$

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$$B = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix} \; ; \; w = \begin{pmatrix} -\lambda^2 & 0 & 0 & \lambda^2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \lambda^2 & 0 & 0 & -\lambda^2 \end{pmatrix} \; ; \; \overline{B} = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}$$

$End(\mathbb{C}^2)$ $\qquad$ $End(\mathbb{C}^2 \otimes \mathbb{C}^2)$ $\qquad$ $End(\mathbb{C}^2)$

The subscript index indicate on which sites the operators are acting

The Markov matrix can be written in a more explicit way:

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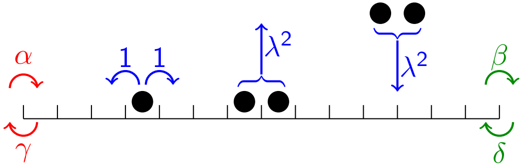
where

$$B = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix} \; ; \; w = \begin{pmatrix} -\lambda^2 & 0 & 0 & \lambda^2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \lambda^2 & 0 & 0 & -\lambda^2 \end{pmatrix} \; ; \; \overline{B} = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}$$

$End(\mathbb{C}^2)$ $\quad End(\mathbb{C}^2 \otimes \mathbb{C}^2)$ $\quad End(\mathbb{C}^2)$

The subscript index indicate on which sites the operators are acting



Stationary state and Matrix Ansatz

Recall

We want to find the steady state $|\mathcal{S}\rangle$ such that $M|\mathcal{S}\rangle = 0$.

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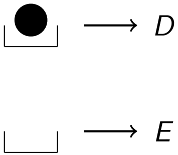

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$\longrightarrow D$

$\longrightarrow E$

$\mathcal{S}($  $) = \dfrac{\langle\langle W|EDEDD|V\rangle\rangle}{Z_5}$

$$Z_L = \langle\langle W|(D+E)^L|V\rangle\rangle$$

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$$|\mathcal{S}\rangle = \begin{pmatrix} \mathcal{S}(\ (0,\dots,0,0,0)\ ) \\ \mathcal{S}(\ (0,\dots,0,0,1)\ ) \\ \vdots \\ \mathcal{S}(\ (1,\dots,1,1,1)\ ) \end{pmatrix} = \frac{1}{Z_L} \begin{pmatrix} \langle\langle W|EE\dots EE|V\rangle\rangle \\ \langle\langle W|EE\dots ED|V\rangle\rangle \\ \vdots \\ \langle\langle W|DD\dots DD|V\rangle\rangle \end{pmatrix}.$$

The vector computed using this ansatz can be written

$$|\mathcal{S}\rangle = \frac{1}{Z_L}\langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle$$

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Recall that we want

$$M|\mathcal{S}\rangle = \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L \right) |\mathcal{S}\rangle = 0.$$

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Assume that

$$w \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} = \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} - \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix}$$

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and

$$\langle\langle W|B \begin{pmatrix} E \\ D \end{pmatrix} = \langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix},$$

The vector computed using this ansatz can be written

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and

$$\langle\!\langle W| B \begin{pmatrix} E \\ D \end{pmatrix} = \langle\!\langle W| \begin{pmatrix} -H \\ H \end{pmatrix}, \quad \overline{B} \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\!\rangle = -\begin{pmatrix} -H \\ H \end{pmatrix} |V\rangle\!\rangle.$$

$$M|\mathcal{S}\rangle = \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L \right) \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle$$

$$
\begin{aligned}
M|\mathcal{S}\rangle &= \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L\right) \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle \\
&= \langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \Big\} B_1|\mathcal{S}\rangle
\end{aligned}
$$

$$
\begin{aligned}
M|\mathcal{S}\rangle &= \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L\right) \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle \\
&= \left.\langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \right\} B_1|\mathcal{S}\rangle \\
&\quad \left.\begin{aligned} &- \langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \\ &+ \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \end{aligned}\right\} w_{1,2}|\mathcal{S}\rangle
\end{aligned}
$$

$$
\begin{aligned}
M|\mathcal{S}\rangle &= \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L \right) \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle \\
&= \left. \langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \right\} B_1|\mathcal{S}\rangle \\
&\quad \left. \begin{aligned} &- \langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \\ &+ \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \end{aligned} \right\} w_{1,2}|\mathcal{S}\rangle
\end{aligned}
$$

