PARAMETER-FREE SOLUTIONS OF THE BOHR MODEL WITH MODIFIED SHAPE PHASE SPACE

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The Bohr-Mottelson (BM) Hamiltonian:

$$H_{BM} = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right] + V(\beta,\gamma)$$

- The situation when the potential is γ independent, is referred to as the γ-unstable.
- The situation when the potential has a single localized minimum in γ, is referred to as the γ-stable.
- The exact solvability of a model is directly related to the presence of an underlying symmetry ($U(5), SU(3), O(6), E(5)$).

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Modified through: a) Kinetic energy; b) Potential

The usual five-dimensional kinetic operator of a γ-soft Bohr Hamiltonian

$$\hat{T}_s = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right]$$

$Q_k(k=1,2,3)$ angular momentum projections on the principal axes of the intrinsic reference frame.

The prolate γ-rigid kinetic operator

$$\hat{T}_r = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^2\frac{\partial}{\partial\beta} - \frac{Q_1^2+Q_2^2}{3\beta^2}\right]$$

- is defined only in a three-dimensional space of the β shape variable, and only two Euler angles. [Bonatsos, Lenis, Petrellis, Terziev, Yigitoglu, PLB **632** (2006) 238]

A hybrid model is realized through a coherent coupling of these operators:

$$H = \chi\hat{T}_r + (1-\chi)\hat{T}_s + V(\beta,\gamma), \quad \chi - \gamma\text{-rigidity}$$

- The associated Hamiltonian acts in a mixed shape phase space with a χ**-weighted integration metric:** $\beta^4 d\beta \quad \longrightarrow \quad \beta^{4-2\chi}d\beta.$

[Budaca&Budaca, JPG **42** (2015) 085103; EPJA **51** (2015) 126]

In order to study critical point nuclei, the mixed Schrödinger equation is treated as in the well known $X(5)$ model [Iachello, PRL **87** (2001) 052502], where an approximate separation of β and γ-angular variables is achieved through

$\Downarrow$

- a small angle approximation
- an adiabatic decoupling of β and γ shape fluctuations

Leaving aside the additive contribution of the γ vibrational energy, the radial-like equation for the β shape variable reads as

$$\left[-\frac{\partial^2}{\partial\beta^2} - \frac{2(2-\chi)}{\beta}\frac{\partial}{\partial\beta} + \frac{L(L+1)}{3\beta^2} + u(\beta)\right]\xi(\beta) = \varepsilon\xi(\beta)$$

Considering $u(\beta) = \begin{cases} 0, & \beta \leqslant \beta_W \\ \infty, & \beta > \beta_W \end{cases}$ the mixing will take place between:

$X(5)$ model $\chi = 0$ [Iachello, PRL **87** (2001) 052502]

$X(3)$ model $\chi \to 1$ [Bonatsos, Lenis, Petrellis, Terziev, Yigitoglu, PLB **632** (2006) 238]

The boundary condition $f(\beta_W) = 0$ gives the β energy spectrum in terms of the s-th zero $x_{s,\nu}$ of the Bessel function $J_\nu(x_{s,\nu}\beta/\beta_W)$, where

$$\nu = \sqrt{\frac{L(L+1)}{3} + \left(\frac{3}{2} - \chi\right)^2} \quad \text{and} \quad s = n_\beta - 1$$

Bohr Hamiltonian solutions with ISW β potential are closely related to the 2nd order Casimir operator of the Euclidean group $E(D) = T_D \bigoplus_S SO(D)$:

$$\left[-\frac{1}{\beta^{D-1}}\frac{\partial}{\partial\beta}\beta^{D-1}\frac{\partial}{\partial\beta} + \frac{\omega(\omega+D-2)}{\beta^2}\right]F(\beta) = k^2F(\beta),\ \ \omega - SO(D) \text{ quantum number}$$

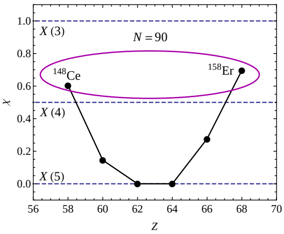
The β equation of the $X(D)$ hybrid model can be easily brought to a similar form bearing the correspondences:

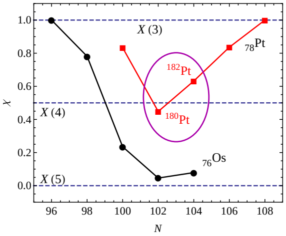
$$\text{Integer dimension}\quad D = 5-2\chi = \begin{cases} 3,\ \chi=1 \ \Rightarrow\ X(3) \\ 4,\ \chi=0.5 \ \Rightarrow\ \textbf{new CPS}\ X(4) \\ 5,\ \chi=0 \ \Rightarrow\ X(5) \end{cases}$$

$$\omega(\omega+D-2) = \frac{L(L+1)}{3} \quad \text{satisfied for} \quad \begin{cases} L=\omega=0, \\ \text{[Bonatsos, McCutchan, Casten, PRL \textbf{101} (2008) 022501]} \\ L = 2\omega = 2(3D-8) \\ \text{[Budaca\&Budaca, PLB \textbf{759} (2016) 349]} \end{cases}$$

- The fact that its 0^+ and 8^+ states follow exactly the $E(4)$ symmetry makes $X(4)$ model a **Partial Dynamical Symmetry of type I**. [Leviatan, PRL **98** (2007) 242502]

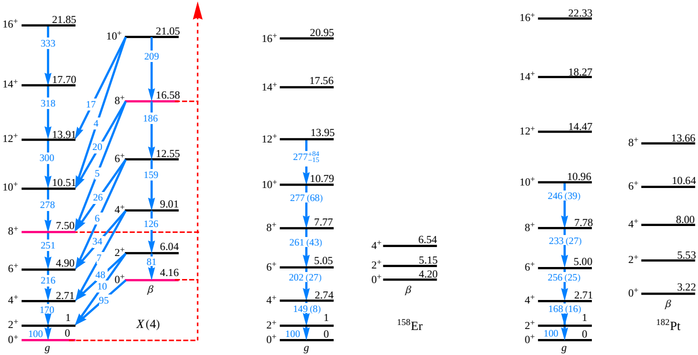
X(4) NUMERICAL APPLICATION





Budaca&Budaca
PRC 94 (2016) 054306

Exact realisations of the $E(4)$ Dynamical Symmetry



Best $X(4)$ candidates [Budaca&Budaca, PLB **759** (2016) 349]

The canonical form of the β equation for γ-unstable case

$$\left[-\frac{\partial^2}{\partial\beta^2}+\frac{(\tau+1)(\tau+2)}{\beta^2}+\underbrace{v(\beta,\epsilon)}\right]f(\beta)=\epsilon f(\beta), \quad \left| \begin{array}{l} f(\beta)=\beta^2 F(\beta) \\ \tau\ -\ \text{seniority} \end{array} \right.$$

Harmonic Oscillator **Coulomb-like**

$$v_{HO}(\beta,\epsilon)=(1+a\epsilon)\beta^2 \qquad v_C=-\frac{1+c\epsilon}{\beta}$$

The energy spectra are determined from the quadratic equations:

$$\epsilon=\sqrt{1+a\epsilon}\left(2n_\beta+\tau+\frac{5}{2}\right), \quad \text{for HO}$$

$$\epsilon=-\left[\frac{1+c\epsilon}{2(n_\beta+\tau+2)}\right]^2, \quad \text{for Coulomb}$$

Physically meaning solutions:

$$\epsilon_N^{HO}=\left(N+\frac{5}{2}\right)\left[\left(N+\frac{5}{2}\right)\frac{a}{2}+\sqrt{1+\left(N+\frac{5}{2}\right)^2\frac{a^2}{4}}\right], \quad N=2n_\beta+\tau$$

$$\epsilon_{N'}^{C}=\frac{1}{c^2}\left[-2(N'+2)^2-c+2(N'+2)\sqrt{(N'+2)^2+c}\right], \quad N'=n_\beta+\tau$$

n_β - quantum number of β excitations

- Due to the energy dependence of the potential, the scalar product is modified as
  Formanek, Lombard, Mares, Czech J. Phys. 54 (2004) 289

$$\beta^4 d\beta \quad \longrightarrow \quad \left[1 - \frac{\partial v(\beta,\epsilon)}{\partial \epsilon}\right] \beta^4 d\beta = \begin{cases} (1 - a\beta)\beta^4 d\beta, & \text{for HO} \\ \left(1 + \frac{c}{\beta}\right)\beta^4 d\beta, & \text{for Coulomb} \end{cases}$$

  in order to satisfy the continuity equation.