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&= \left.\langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \right\} B_1|\mathcal{S}\rangle \\
&\quad \left.\begin{aligned} &- \langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \\ &+ \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \end{aligned}\right\} w_{1,2}|\mathcal{S}\rangle \\
&\quad \left.\begin{aligned} &- \langle\langle W| \quad \ldots \quad \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad |V\rangle\rangle \\ &+ \langle\langle W| \quad \ldots \quad \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \quad \ldots \quad |V\rangle\rangle \end{aligned}\right\} w_{2,3}|\mathcal{S}\rangle
\end{aligned}
$$

$$M|\mathcal{S}\rangle = \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L\right) \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle$$

$$= \left.\langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \cancel{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots} \quad \ldots \quad |V\rangle\rangle \right\} B_1|\mathcal{S}\rangle$$

$$\left.\begin{aligned} &- \langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \cancel{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots} \quad \ldots \quad |V\rangle\rangle \\ &+ \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cancel{\begin{pmatrix} -H \\ H \end{pmatrix} \otimes \quad \ldots \quad \ldots} \quad \ldots \quad |V\rangle\rangle \end{aligned}\right\} w_{1,2}|\mathcal{S}\rangle$$

$$\left.\begin{aligned} &- \langle\langle W| \quad \ldots \quad \otimes \cancel{\begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix}} \otimes \quad \ldots \quad |V\rangle\rangle \\ &+ \langle\langle W| \quad \ldots \quad \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \quad \ldots \quad |V\rangle\rangle \end{aligned}\right\} w_{2,3}|\mathcal{S}\rangle$$

$$
\begin{aligned}
M|\mathcal{S}\rangle &= \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L\right) \langle\!\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\!\rangle \\
&= \left.\langle\!\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\!\rangle \right\} B_1|\mathcal{S}\rangle \\
&\left.\begin{aligned}
- \; &\langle\!\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\!\rangle \\
+ \; &\langle\!\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\!\rangle
\end{aligned}\right\} w_{1,2}|\mathcal{S}\rangle \\
&\left.\begin{aligned}
- \; &\langle\!\langle W| \quad \ldots \quad \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad |V\rangle\!\rangle \\
+ \; &\langle\!\langle W| \quad \ldots \quad \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \quad \ldots \quad |V\rangle\!\rangle
\end{aligned}\right\} w_{2,3}|\mathcal{S}\rangle \\
&\qquad\qquad\vdots
\end{aligned}
$$

$$
\begin{aligned}
M|\mathcal{S}\rangle &= \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L \right) \langle\!\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\!\rangle \\
&= \left.\langle\!\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \ \cdots \ \cdots \ \cdots \ |V\rangle\!\rangle \right\} B_1|\mathcal{S}\rangle \\
&\left.\begin{aligned}
&- \langle\!\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \ \cdots \ \cdots \ \cdots \ |V\rangle\!\rangle \\
&+ \langle\!\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \ \cdots \ \cdots \ \cdots \ |V\rangle\!\rangle
\end{aligned}\right\} w_{1,2}|\mathcal{S}\rangle \\
&\left.\begin{aligned}
&- \langle\!\langle W| \ \cdots \ \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \ \cdots \ |V\rangle\!\rangle \\
&+ \langle\!\langle W| \ \cdots \ \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \ \cdots \ |V\rangle\!\rangle
\end{aligned}\right\} w_{2,3}|\mathcal{S}\rangle \\
&\qquad\qquad\qquad\vdots \\
&\left.\begin{aligned}
&- \langle\!\langle W| \ \cdots \ \cdots \ \cdots \ \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\!\rangle \\
&+ \langle\!\langle W| \ \cdots \ \cdots \ \cdots \ \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} |V\rangle\!\rangle
\end{aligned}\right\} w_{L-1,L}|\mathcal{S}\rangle
\end{aligned}
$$

$$
\begin{aligned}
M|\mathcal{S}\rangle &= \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L \right) \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle \\
&= \quad \langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \Big\} \; B_1|\mathcal{S}\rangle \\
& \left.\begin{aligned}
&- \langle\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle \\
&+ \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \quad \ldots \quad \ldots \quad \ldots \quad |V\rangle\rangle
\end{aligned}\right\} w_{1,2}|\mathcal{S}\rangle \\
& \left.\begin{aligned}
&- \langle\langle W| \quad \ldots \quad \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \quad \ldots \quad |V\rangle\rangle \\
&+ \langle\langle W| \quad \ldots \quad \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \quad \ldots \quad |V\rangle\rangle
\end{aligned}\right\} w_{2,3}|\mathcal{S}\rangle \\
& \qquad\qquad\qquad \vdots \\
& \left.\begin{aligned}
&- \langle\langle W| \quad \ldots \quad \ldots \quad \ldots \quad \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle \\
&+ \langle\langle W| \quad \ldots \quad \ldots \quad \ldots \quad \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} |V\rangle\rangle
\end{aligned}\right\} w_{L-1,L}|\mathcal{S}\rangle \\
& - \langle\langle W| \quad \ldots \quad \ldots \quad \ldots \quad \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} |V\rangle\rangle \Big\} \; \overline{B}_L|\mathcal{S}\rangle
\end{aligned}
$$

$$
\begin{array}{rcl}
M|\mathcal{S}\rangle & = & \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L\right) \langle\!\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\!\rangle \\
& = & \left.\langle\!\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \;\cdots\;\cdots\;\cdots\; |V\rangle\!\rangle \right\} B_1|\mathcal{S}\rangle \\
& & \left.\begin{array}{l}
- \langle\!\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \;\cdots\;\cdots\;\cdots\; |V\rangle\!\rangle \\
+ \langle\!\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \;\cdots\;\cdots\;\cdots\; |V\rangle\!\rangle
\end{array}\right\} w_{1,2}|\mathcal{S}\rangle \\
& & \left.\begin{array}{l}
- \langle\!\langle W| \;\cdots\; \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \;\cdots\; |V\rangle\!\rangle \\
+ \langle\!\langle W| \;\cdots\; \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \;\cdots\; |V\rangle\!\rangle
\end{array}\right\} w_{2,3}|\mathcal{S}\rangle \\
& & \vdots \\
& & \left.\begin{array}{l}
- \langle\!\langle W| \;\cdots\;\cdots\;\cdots\; \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\!\rangle \\
+ \langle\!\langle W| \;\cdots\;\cdots\;\cdots\; \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} |V\rangle\!\rangle
\end{array}\right\} w_{L-1,L}|\mathcal{S}\rangle \\
& & \left. - \langle\!\langle W| \;\cdots\;\cdots\;\cdots\; \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} |V\rangle\!\rangle \right\} \overline{B}_L|\mathcal{S}\rangle
\end{array}
$$

$$
\begin{aligned}
M|\mathcal{S}\rangle &= \left(B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L\right) \langle\!\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\!\rangle \\
&= \left.\langle\!\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \ \ldots \ \ldots \ \ldots \ |V\rangle\!\rangle \right\} B_1|\mathcal{S}\rangle \\
&\quad \left.\begin{aligned}
&- \langle\!\langle W| \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \ \ldots \ \ldots \ \ldots \ |V\rangle\!\rangle \\
&+ \langle\!\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \ \ldots \ \ldots \ \ldots \ |V\rangle\!\rangle
\end{aligned}\right\} w_{1,2}|\mathcal{S}\rangle \\
&\quad \left.\begin{aligned}
&- \langle\!\langle W| \ \ldots \ \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \ \ldots \ |V\rangle\!\rangle \\
&+ \langle\!\langle W| \ \ldots \ \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \ \ldots \ |V\rangle\!\rangle
\end{aligned}\right\} w_{2,3}|\mathcal{S}\rangle \\
&\qquad\qquad\vdots \\
&\quad \left.\begin{aligned}
&- \langle\!\langle W| \ \ldots \ \ldots \ \ldots \ \otimes \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\!\rangle \\
&+ \langle\!\langle W| \ \ldots \ \ldots \ \ldots \ \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} |V\rangle\!\rangle
\end{aligned}\right\} w_{L-1,L}|\mathcal{S}\rangle \\
&\quad \left. - \langle\!\langle W| \ \ldots \ \ldots \ \ldots \ \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} |V\rangle\!\rangle \right\} \overline{B}_L|\mathcal{S}\rangle \\
&= \mathbf{0}\ !
\end{aligned}
$$