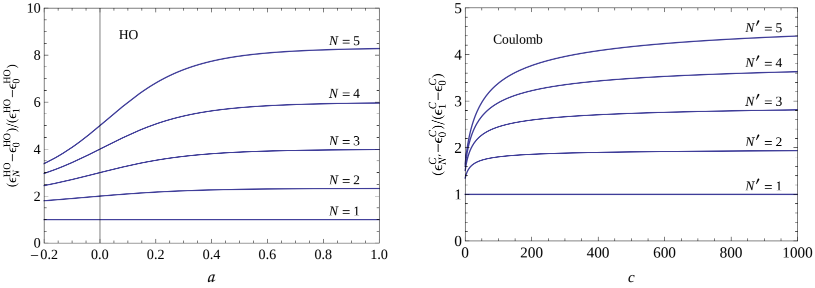
- The linear energy dependence leads to a state independent integration metric and to a coherent quantum theory (**Special case**).

- Consequently, the model must have positive definite norms, averages and density distribution:

$$\rho^i_{n_\beta \tau}(\beta) = \left|F^i_{n_\beta \tau}(\beta)\right|^2 \left[1 - \frac{\partial v^i(\beta,\epsilon)}{\partial \epsilon}\right] \beta^4 > 0, \;\; i = HO, C.$$

  This is exactly satisfied only if $a < 0$ and $c > 0$ and approximately for $a \longrightarrow \infty$.

ASYMPTOTIC LIMIT



- Only the asymptotic limits have a practical applicability for the collective nuclear phenomena $\left(\dfrac{E(4^+)}{E(2^+)} \geq 2\right)$.

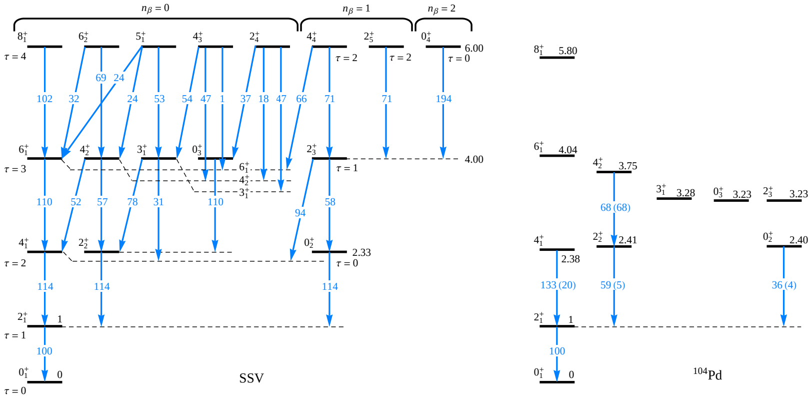
In the asymptotic limit of a and c the corresponding solutions become fully scalable:

$$
\begin{aligned}
\lim_{a\to\infty} \epsilon_N^{HO} &= \frac{a}{2}\left(N + \frac{5}{2}\right)^2, \quad \text{Stffening Spherical Vibrator } (\textbf{SSV}) \\
\lim_{c\to\infty} \epsilon_{N'}^{C} &= -\frac{1}{c} + \frac{2}{c^{3/2}}\left(N' + 2\right), \quad \text{Asymptotic Energy Dependent Coulomb } (\textbf{AEDC})
\end{aligned}
$$

The formula for $E2$ transition probabilities contains:

$SO(5)$ **Clebsh-Gordan coefficient** dictates the angular momentum and seniority selection rules.

integral over the β variable is numerically evaluated for successive high values of the parameter a, retaining the value when a reliable degree of convergence is achieved.



Experimental realizations [Budaca, PLB **751** (2015) 39]

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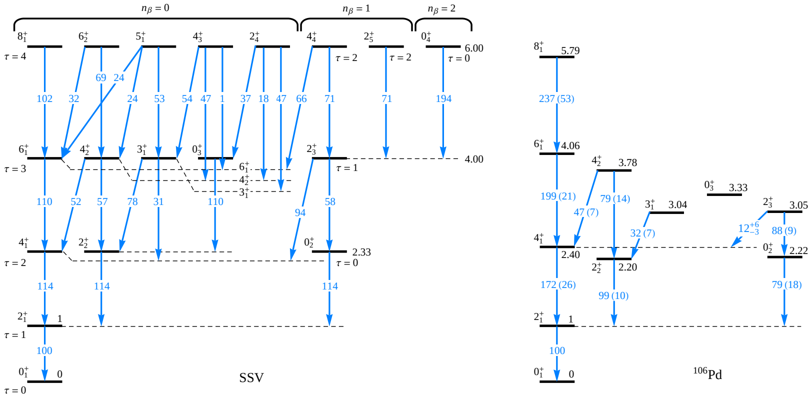

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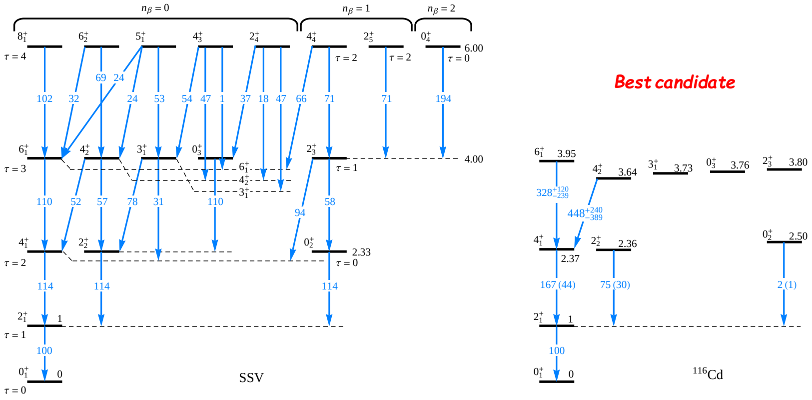

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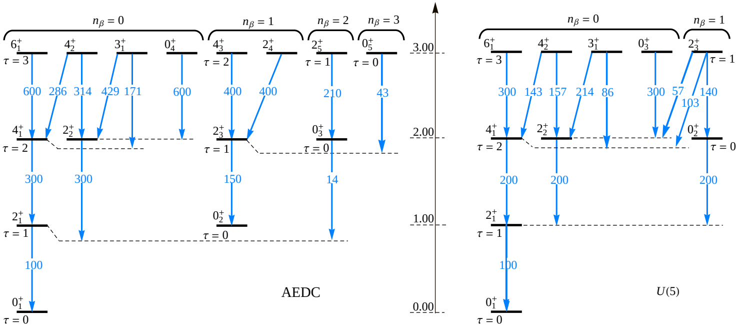

Experimental realizations [Budaca, EPJA **52** (2016) 314]

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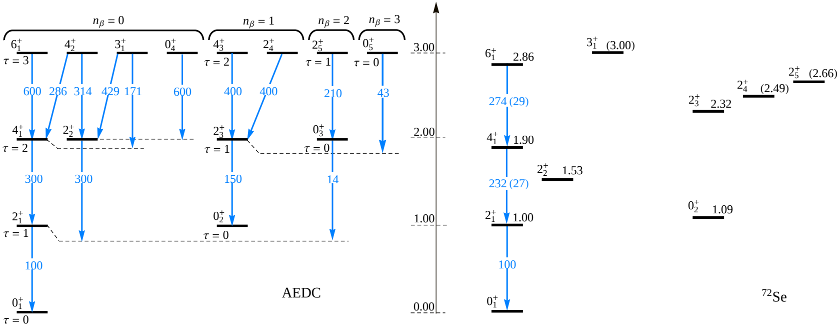

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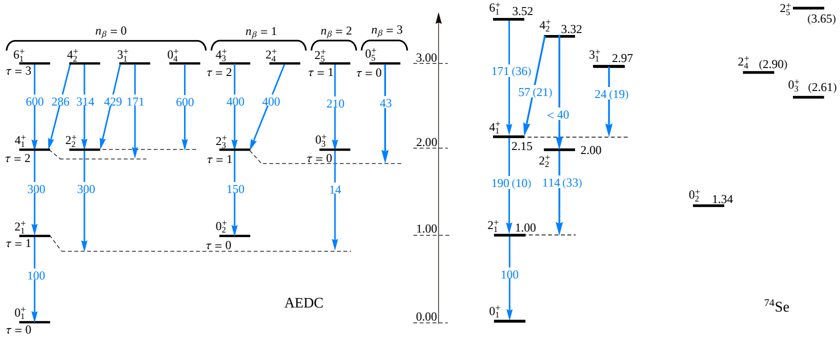