The previous relations are fulfilled if and only if the matrices E, D and H satisfy the algebraic relations

Algebraic relations

$$\begin{aligned} DE - ED &= EH + HD, \\ \lambda^2(D^2 - E^2) &= HE - EH = HD - DH \\ (\delta E - \beta D)|V\rangle\rangle &= -H|V\rangle\rangle \\ \langle\langle W|(\alpha E - \gamma D) &= \langle\langle W|H \end{aligned}$$

Can we compute something interesting with this algebra?

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with much simpler commutation relations:

$$G_3 G_1 = G_1 G_3, \quad G_2 G_1 = \phi G_1 G_2, \quad G_3 G_2 = \phi G_2 G_3,$$

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$$G_3 G_1 = G_1 G_3, \quad G_2 G_1 = \phi G_1 G_2, \quad G_3 G_2 = \phi G_2 G_3,$$

and relations on the boundaries:

$$\langle\langle W | G_1 = \langle\langle W | (a G_3 + c G_2), \quad G_3 | V \rangle\rangle = (b G_1 + d G_2) | V \rangle\rangle$$

Can we compute something interesting with this algebra? Yes!

Change of generators basis in the algebra $\{E, D, H\} \rightarrow \{G_1, G_2, G_3\}$

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Mean particle density at site i:

$$\langle \tau_i \rangle = \frac{\langle\langle W|(E+D)^{i-1}D(E+D)^{L-i}|V\rangle\rangle}{\langle\langle W|(E+D)^{L}|V\rangle\rangle}$$

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Hence

$$\langle\langle W|G_2^{i-1}G_1G_2^{L-i}|V\rangle\rangle = \frac{\phi^{i-1}(c+ad\phi^{L-1})}{1-ab\phi^{2L-2}}\langle\langle W|G_2^{L}|V\rangle\rangle.$$

- Mean particle density at site i:

$$\langle \tau_i \rangle = \frac{1}{2}\left(1 - \frac{\phi^{i-1}(c + ad\phi^{L-1}) + \phi^{L-i}(d + bc\phi^{L-1})}{1 - ab\phi^{2L-2}}\right).$$

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- Mean particle diffusion current between sites i and $i+1$:

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Thermodynamical limit

We want to keep a competition between the evaporation/condensation process and the diffusion process as $L \to \infty$.

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We have to take

$$\lambda = \frac{\lambda_0}{L}.$$

In the stationary state, the mean particle density is given by

$$\begin{aligned}\langle \rho(x) \rangle &:= \lim_{L\to\infty} \langle n_{Lx} \rangle \\ &= \frac{1}{2} + \frac{1}{2\sinh 2\lambda_0}\left(q_1 e^{-2\lambda_0(x-1/2)} + q_2 e^{2\lambda_0(x-1/2)}\right)\end{aligned}$$

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with

$$\begin{aligned}q_1 &= \left(\rho_a-\frac{1}{2}\right)e^{\lambda_0}-\left(\rho_b-\frac{1}{2}\right)e^{-\lambda_0}\\ q_2 &= \left(\rho_b-\frac{1}{2}\right)e^{\lambda_0}-\left(\rho_a-\frac{1}{2}\right)e^{-\lambda_0}.\end{aligned}$$