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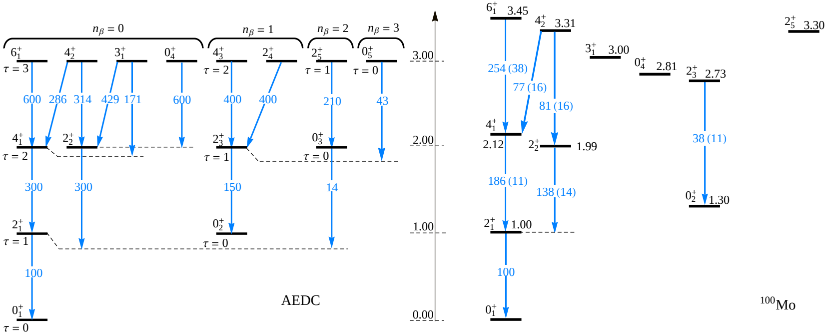

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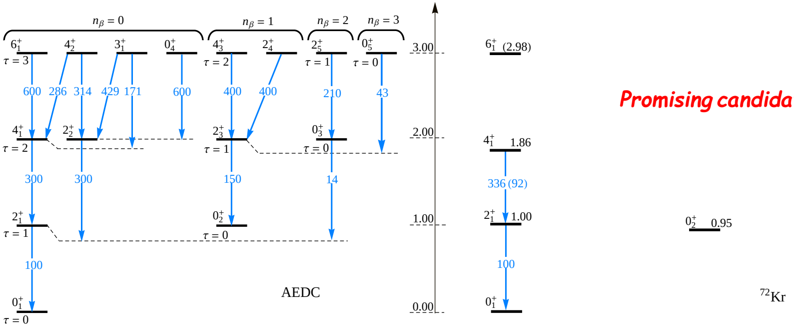

Promising candidate

Experimental realizations [Budaca, EPJA **52** (2016) 314]

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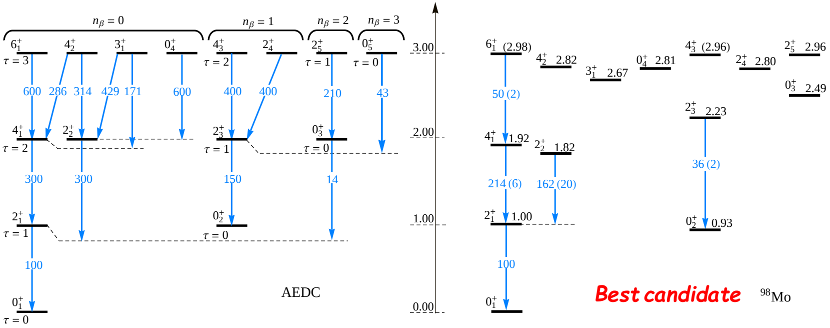


Experimental realizations [Budaca, EPJA **52** (2016) 314]

- The equal shape phase space mixing between γ-stable conditions of the $X(5)$ model and the γ-rigid ones of the $X(3)$ solution, lead to a new four-dimensional CPS called $X(4)$.
- $X(4)$ exhibits properties of the Euclidean dynamical symmetry in four dimensions, *i.e.* 0^+ and 8^+ states satisfy exactly the $E(4)$ differential equation.
- New collective spectra and associated phenomenologies are obtained with energy dependent potentials in the mainframe of the Bohr-Mottelson model.
- By considering a fast stiffening spherical harmonic oscillator potential one attained an exactly solvable collective model associated to a near β-rigid spherical vibrator.
- Similarly, an asymptotically increasing coupling constant for a γ-unstable Coulomb-like potential leads to a model associated to an extremely β soft nucleus, whose wave functions are found to exhibit properties that pertain to shape coexistence.

The solutions are parameter free

representing thus useful references along with the cornerstone dynamical symmetries.

The kinetic energy of the collective Hamiltonian is a Laplacian operator in curvilinear coordinates [Prochniak&Rohozinski, JPG 36 (2009) 123101]:

$$\hat{T} = -\frac{\hbar^2}{2}\nabla^2 = -\frac{\hbar^2}{2}\sum_{lm}\frac{1}{J}\frac{\partial}{\partial x^l}J\bar{G}^{lm}\frac{\partial}{\partial x^m}.$$