$$\rho_a < \frac{1}{2}, \quad \rho_b < \frac{1}{2}.$$

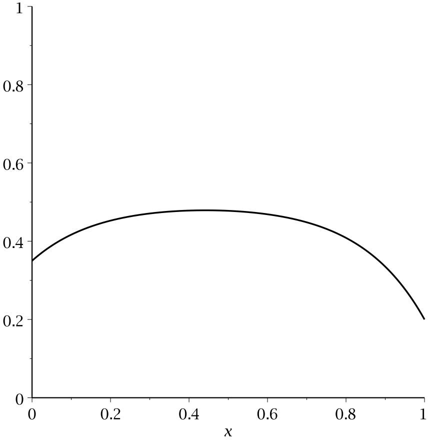

Figure : Plot of the density for $\rho_a = 0.35$, $\rho_b = 0.2$ and $\lambda_0 = 3$.

$$\rho_a > \frac{1}{2}, \qquad \rho_b > \frac{1}{2}.$$

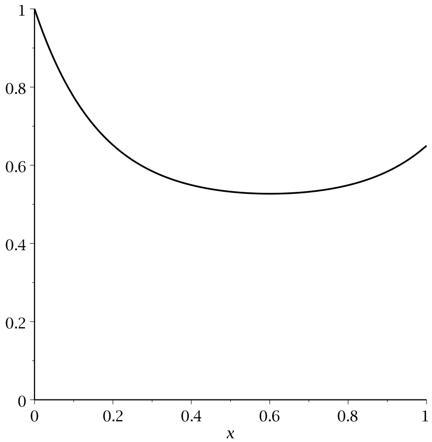

Figure : Plot of the density for $\rho_a = 1$, $\rho_b = 0.65$ and $\lambda_0 = 3$.

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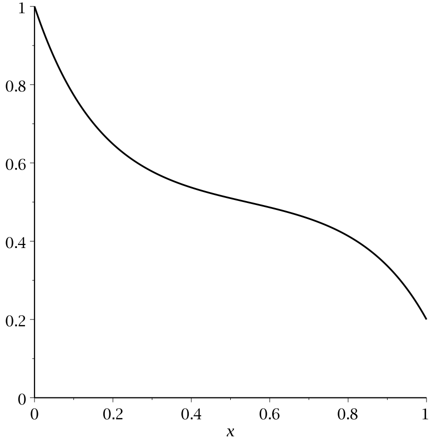

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We can also compute the mean particle current on the lattice

$$\begin{aligned}\langle j^{lat}(x)\rangle &:= \lim_{L\to\infty} L \times \langle J^{lat}_{Lx\to Lx+1}\rangle \\ &= \frac{\lambda_0}{\sinh 2\lambda_0}\left(q_1 e^{-2\lambda_0(x-1/2)} - q_2 e^{2\lambda_0(x-1/2)}\right),\end{aligned}$$

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and the mean particle condensation current

$$\begin{aligned} \langle j^{cond}(x)\rangle &:= \lim_{L\to\infty} L^2 \times \langle J^{cond}_{Lx,Lx+1}\rangle \\ &= \frac{-\lambda_0^2}{\sinh 2\lambda_0}\left(q_1 e^{-2\lambda_0(x-1/2)} + q_2 e^{2\lambda_0(x-1/2)}\right) \end{aligned}$$

We can also compute exactly in the thermodynamical limit the variance of the lattice current