G_{lm} - symmetric positive-definite bitensor matrix

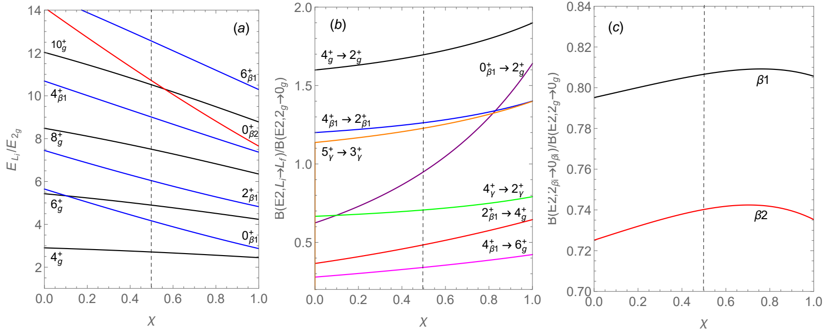
$J = \sqrt{det(g)}$ - Jacobian of the transformation $\{q_k\} \longrightarrow \{x^l\} = \{\beta, \gamma, \theta_1, \theta_2, \theta_3\}$

$g_{lm} = \sum_k \frac{\partial q_k}{\partial x^l}\frac{\partial q_k}{\partial x^m}$ metric tensor of the transformation

- In the general five-dimensional Bohr model, $G_{lm} \sim g_{lm}$ and the kinetic operator acquires the well known form of the Laplace-Beltrami operator.
- This is no longer valid if we want to introduce the rigidity dependence.
- The χ dependent weighting factor arises naturally if we consider the following mass tensor components:

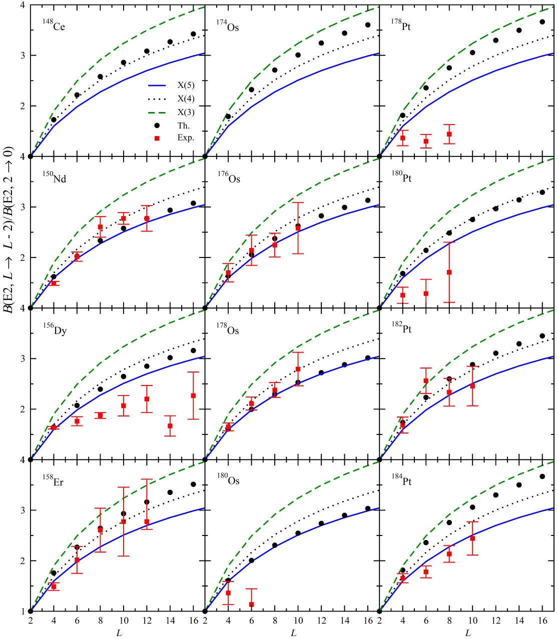
$$\begin{aligned} G_{lm} &= 0,\ l \neq m,\ G_{\beta\beta} = B,\ G_{\gamma\gamma} = \frac{B}{1-\chi}, \\ G_{kk} &= \frac{4B\beta^2}{1-\chi\delta_{k,3}}\sin^2\gamma_k,\ \gamma_k = \gamma - \frac{2k\pi}{3},\ k = 1, 2, 3. \end{aligned}$$

☞ Infinite inertial parameters for the conjugate momentum of the γ shape variable and the angular velocity $\omega_3 = \dot{\theta}_3$ around the third intrinsic axis in the γ-rigid limit.

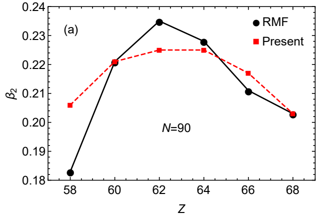


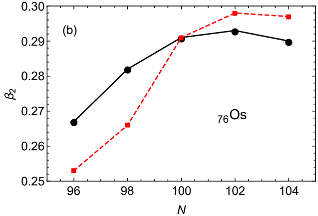
(a) The low-lying energy spectrum of ground and first two β excited bands

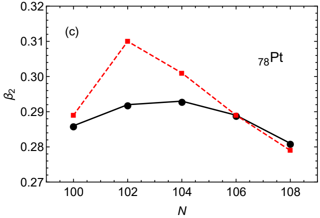
(b) and (c) Few $\Delta K = 0$ $B(E2)$ transition probabilities



^{148}Ce
^{174}Os
^{178}Pt
^{150}Nd
^{176}Os
^{180}Pt
^{156}Dy
^{178}Os
^{182}Pt
^{158}Er
^{180}Os
^{184}Pt
X(5)
X(4)
X(3)
Th.
Exp.
$B(\mathrm{E2}, L \to L-2)/B(\mathrm{E2}, 2 \to 0)$
L