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$$\begin{aligned}
\mu_2(x) &= 2q_1\, q_2\, \lambda_0^2 \left\{ (2x-1)\, \frac{\sinh\Big(2\,\lambda_0\,(2\,x-1)\Big)}{\Big(\sinh(2\,\lambda_0)\Big)^3} - \frac{\cosh(2\,\lambda_0)\,\cosh\Big(2\,\lambda_0\,(2x-1)\Big)+1}{\Big(\sinh(2\,\lambda_0)\Big)^4} \right\} \\
&- q_2^2\lambda_0\, \frac{e^{4\,\lambda_0\,x}+e^{-4\,\lambda_0\,(1-x)}-e^{4\,\lambda_0\,(2\,x-1)}+3}{4\Big(\sinh(2\,\lambda_0)\Big)^3} - q_1^2\lambda_0\, \frac{e^{4\,\lambda_0\,(1-x)}+e^{-4\,\lambda_0\,x}-e^{4\,\lambda_0\,(1-2\,x)}+3}{4\Big(\sinh(2\,\lambda_0)\Big)^3} \\
&+ \frac{\lambda_0\,\cosh\Big(2\,\lambda_0\,x\Big)\cosh\Big(2\,\lambda_0\,(1-x)\Big)}{\sinh(2\,\lambda_0)}.
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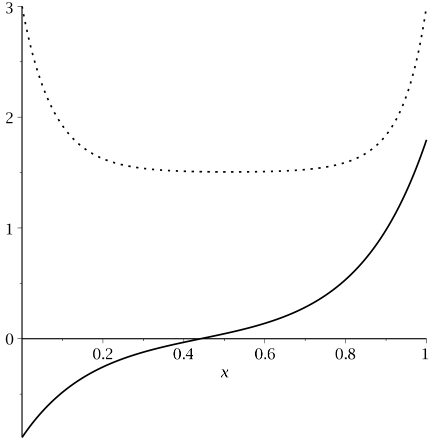

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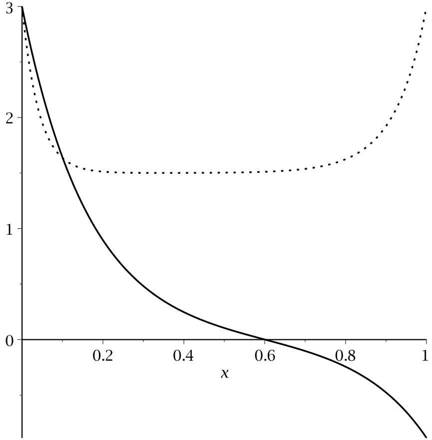

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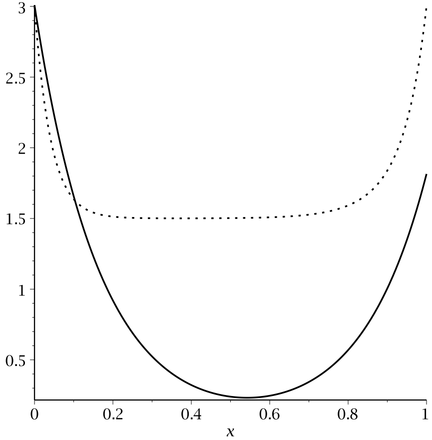

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General framework in the thermodynamical limit: Macroscopic Fluctuation Theory

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Allows to compute fluctuations of the density and currents profiles $\rho(x,t)$, $j^{lat}(x,t)$ and $j^{cond}(x,t)$ around their mean values.

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Main idea

$$\mathbb{P}_{[0,T]}\left(\{\rho, j^{lat}, j^{cond}\}\right) \sim \exp\left[-L\mathcal{I}_{[0,T]}(\rho, j^{lat}, j^{cond})\right]$$
$$\sim \text{''} \exp\left[-\mathcal{A}\right] \text{''}$$

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- Minimizing this large deviation functional (over the three fields) gives the "equations of motion" that is the hydrodynamic equation satisfied by the mean values of the fields.

The large deviation functional is given by (Bodineau, Lagouge, 2009)

$$\mathcal{I}_{[0,T]}(\rho,j^{lat},j^{cond})=\int_0^T dt \int_0^1 dx \left\{ \frac{(j^{lat}+D(\rho)\partial_x\rho)^2}{2\sigma(\rho)}+\Phi\left(\rho,j^{cond}\right)\right\},$$

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- Only 4 relevant parameters: the diffusion coefficient $D(\rho)$, the conductivity $\sigma(\rho)$, the creation and annihilation rates $C(\rho)$ and $A(\rho)$.
- The action vanishes (is minimal) when

$$j^{lat} = D(\rho)\partial_x \rho, \qquad j^{cond} = C(\rho) - A(\rho).$$

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- The solution matches exactly the value computed previously from the finite size lattice: this provides a check of the MFT!!!

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- Construct the excited states in a matrix product form.
- Solve more complicated models: for instance a 2-species TASEP with boundaries (work in progress).

Thank you!