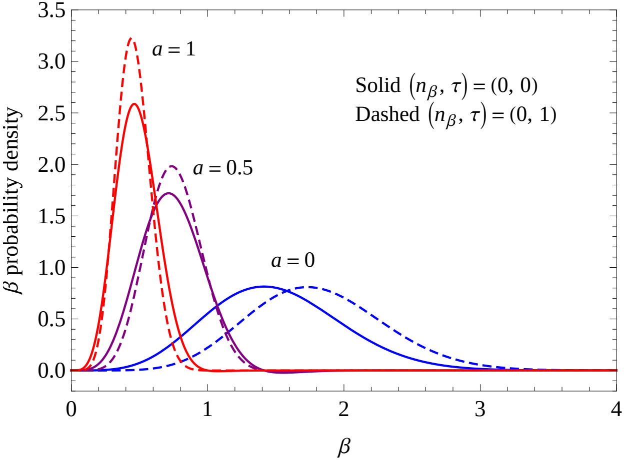


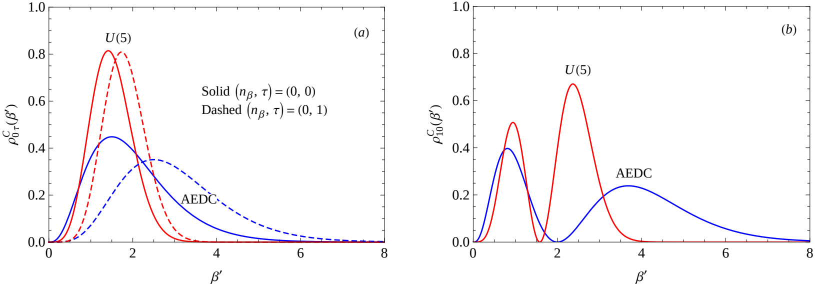
The quadrupole deformation β_2 calculated with relativistic mean-field theory [Lalazissis&Raman, ADNDT **71** (1999) 1] and the scaled ground state average of β.

Everything **OK** in the asymptotic limit of the a parameter

$$\lim_{a\to\infty} \rho(\beta) \sim \delta(\beta - \beta_0), \;\; \beta_0 \to 0.$$

$\delta(x)$ - positive definite.

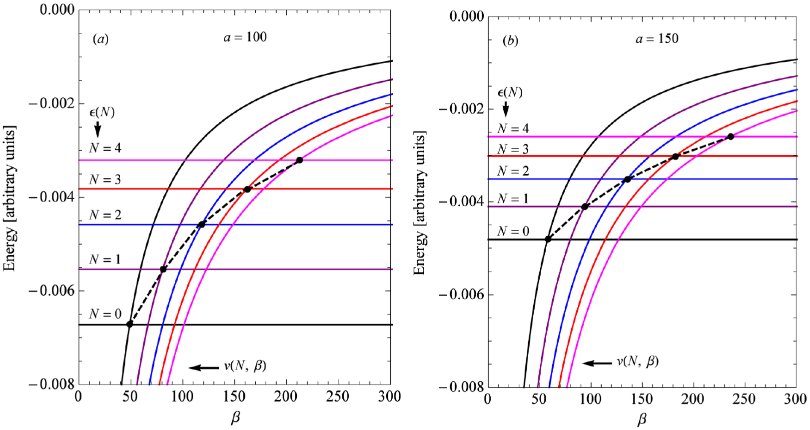




(a) Ground state and $\tau = 1$ excited state β density probability in the present and $U(5)$ cases as a function of β'. For $U(5)$ $\beta' = \beta$, while for AEDC $\beta' = \beta/\sqrt{a}$.

(b) The same but for the first β excited state density probability.

$$\frac{\langle\beta'\rangle_{1,0} - \langle\beta'\rangle_{0,0}}{\langle\beta'\rangle_{0,0}} = \begin{cases} 0.3, & \text{for } U(5) \\ 0.5, & \text{for AEDC.} \end{cases}$$



- Energy levels for the first few states are visualized together with their associated state-dependent potentials $v(\epsilon(N), \beta) = v(N, \beta)$ for a = 100 (a) and a = 150 (b). For clarity, the corresponding intersections are marked with expanded dots which are linked by straight lines in order to simulate a smooth evolution.